An Interesting Integral

Edgar Valdebenito

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Abstract. This note presents a definite integral related with the Euler-Mascheroni constant.

1. Introduction

Notation:

- Euler-Mascheroni constant :

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...+\frac{1}{n}-\ln n\right)=0.577215... \tag{1}$$

- Bessel function:

$$J_0(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!^2}\left(\frac{x}{2}\right)^{2n} \tag{2}$$

2. Integral

$$\int_1^{\infty}\frac{J_0^2(x)-e^{-x^2}\cos x}{x}dx = \ln 2-\frac{\gamma}{2}+ + \frac{1}{4}\,{}_2F_2\left(1,1;\frac{3}{2},2;-\frac{1}{4}\right)-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{k=0}^{n}\left(\frac{1}{(2n-2k)!k!}-\frac{2^{-2n}}{((n-k)!k!)^2}\right) \tag{3}$$

Remarks:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{k=0}^{n}\frac{1}{k!(2n-2k)!}=-\sum_{n=1}^{\infty}\frac{2^{2n}}{n(2n)!}U\left(-n,\frac{1}{2},-\frac{1}{4}\right) \tag{4}$$

- $U(a,b,z)$ is the confluent hypergeometric function:

$$U(a,b,z)=\frac{1}{\Gamma(a)}\int_0^{\infty}e^{-zt}t^{a-1}(1+t)^{b-a-1}dt \quad ,\mathrm{Re}(a)>0 \tag{5}$$

$$U(-n,b,z)=(-1)^n\sum_{k=0}^{n}\binom{n}{k}(b+k)_{n-k}(-z)^k \tag{6}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{k=0}^{n}\frac{2^{-2n}}{(k!(n-k)!)^2}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{-2n}}{n(n!)^2}\binom{2n}{n}=\frac{1}{2}\,{}_3F_4\left(1,1,\frac{3}{2};2,2,2,2;-1\right) \tag{7}$$

$$\int_1^{\infty} \frac{J_0^2(x)}{x} dx = -\gamma + \ln 2 + \frac{1}{4} {}_3F_4\left(1,1,\frac{3}{2};2,2,2,2;-1\right) \tag{8}$$

$$\gamma + 2\int_1^{\infty} \frac{e^{-x^2}}{x} dx = \\ = e^{-1}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{(2n)!}\sum_{k=0}^{n}\frac{1}{k!} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{k=0}^{n}\frac{1}{k!(2n-2k)!} - \frac{1}{2} {}_2F_2\left(1,1;\frac{3}{2},2;-\frac{1}{4}\right) \tag{9}$$

$$2\int_1^{\infty} \frac{e^{-x^2}}{x} dx = \int_1^{\infty} \frac{e^{-x}}{x} dx \tag{10}$$

- ${}_pF_q\left(a_1,...,a_p;b_1,...,b_q;z\right)$ is the classical hypergeometric function:

$${}_pF_q\left(a_1,...,a_p;b_1,...,b_q;z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n ...(a_p)_n}{(b_1)_n ...(b_q)_n} \frac{z^n}{n!} \tag{11}$$

with $(a)_n = a(a+1)...(a+n-1), (a)_0 = 1$.

References

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