Cosine fixed point constant and related sequences

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Abstract: This note presents some sequences related with the cosine fixed point constant.

1. Introduction. The cosine fixed point constant d is defined by

$$d = \lim_{n\to\infty} \cos^{[n]}(x) = \lim_{n\to\infty} \underbrace{\cos\left(\cos\left(\cos\left(\ldots\left(\cos x\right)\right)\right)\right)}_{n} = 0.739085\ldots \qquad (1)$$

The limit (1) is independent of x .
Equation for d :

$$\cos d = d \qquad (2)$$

The cosine fixed point constant is the unique real root of $\cos x = x$.

2. Related Sequences.

- Sequence 1. (Convergent)

$$x_{n+1} = \cos x_n \quad , x_0 = 1 \Rightarrow x_n \to d \qquad (3)$$

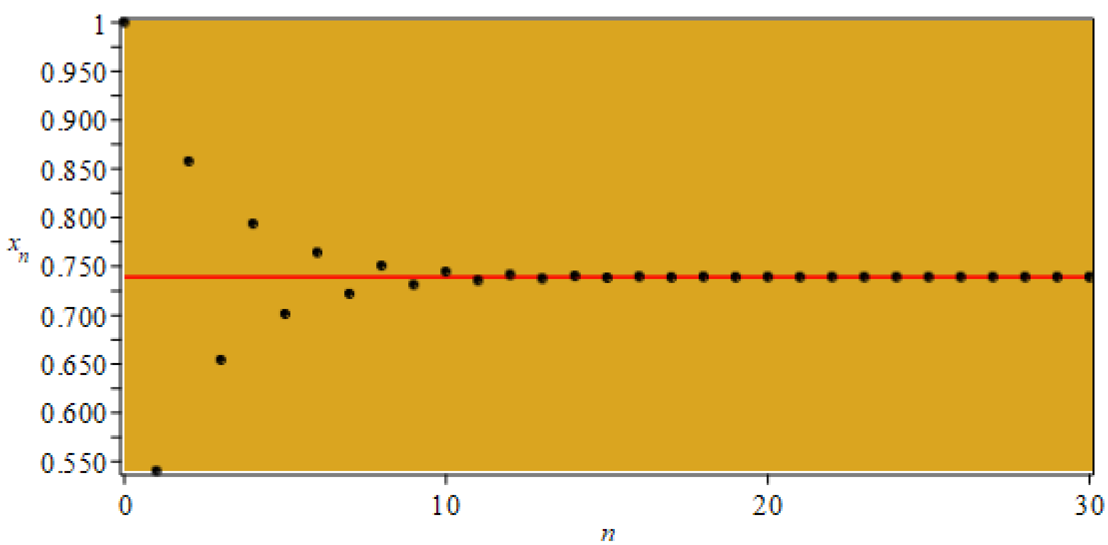

Figure 1. • x_n , • d

- Sequence 2. (Divergent, Oscillation)

$$x_{n+1} = \frac{1}{2}\sin^{-1}\left(2x_n\sqrt{1-x_n^2}\right) \quad ,0 < x_0 < 1 \tag{4}$$

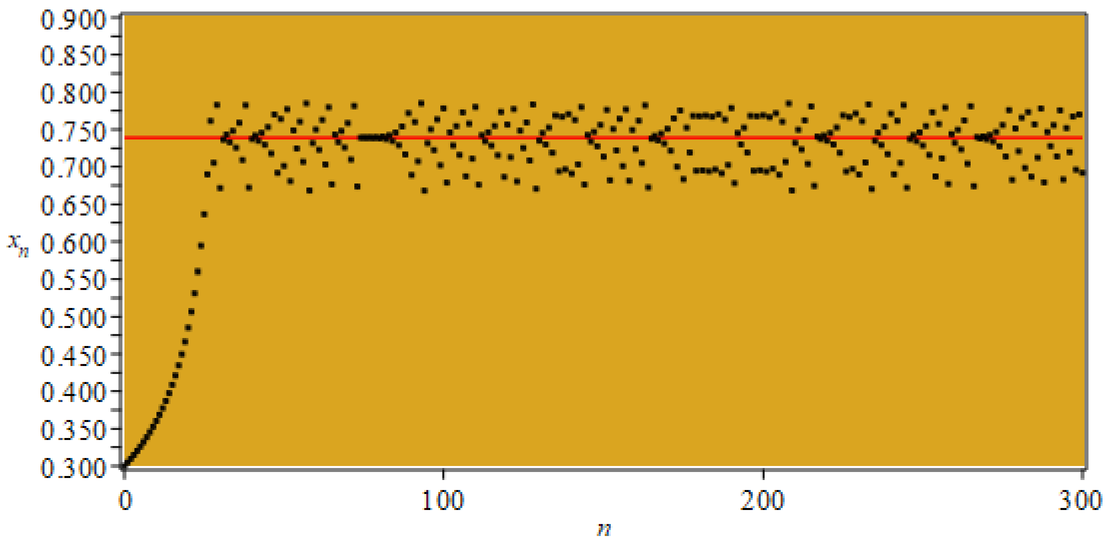

Figure 2. $x_0 = 0.3$, • x_n , • d

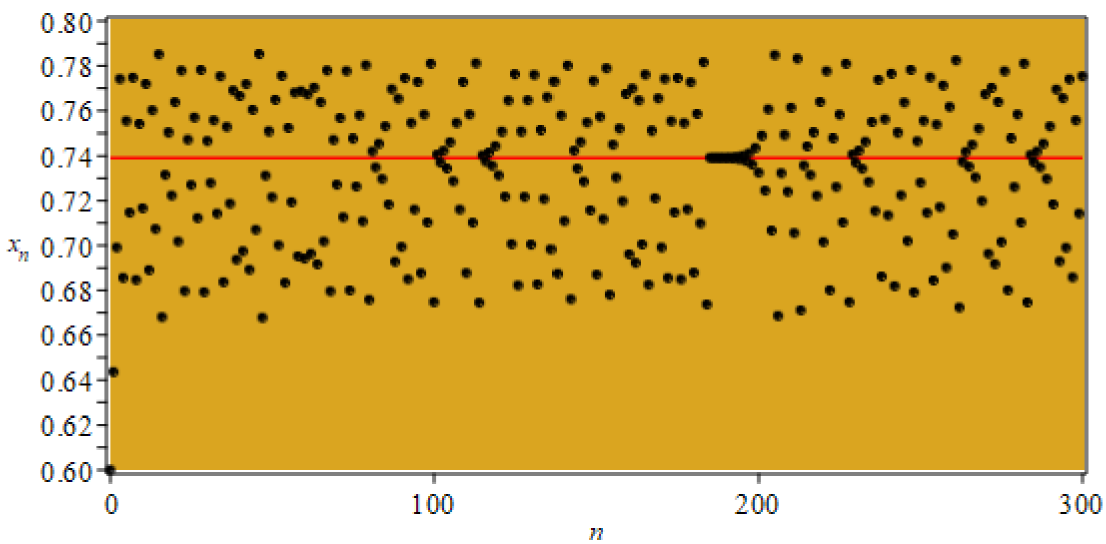

Figure 3. $x_0 = 0.6$, • x_n , • d

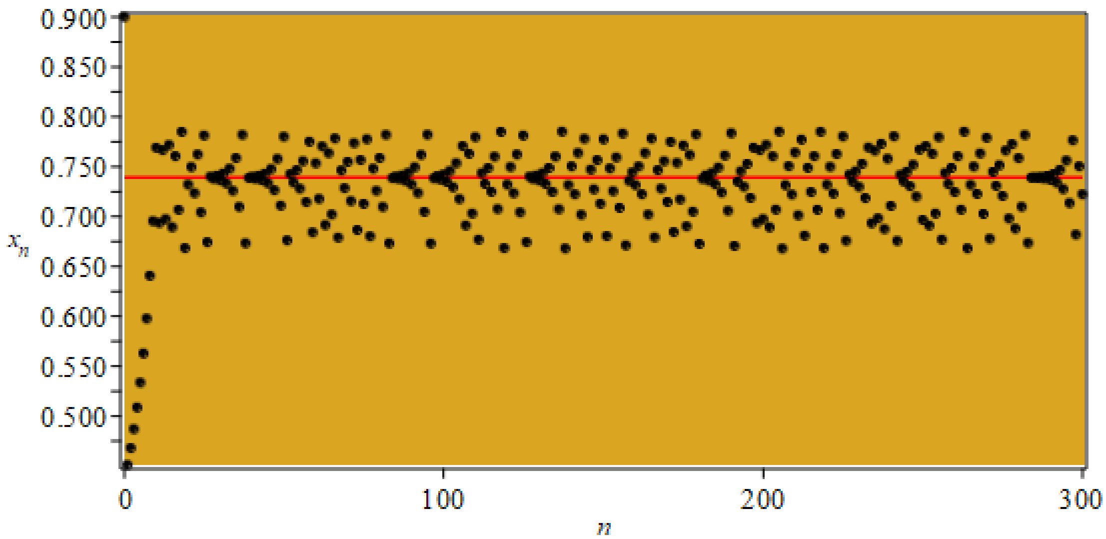

Figure 4. $x_0 = 0.9$, • x_n , • d

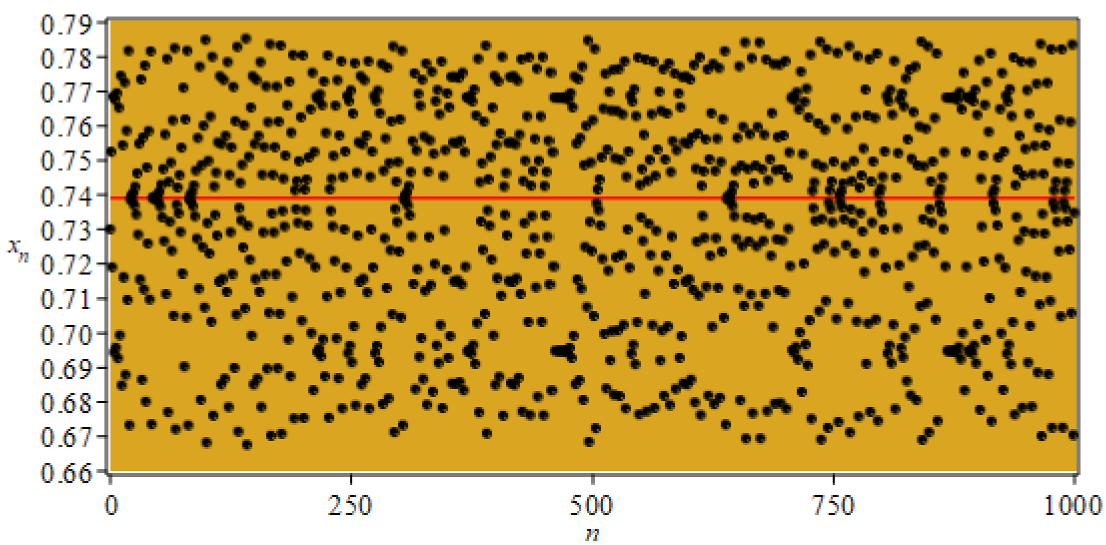

Figure 5. $x_0 = 0.73, n = 1000$, • x_n , • d

- Sequence 3. (Convergent)

$$x_{n+2}=\frac{1}{2}f\left(x_{n+1}\right)+\frac{1}{2}f\left(x_n\right)\quad ,x_0,x_1\in(0,1)\Rightarrow x_n\to d \tag{5}$$

$$f\left(x\right)=\frac{1}{2}\sin^{-1}\left(2x\sqrt{1-x^2}\right) \tag{6}$$

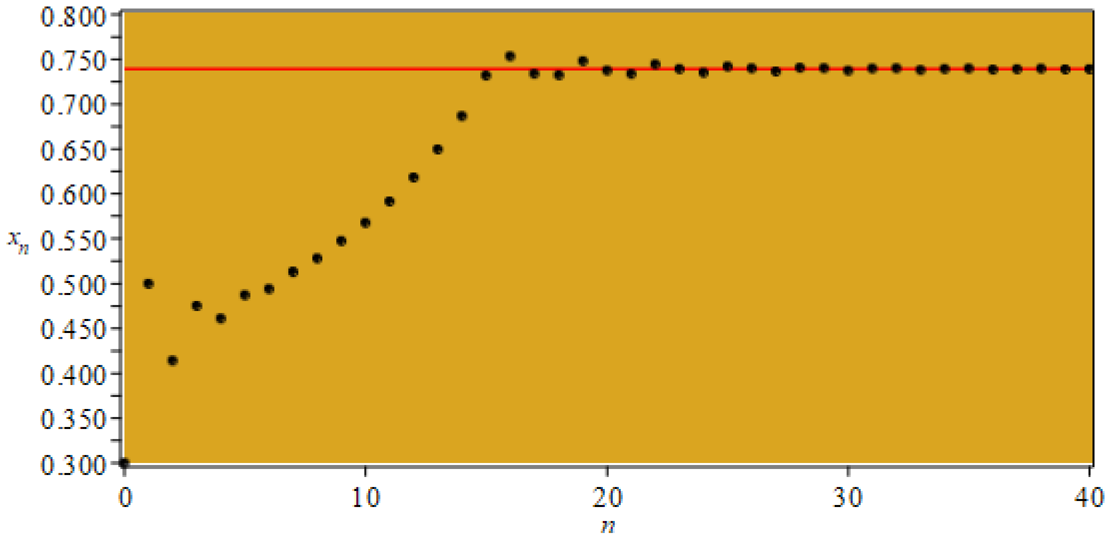

Figure 6. $x_0=0.3, x_1=0.5$, $n=40$, $\bullet\, x_n$, $\bullet\, d$

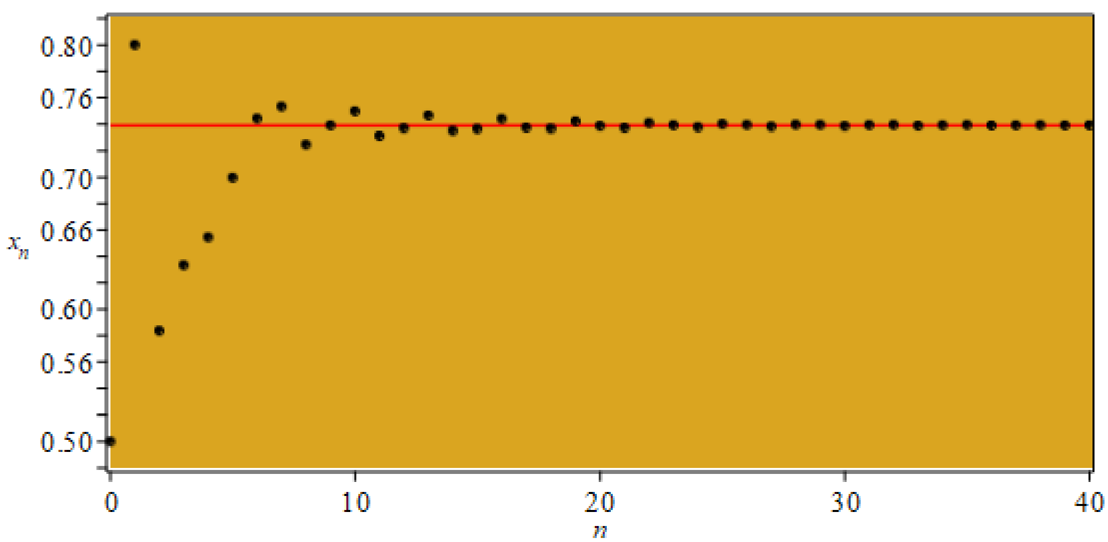

Figure 7. $x_0=0.5, x_1=0.8$, $n=40$, $\bullet\, x_n$, $\bullet\, d$

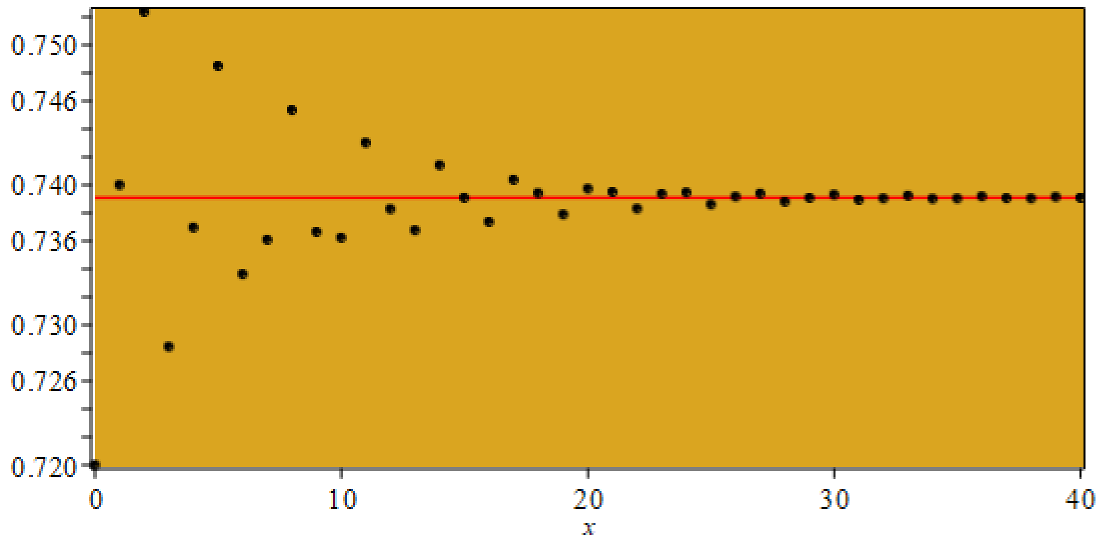

Figure 8. $x_0 = 0.72, x_1 = 0.74$, $n = 40$, • x_n , • d

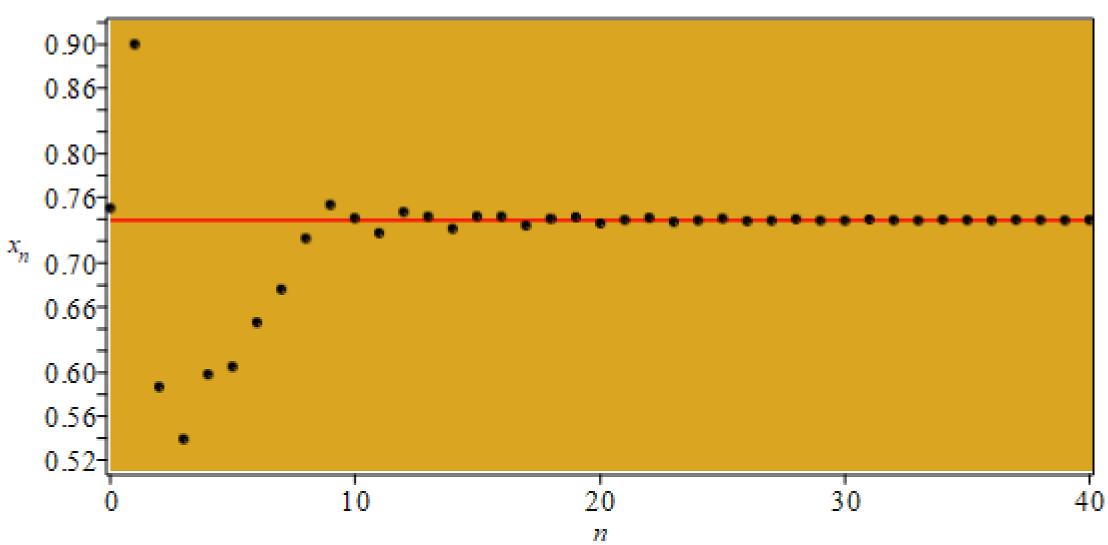

Figure 9. $x_0 = 0.75, x_1 = 0.9$, $n = 40$, • x_n , • d

3. Some Remaks:

Let $f(x) = \frac{1}{2}\sin^{-1}\left(2x\sqrt{1-x^2}\right) \quad , x \in [0,1]$, then

- $f([0,1]) \subset [0,1]$
- $|f'(x)| \geq 1 \quad , x \in [0,1]$

$$f'(x) = \frac{1-2x^2}{\sqrt{1-x^2}\,|1-2x^2|} = \frac{sign(1-2x^2)}{\sqrt{1-x^2}} \quad , 0 \leq x \leq 1 \tag{7}$$

$$sign(x) = \begin{cases} +1 & , x > 0 \\ 0 & , x = 0 \\ -1 & , x < 0 \end{cases} \tag{8}$$

- If $x_0 \in [0,1], x_{n+1} = f(x_n)$, then $0 \leq x_n \leq \pi/4, n \in \mathbb{N}$.
- $0 < x_0 < 1, x_0 \neq d, x_{n+1} = f(x_n) \Rightarrow x_n$ is Divergent (oscillation).
- If $n \in \mathbb{N} \cup \{0\}, x_0, x_1 \in (0,1), x_{n+2} = \frac{f(x_{n+1}) + f(x_n)}{2}$, then

$$|x_{n+3} - x_{n+2}| \leq \frac{2}{\sqrt{16-\pi^2}} |x_{n+2} - x_n| \tag{9}$$

- $x, y \in (0,1) \Rightarrow 0 < \frac{f(x) + f(y)}{2} \leq \pi/4$.
- If $n \in \mathbb{N} \cup \{0\}, x_0, x_1 \in (0,1), x_{n+2} = \frac{f(x_{n+1}) + f(x_n)}{2}$, then

$$|x_{n+2} - d| \leq \frac{2}{\sqrt{16-\pi^2}} \left(|x_{n+1} - d| + |x_n - d|\right) \quad , n = 2,3,4,... \tag{10}$$

References

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