Entanglement condition for W type multimode states and a scheme for experimental realization

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Abstract

We derive a class of inequality relations, using both the sum uncertainty relations of $su(2)$ algebra operators and the Schrodinger-Robertson uncertainty relation of partially transposed $su(1,1)$ algebra operators, to detect the three-mode entanglement of non-Gaussian states of electromagnetic field. These operators are quadratic in mode creation and annihilation operators. The inseparability condition obtained using $su(2)$ algebra operators is shown to guarantee the violation of stronger separability condition provided by Schrodinger-Robertson uncertainty relation of partially transposed $su(1,1)$ algebra operators. The obtained inseparability condition is also shown to be a necessary condition for W type entangled states and it is used to derive the general form for a family of such inseparability conditions. An experimental scheme is proposed to test the violation of separability condition. The results derived for three-mode systems are generalized to multimode systems.

Introduction

Quantum entanglement, a nonlocal trait of composite quantum systems, is of fundamental interest in quantum information theory. Both the discrete and continuous variable entanglements play vital role in quantum information processing. However, continuous variable (CV) quantum information processing protocols can readily be implemented on optical systems. Quantum teleportation [1, 2], super dense coding [3, 4], quantum cryptography [5, 6] and quantum error corrections [7, 8] were demonstrated using Gaussian entangled states. Advanced quantum information processing protocols, such as universal quantum computation, were proposed using non-Gaussian entangled states [9, 10]. For complete review see [11, 12]. CV entanglement were even shown to improve the resolution in quantum metrology [13, 14, 15]. Recently the cat code has been developed with CV states [16, 17, 18]. Despite many applications of quantum entanglement, the basic problem of detection and quantification of entanglement remains largely unexplored in the multipartite scenario [19] and references therein. For discrete variable quantum systems, entanglement measures were introduced to quantify the entanglement of bipartite and multipartite systems [20, 21, 22, 23, 24].

Uncertainty relations have been used to derive separability conditions for bipartite CV systems. Two different approaches were used: The idea behind the first approach is that the sum of variances of two or more non-commuting observables admits a state-independent lower bound due to their algebraic structure [25],[26] and references therein. Any quantum state has to satisfy this lower bounds, however, the separable states tend to impose a stronger lower bound and finding this stronger lower bound provides a separability condition for bipartite CV systems. This approach was used to derive the separability criteria for bipartite Gaussian states [27, 28, 29]. In the second approach, the uncertainty relations and other necessary conditions for a quantum state were used in conjunction with partial transposition (PT). Under PT, the density matrix of a separable bipartite system transforms into a valid density matrix whereas at least one eigenvalue of the density matrix of entangled systems becomes negative. This positivity under partial transposition (PPT) is a necessary and sufficient separability condition of the (2×2)- and (2×3)-dimensional systems [30, 31]. Simon showed that the PT of density matrix corresponds to the time reversal operation in the subspace of bipartite CV system and also the PPT is a necessary and sufficient condition for the separability of any bipartite Gaussian states [32]. Violation of uncertainty principle, which normally holds for any density matrix, under PT provides a stronger inseparability condition. Many of these criteria naturally manifest in optical systems.

Operators describing non-linear optical processes form representations of different Lie algebras and the coherent states of such Lie algebras are non-Gaussian states. Coherent states of different Lie algebras and their applications to non-linear optics have been studied extensively [33, 34, 35]. Agarwal *et al* were the first to point out the necessity of higher order correlations to detect the entanglement of bipartite non-Gaussian states and derived an inseparability condition by using the Heisenberg uncertainty relation of $su(2)$ and $su(1,1)$ algebra operators in conjunction with PT [36]. These $su(2)$ and $su(1,1)$ algebra operators were shown to describe the difference and sum squeezing processes [37]. With the same sets of operators, Hillery *et al* obtained another pair of inseparability conditions for bipartite non-Gaussian states using sum uncertainty relations [38]. However, the result of Agarwal *et al* was shown to be the stronger condition for detecting the entanglement of $su(2)$ minimum-uncertainty states [39]. Much stronger separability condition was obtained from the same $su(2)$ and $su(1,1)$ algebra operators using the Schrodinger-Robertson uncertainty relation (SRUR) in conjunction with PT [40].

Existence of different hierarchy of non-Gaussian multipartite CV entangled states increases the complexity in the detection of multipartite CV entanglement. The method of sum uncertainty relations was used to derive inseparability conditions for detecting biseparable and fully entangled multimode states [41]. Different ways to derive separability conditions for multipartite CV systems using PT along with necessary conditions of quantum states was emphasized in [42]. Along these lines, the violation of uncertainty relations by entangled multipartite CV states under PT was proposed as inseparability criterion [43, 44]. However, these multipartite inseparability criteria do not distinguish the genuine W type entangled states from the separable states. The characteristic of n-partite W type entangled states is that any two arbitrary subsystems obtained by tracing out other $(n-2)$ subsystems remain entangled [45, 46]. Different schemes were proposed to prepare single photon and multiphoton n-mode W type states and the presence of W type entanglement was demonstrated only by detecting the pairwise entanglement of each of $n(n-1)/2$ pairs of modes [47, 48]. Hence, an explicit inseparability condition to detect the genuine W type CV entangled states needs to be derived.

In this paper, we derive a class of inequality relations to detect the entanglement of multimode non-Gaussian states. In contrary to other n-mode inseparability conditions [41, 43, 44], where the chosen operators are n^{th} order in mode creation and annihilation operators, we consider two sets of quadratic operators satisfying $su(2)$ and $su(1,1)$ algebras. We derive an inseparability condition for three-mode states by using two different approaches discussed above. First we consider the sum of uncertainties of $su(2)$ algebra operators to derive the inseparability condition and show it to be a necessary condition for W type entangled states. The same inseparability condition is shown to be a sufficient condition to violate the SRUR of partially transposed $su(1,1)$ algebra operators. An experimental scheme is proposed to test this violation using linear optical setup. Finally, we provide a family of inseparability conditions for three-mode states and generalize it to multimode states. These family of inseparability conditions along with other inseparability conditions proposed earlier [41, 43] can be used to characterize the entanglement of multipartite CV systems completely.

Inseparability condition for three-mode states

Let $\hat{a}$, $\hat{b}$ and $\hat{c}$ be the annihilation operators of three modes of electromagnetic field. We define the following set of three operators:

$$\begin{aligned}\hat{L}_1 &= \frac{1}{\sqrt{2}}\big[(\hat{a}+\hat{b})\hat{c}^\dagger + (\hat{a}+\hat{b})^\dagger\hat{c}\big],\\ \hat{L}_2 &= \frac{i}{\sqrt{2}}\big[(\hat{a}+\hat{b})\hat{c}^\dagger - (\hat{a}+\hat{b})^\dagger\hat{c}\big],\\ \hat{L}_3 &= \hat{N}_{a+b} - \hat{N}_c,\end{aligned} \tag{1}$$

where $\hat{N}_{a+b} = \dfrac{(\hat{a}+\hat{b})^\dagger(\hat{a}+\hat{b})}{2}$.

These three operators satisfy a $su(2)$ algebra, $[\hat{L}_p, \hat{L}_q] = 2i\epsilon_{pqr}\hat{L}_r$. Calculating the variances of $\hat{L}_1$ and $\hat{L}_2$ and adding them,

$$(\Delta\hat{L}_1)^2 + (\Delta\hat{L}_2)^2 = 2\big[\langle\hat{N}_{a+b}(\hat{N}_c+1)\rangle + \langle(\hat{N}_{a+b}+1)\hat{N}_c\rangle - |\langle(\hat{a}+\hat{b})\hat{c}^\dagger\rangle|^2\big]. \tag{2}$$

If the state under consideration is a pure product state then the sum of variances become,

$$(\Delta\hat{L}_1)^2 + (\Delta\hat{L}_2)^2 = 2\big[\langle\hat{N}_{a+b}\rangle\langle\hat{N}_c+1\rangle + \langle\hat{N}_{a+b}+1\rangle\langle\hat{N}_c\rangle - |\langle(\hat{a}+\hat{b})\rangle\langle\hat{c}^\dagger\rangle|^2\big]. \tag{3}$$

The Schwarz inequality, $|\langle\hat{a}\rangle|^2 \le \langle\hat{N}_a\rangle$, $|\langle\hat{b}\rangle|^2 \le \langle\hat{N}_b\rangle$ and $|\langle\hat{c}^\dagger\rangle|^2 \le \langle\hat{N}_c\rangle$, implies the following inequality relation for the pure product states,

$$(\Delta\hat{L}_1)^2 + (\Delta\hat{L}_2)^2 \;\ge\; 2\big[\langle\hat{N}_{a+b}\rangle + \langle\hat{N}_c\rangle\big]. \tag{4}$$

This inequality can be extended to any separable state as shown in [38]. Hence, an inseparable state should violate this inequality. From eqs. (2) and (4), a three-mode state is entangled, if it satisfies the following condition:

$$|\langle(\hat{a}+\hat{b})\hat{c}^{\dagger}\rangle|^2 \;>\; 2\langle\hat{N}_{a+b}\hat{N}_c\rangle. \tag{5}$$

There exists states that satisfy this inequality relation. To show this, we consider the Heisenberg uncertainty relation of $\hat{L}_1$ and $\hat{L}_2$.

$$(\Delta\hat{L}_1)(\Delta\hat{L}_2) \;\geq\; \frac{1}{2}|[\hat{L}_1,\hat{L}_2]|. \tag{6}$$

This implies,

$$(\Delta\hat{L}_1)^2+(\Delta\hat{L}_2)^2 \;\geq\; 2|\langle\hat{N}_{a+b}-\hat{N}_c\rangle|. \tag{7}$$

This relation holds for any state. This lower bound of $(\Delta L_1)^2+(\Delta L_2)^2$ is smaller than the lower bound set by separable states. Hence there are states that satisfy eq. (7) and violate eq. (4). Such states are entangled and they obey eq. (5).

The inequality relation represented by eq. (5) is a necessary condition for W type entangled states. To show this, we consider a density matrix $\hat{\rho}$ representing the state of three modes of electromagnetic field. If the reduced density matrix $\hat{\rho}_{ac}$ satisfies the bipartite inseparability condition,

$$|\langle\hat{a}\hat{c}^{\dagger}\rangle_{\hat{\rho}_{ac}}|^2 \;>\; \langle\hat{N}_a\hat{N}_b\rangle_{\hat{\rho}_{ac}}, \tag{8}$$

then $\hat{\rho}$ cannot be written as either $\sum_j p_j\hat{\rho}_{a_j}\otimes\hat{\rho}_{bc_j}$ or $\sum_j p_j\hat{\rho}_{ab_j}\otimes\hat{\rho}_{c_j}$. Similarly, if the reduced density matrix $\hat{\rho}_{bc}$ satisfies the condition,

$$|\langle\hat{b}\hat{c}^{\dagger}\rangle_{\hat{\rho}_{bc}}|^2 \;>\; \langle\hat{N}_b\hat{N}_c\rangle_{\hat{\rho}_{bc}}, \tag{9}$$

then $\hat{\rho}$ cannot be written as $\sum_j p_j\hat{\rho}_{ac_j}\otimes\hat{\rho}_{b_j}$.

Thus if the reduced density matrices, $\hat{\rho}_{ac}$ and $\hat{\rho}_{bc}$ satisfy eqs. (8) and (9) respectively, then $\hat{\rho}$ represents a genuinely entangled W type state [49] and it satisfies the following condition,

$$|\langle\hat{a}\hat{c}^{\dagger}\rangle_{\hat{\rho}}|^2+|\langle\hat{b}\hat{c}^{\dagger}\rangle_{\hat{\rho}}|^2 \;>\; \langle\hat{N}_a\hat{N}_c\rangle_{\hat{\rho}}+\langle\hat{N}_b\hat{N}_c\rangle_{\hat{\rho}}. \tag{10}$$

Since, for separable states, $\langle\hat{a}\hat{c}^{\dagger}\rangle\langle\hat{b}^{\dagger}\hat{c}\rangle \;\leq\; \langle\hat{a}\hat{b}^{\dagger}\hat{N}_c\rangle$ and $\langle\hat{b}\hat{c}^{\dagger}\rangle\langle\hat{a}^{\dagger}\hat{c}\rangle \;\leq\; \langle\hat{b}\hat{a}^{\dagger}\hat{N}_c\rangle$, W type states satisfy eq. (5). Hence the inequality relation represented by eq. (5) is a necessary condition for W type entangled states.

Method based on partial transposition (PT)

For two non-commuting observables $\hat{A}$ and $\hat{B}$, the Schrodinger-Robertson uncertainty relation (SRUR) [43, 50] is written as,

$$(\Delta\hat{A})^2(\Delta\hat{B})^2 \;\geq\; \frac{1}{4}\Big|\langle[\hat{A},\hat{B}]\rangle\Big|^2+\left|\frac{\langle\hat{A}\hat{B}+\hat{B}\hat{A}\rangle}{2}-\langle\hat{A}\rangle\langle\hat{B}\rangle\right|^2. \tag{11}$$

This inequality relation is valid for any quantum state. Under partial transposition of one of the subsystem, the violation of this inequality relation implies the entanglement of partially transposed subsystem with the rest of the subsystems.

Now we consider the following set of operators:

$$\hat{K}_1=\frac{1}{\sqrt{2}}[(\hat{a}+\hat{b})\hat{c}+(\hat{a}+\hat{b})^{\dagger}\hat{c}^{\dagger}],$$

$$\hat{K}_2=\frac{i}{\sqrt{2}}[(\hat{a}+\hat{b})\hat{c}-(\hat{a}+\hat{b})^{\dagger}\hat{c}^{\dagger}],$$

$$\hat{K}_3=\hat{N}_{a+b}+\hat{N}_c+1. \tag{12}$$

These operators satisfy a $su(1,1)$ algebra, $[\hat{K}_1,\hat{K}_2]=-2i\hat{K}_3$, $[\hat{K}_2,\hat{K}_3]=2i\hat{K}_1$ and $[\hat{K}_3,\hat{K}_1]=2i\hat{K}_2$. Under the partial transposition of the third mode ($\hat{c}$), the SRUR of $\hat{K}_1$ and $\hat{K}_2$ can be written as

$$(\Delta\hat{K}_1)^2_{\rho^{PT}}(\Delta\hat{K}_2)^2_{\rho^{PT}} \;\geq\; |\langle\hat{K}_3\rangle_{\rho^{PT}}|^2+\left|\frac{\langle\hat{K}_1\hat{K}_2+\hat{K}_2\hat{K}_1\rangle_{\rho^{PT}}}{2}-\langle\hat{K}_1\rangle_{\rho^{PT}}\langle\hat{K}_2\rangle_{\rho^{PT}}\right|^2. \tag{13}$$

We have the following relations:

$$(\Delta\hat{K}_1)^2_{\rho^{PT}}=1+(\Delta\hat{L}_1)^2,$$

$$(\Delta\hat{K}_2)^2_{\rho^{PT}} = 1 + (\Delta\hat{L}_2)^2,$$

$$\langle\hat{K}_3\rangle_{\rho^{PT}} = \langle\hat{K}_3\rangle = \langle 1 + \hat{N}_{a+b} + \hat{N}_c\rangle, \tag{14}$$

Hence,

$$\left|\frac{\langle\hat{K}_1\hat{K}_2 + \hat{K}_2\hat{K}_1\rangle_{\rho^{PT}}}{2} - \langle\hat{K}_1\rangle_{\rho^{PT}}\langle\hat{K}_2\rangle_{\rho^{PT}}\right|^2 = \left|\frac{\langle\hat{L}_1\hat{L}_2 + \hat{L}_2\hat{L}_1\rangle}{2} - \langle\hat{L}_1\rangle\langle\hat{L}_2\rangle\right|^2 = \left[\Im\left\{[\Delta(\hat{a}+\hat{b})^\dagger\hat{c}]^2\right\}\right]^2. \tag{15}$$

Substituting these expressions in eq. (13) we get

$$[1 + (\Delta\hat{L}_1)^2][1 + (\Delta\hat{L}_2)^2] \ \geq\ [1 + \langle\hat{N}_{a+b}\rangle + \langle\hat{N}_c\rangle]^2 + \left[\Im\left\{[\Delta(\hat{a}^\dagger + \hat{b}^\dagger)\hat{c}]^2\right\}\right]^2. \tag{16}$$

This inequality relation represents a stronger separability condition than the one represented by eq. (4). To experimentally verify the violation of this separability condition, one has to measure the operators $\hat{L}_1$, $\hat{L}_2$, $\hat{L}_1\hat{L}_2 + \hat{L}_2\hat{L}_1$ and $\hat{N}_{a+b} + \hat{N}_c$. An experimental scheme to measure these operators is demonstrated in the next section.

Substituting the expressions of $[1 + (\Delta\hat{L}_1)^2]$ and $[1 + (\Delta\hat{L}_2)^2]$ in eq. (15) leads to the following inequality relation:

$$\left[2\langle\hat{N}_{a+b}\hat{N}_c\rangle - |\langle(\hat{a}+\hat{b})\hat{c}^\dagger\rangle|^2\right]\left[2\langle(\hat{N}_{a+b}+1)(\hat{N}_c+1)\rangle - |\langle(\hat{a}+\hat{b})\hat{c}^\dagger\rangle|^2\right] \ \geq\ \left[\Re\left\{[\Delta(\hat{a}+\hat{b})^\dagger\hat{c}]^2\right\}\right]^2 + \left[\Im\left\{[\Delta(\hat{a}+\hat{b})^\dagger\hat{c}]^2\right\}\right]^2. \tag{17}$$

If we use Heisenberg uncertainty relation (HUR) instead of SRUR then we get,

$$\left[2\langle\hat{N}_{a+b}\hat{N}_c\rangle - |\langle(\hat{a}+\hat{b})\hat{c}^\dagger\rangle|^2\right]\left[2\langle(\hat{N}_{a+b}+1)(\hat{N}_c+1)\rangle - |\langle(\hat{a}+\hat{b})\hat{c}^\dagger\rangle|^2\right] \ \geq\ \left[\Re\left\{[\Delta(\hat{a}+\hat{b})^\dagger\hat{c}]^2\right\}\right]^2. \tag{18}$$

Comparing the right sides of eqs. (16) and (17), it is clear that SRUR provides the stronger separability condition than HUR. Since the right sides of both eqs. (16) and (17) are positive quantities, a sufficient condition for entangled states can be written as

$$\left[2\langle\hat{N}_{a+b}\hat{N}_c\rangle - |\langle(\hat{a}+\hat{b})\hat{c}^\dagger\rangle|^2\right]\left[2\langle(\hat{N}_{a+b}+1)(\hat{N}_c+1)\rangle - |\langle(\hat{a}+\hat{b})\hat{c}^\dagger\rangle|^2\right] \ <\ 0. \tag{19}$$

This condition is satisfied if, and only if, the following two inequality relations are satisfied simultaneously,

$$|\langle(\hat{a}+\hat{b})\hat{c}^\dagger\rangle|^2 \ >\ 2\langle\hat{N}_{a+b}\hat{N}_c\rangle,$$

$$|\langle(\hat{a}+\hat{b})\hat{c}^\dagger\rangle|^2 \ <\ 2\langle(\hat{N}_{a+b}+1)(\hat{N}_c+1)\rangle. \tag{20}$$

The first condition is the inseparability condition [eq. (5)] obtained in the first section. From Schwarz inequality we have

$$|\langle(\hat{a}+\hat{b})\hat{c}^\dagger\rangle|^2 \ \leq\ 2\langle\hat{N}_{a+b}(\hat{N}_c+1)\rangle. \tag{21}$$

Hence the second condition of eq. (19) holds for any state. If eq. (5) is satisfied, it implies the violation of eq. (15), however, the converse need not be true.

Experimental scheme

Violation of inequality relation represented by eq. (15) can be tested experimentally. An experimental scheme to measure the operators involved in eq. (15) is shown in Figure 1. The modes $\hat{a}$ and $\hat{b}$ are injected into a $50:50$ beam splitter (BS_1). The output mode $\hat{e} = (1/\sqrt{2})(-\hat{a}+\hat{b})$ is sent to a beam dump (BD) and $\hat{d} = (1/\sqrt{2})(\hat{a}+\hat{b})$ is mixed with the mode $\hat{c}$ in the second $50:50$ beam splitter (BS_2). The mode $\hat{c}$ undergoes a phase shift by ϕ before injection. The output modes $\hat{f} = (1/\sqrt{2})(\hat{d} + \hat{c}e^{-i\phi})$ and $\hat{g} = (1/\sqrt{2})(-\hat{d} + \hat{c}e^{-i\phi})$ are detected in photodiodes (PD_1) and (PD_2) respectively.

The operators $\hat{L}_1$ and $\hat{L}_2$ can be measured by detecting the photon number difference at the output:

$$\hat{N}_{\{-,\phi\}} = \hat{f}^\dagger\hat{f} - \hat{g}^\dagger\hat{g} = \frac{(\hat{a}+\hat{b})^\dagger\hat{c}e^{-i\phi} + (\hat{a}+\hat{b})\hat{c}^\dagger e^{i\phi}}{\sqrt{2}}. \tag{22}$$

The choice of $\phi = 0$ gives $\hat{L}_1$. The operator $\hat{L}_2$ can be measured with $\phi = \pi/2$. The operator $\hat{L}_1\hat{L}_2 + \hat{L}_2\hat{L}_1$ can be measured in two steps, by choosing $\phi = \pi/4$ and $\phi = -\pi/4$, as shown below.

$$\hat{N}^2_{\{-,\phi=\pi/4\}} - \hat{N}^2_{\{-,\phi=-\pi/4\}} = i\left[(\hat{a}+\hat{b})^2(\hat{c}^\dagger)^2 - (\hat{a}^\dagger+\hat{b}^\dagger)^2\hat{c}^2\right] = \hat{L}_1\hat{L}_2 + \hat{L}_2\hat{L}_1 \tag{23}$$

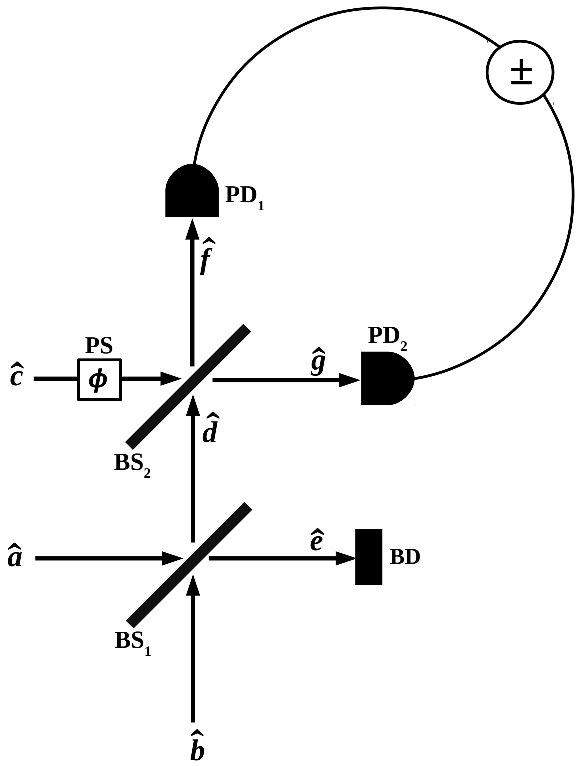

Figure 1: Experimental setup to test the violation of separability condition [eq. (15)]. BS_1 & BS_2, $50:50$ beam splitters; BD, beam dump; PS, phase shifter and PD_1 & PD_2, photodetectors. All the required operators can be measured by detecting the sum and differences of photon numbers at the output with appropriate phase shifts as described in the main text.

Irrespective of the phase shift of mode $\hat{c}$, the sum of photon numbers at the output provides $\hat{N}_{a+b} + \hat{N}_c$. That is,

$$\hat{N}_+ = \hat{f}^\dagger \hat{f} + \hat{g}^\dagger \hat{g} = \hat{N}_{a+b} + \hat{N}_c. \tag{24}$$

Thus the experimental setup shown in Figure 1 can be used to test the violation of separability condition represented by eq. (15).

General form of inseparability condition

The inseparability criterion eq. (5) suggests that there exist a family of such inseparability criteria. They can be obtained by replacing $(\hat{a}+\hat{b})\hat{c}^\dagger$ with $(\hat{a}^p+\hat{b}^q)(\hat{c}^\dagger)^r$. For a pure product state we have,

$$\begin{aligned} |\langle(\hat{a}^p+\hat{b}^q)(\hat{c}^\dagger)^r\rangle|^2 &= |\langle\hat{a}^p\rangle|^2|\langle(\hat{c}^\dagger)^r\rangle|^2 + |\langle\hat{b}^q\rangle|^2|\langle(\hat{c}^\dagger)^r\rangle|^2 + \langle\hat{a}^p\rangle\langle(\hat{b}^\dagger)^q\rangle|\langle\hat{c}^r\rangle|^2 + \langle(\hat{a}^\dagger)^p\rangle\langle\hat{b}^q\rangle|\langle\hat{c}^r\rangle|^2, \\ &\le [\langle(\hat{a}^\dagger)^p\hat{a}^p\rangle + \langle(\hat{b}^\dagger)^q\hat{b}^q\rangle + \langle\hat{a}^p\rangle\langle(\hat{b}^\dagger)^q\rangle + \langle(\hat{a}^\dagger)^p\rangle\langle\hat{b}^q\rangle][\langle(\hat{c}^\dagger)^r\hat{c}^r\rangle]. \end{aligned} \tag{25}$$

Since for a product state $\langle(\hat{a}^\dagger)^p\hat{a}^p\rangle\langle(\hat{c}^\dagger)^q\hat{c}^q\rangle = \langle(\hat{a}^\dagger)^p\hat{a}^p(\hat{c}^\dagger)^q\hat{c}^q\rangle$, we have the following inequality relation,

$$|\langle(\hat{a}^p+\hat{b}^q)(\hat{c}^\dagger)^r\rangle|^2 \le \langle(\hat{a}^p+\hat{b}^q)^\dagger(\hat{a}^p+\hat{b}^q)(\hat{c}^\dagger)^r\hat{c}^r\rangle, \tag{26}$$

holding for any separable state. To show this, we consider the density matrix $\hat{\rho} = \sum_k p_k\hat{\rho}_k$ representing a general separable state, where $\hat{\rho}_k$ is a product state and $\sum_k p_k = 1$. Let us define $\hat{A} = \hat{a}^p$, $\hat{B} = \hat{b}^q$ and $\hat{C} = \hat{c}^r$. Then we have

$$\begin{aligned} |\langle(\hat{A}+\hat{B})\hat{C}^\dagger\rangle| &\le \sum_k p_k|\langle(\hat{A}+\hat{B})\hat{C}^\dagger\rangle_k|, \\ &= \sum_k p_k\sqrt{\langle(\hat{A}+\hat{B})\hat{C}^\dagger\rangle_k\langle(\hat{A}^\dagger+\hat{B}^\dagger)\hat{C}\rangle_k}, \\ &\le \sum_k p_k\sqrt{\langle(\hat{A}+\hat{B})^\dagger(\hat{A}+\hat{B})\hat{C}^\dagger\hat{C}\rangle_k}, \end{aligned}$$

$$\leq \sqrt{\sum_k p_k} \sqrt{\sum_k p_k \langle (\hat{A}+\hat{B})^\dagger (\hat{A}+\hat{B}) \hat{C}^\dagger \hat{C} \rangle_k}.$$

Thus for a general separable state we have

$$|\langle (\hat{A}+\hat{B})\hat{C}^\dagger \rangle| \leq \sqrt{\langle (\hat{A}+\hat{B})^\dagger (\hat{A}+\hat{B}) \hat{C}^\dagger \hat{C} \rangle}. \tag{27}$$

Therefore the general form of inseparability condition can be written as

$$|\langle (\hat{a}^p + \hat{b}^q)(\hat{c}^\dagger)^r \rangle|^2 \ > \ \langle (\hat{a}^p + \hat{b}^q)^\dagger (\hat{a}^p + \hat{b}^q)(\hat{c}^\dagger)^r \hat{c}^r \rangle. \tag{28}$$

This higher order inseparability condition can be used to detect the entanglement of multiphoton three-mode W type states.

Generalization to multimode states

The inseparability criterion derived for three-mode states can be generalized to multimode states. Let $\hat{a}_1$, $\hat{a}_2$,.....,$\hat{a}_n$ be the annihilation operators of n modes of electromagnetic field. The inseparability condition for n-mode states is given by

$$|\langle (\hat{a}_1 + \hat{a}_2 + ... + \hat{a}_{n-1})\hat{a}_n^\dagger \rangle|^2 \ > \ (n-1)\langle \hat{N}_{a_1+a_2+.....+a_{n-1}} \hat{N}_{a_n} \rangle, \tag{29}$$

where $\hat{N}_{a_1+a_2+.....+a_{n-1}} = \dfrac{(\hat{a}_1 + \hat{a}_2 + ... + \hat{a}_{n-1})^\dagger (\hat{a}_1 + \hat{a}_2 + ... + \hat{a}_{n-1})}{(n-1)}$.

The most general form of entanglement condition for n-mode states can be written as

$$|\langle (\hat{a}_1^{p_1} + \hat{a}_2^{p_2} + ... + \hat{a}_{n-1}^{p_{n-1}})(\hat{a}_n^\dagger)^{p_n} \rangle|^2 \ > \ \langle (\hat{a}_1^{p_1} + \hat{a}_2^{p_2} + ... + \hat{a}_{n-1}^{p_{n-1}})^\dagger (\hat{a}_1^{p_1} + \hat{a}_2^{p_2} + ... + \hat{a}_{n-1}^{p_{n-1}})(\hat{a}_n^\dagger)^{p_n} (\hat{a}_n)^{p_n} \rangle. \tag{30}$$

Conclusion

In conclusion, we have presented a sufficient condition of inseparability for three-mode states of electromagnetic field, using $su(2)$ and $su(1,1)$ algebra operators that are quadratic in mode creation and annihilation operators. This inseparability condition is obtained by using two different approches: by considering the sum uncertainty relations of $su(2)$ algebra operators and the SRUR of partially transposed $su(1,1)$ algebra operators. The latter approach provides a stronger separability criterion and the result of former approach is explicitly shown to be a sufficient condition to violate this stronger separability criterion and also a necessary condition for W type entangled states. A general form is provided for a family of such inseparable conditions which are generalized to multimode states. An experimental scheme is proposed to test the violation of the proposed stronger separability condition. Thus, we circumvent the difficulty of testing $n(n-1)/2$ entanglement conditions for each pair of modes of W type states, by proposing a necessary inseparaiblity condition.

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