SOME FINITE SERIES AND THEIR APPLICATION

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$Q.1.$ *Find the sum of* $\binom{2n}{0}^2 - \binom{2n}{1}^2 + \binom{2n}{2}^2 - \cdots - \binom{2n}{2n-1}^2 + \binom{2n}{2n}^2.$

Answer: we know that $(1+x)^{2n} = \binom{2n}{0} + \binom{2n}{1}x + \binom{2n}{2}x^2 + \cdots + \binom{2n}{2n}x^{2n}. \quad \cdots(1)$

and $(1-x)^{2n} = \binom{2n}{0} - \binom{2n}{1}x + \binom{2n}{2}x^2 - \cdots + \binom{2n}{2n}x^{2n}. \quad \cdots(2)$

we can write $\binom{2n}{0}^2 + \binom{2n}{1}^2 + \binom{2n}{2}^2 + \cdots + \binom{2n}{2n}^2 = \binom{2n}{2n} - \binom{2n}{2n-1}\binom{2n}{1} + \binom{2n}{2n-2}\binom{2n}{2} - \cdots - \binom{2n}{1}\binom{2n}{2n-1} + \binom{2n}{2n}.$ *so, the coefficient of* x^{2n} *in* $(1+x)^{2n} \cdot (1-x)^{2n}.$

$= $ *the coefficient of* x^{2n} *in* $(1-x^2)^{2n}.$

using binomial expansion we get the coefficient of x^{2n} *in* $(1-x^2)^{2n}$ *and that is* $(-1)^n\binom{2n}{n}.$

$Q.2.$ *Let* $S_n, n \geq 1,$ *be the sets defined as follows :* $S_1 = \{0\}, S_2 = \left\{\frac{3}{2}, \frac{5}{2},\right\}, S_3 = \left\{\frac{8}{3}, \frac{11}{3}, \frac{14}{3}\right\}, S_4 = \left\{\frac{15}{4}, \frac{19}{4}, \frac{23}{4}, \frac{27}{4}\right\}$ *and so on. Then, find the sum of the elements of* $S_{20}.$

Answer: From the pattern of the sets we get, $S_{20} = \left\{\frac{399}{20}, \frac{419}{20}, \cdots, \frac{779}{20}\right\}.$ *Because the first term of every set is* $\frac{n^2-1}{n}, n \geq 1.$ *so, the sum of the all elements in* S_{20} *is* $\frac{20\left\{\frac{399+779}{20}\right\}}{2}.$

$$= \frac{1178}{2}.$$

$$= 589.$$

$Q.3.$ *For any positive integer* $k,$ *compute the integral part of* $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{10,000}}$

using the property that $2(\sqrt{k+1} - \sqrt{k}) < \frac{1}{\sqrt{k}} < 2(\sqrt{k} - \sqrt{k-1}).$

Answer: we know that $2(\sqrt{k+1} - \sqrt{k}) < \frac{1}{\sqrt{k}} < 2(\sqrt{k} - \sqrt{k-1}), \quad k = 1,2, \ldots$

after putting $k = 2$ *we get,* $2(\sqrt{3} - \sqrt{2}) < \frac{1}{\sqrt{2}} < 2(\sqrt{2} - 1).$

after putting $k = 3$ *we get,* $2(\sqrt{4} - \sqrt{3}) < \frac{1}{\sqrt{3}} < 2(\sqrt{3} - \sqrt{2}).$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

after putting $k = 10,000$ *we get,* $2(\sqrt{10,001} - \sqrt{10,000}) < \frac{1}{\sqrt{10,000}} < 2(\sqrt{10,000} - \sqrt{9999}).$

$after\ adding\ all\ inequalities\ we\ get,$ $2\left(\sqrt{10{,}001}-\sqrt{2}\right) < \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{10{,}000}} < 2\left(\sqrt{10{,}000}-1\right).$

$$or, 197.181 < \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{10{,}000}} < 198.$$

$so, the\ integral\ part\ is$ 197.

$Q.4.$ $Find\ the\ sum\ of\ the\ series\ of$ $1+\binom{n}{1}\cos\theta+\binom{n}{2}\cos 2\theta+\cdots+\binom{n}{n}\cos n\theta.$

$Answer: we\ know\ that$ $(1+a)^n$

$$= 1+\binom{n}{1}a+\binom{n}{2}a^2+\cdots+\binom{n}{n}a^n \text{ , where a is real number or complex number.}$$

$if\ we\ put$ $a=\cos\theta+i\sin\theta$ $we\ get,$ $1+\binom{n}{1}(\cos\theta+i\sin\theta)+\binom{n}{2}(\cos\theta+i\sin\theta)^2+\cdots+\binom{n}{n}(\cos\theta+i\sin\theta)^n.$

$so, this\ is\ implies\ that$ $\{(1+\cos\theta)+i\sin\theta\}^n = 1+\binom{n}{1}\cos\theta+\binom{n}{2}\cos 2\theta+\cdots+\binom{n}{n}\cos n\theta + i\binom{n}{1}\sin\theta+i\binom{n}{2}\sin 2\theta+\cdots+i\binom{n}{n}\sin n\theta\,.\cdots(1)$

$and\ if\ we\ put$ $a=\cos\theta-i\sin\theta$ $we\ get,$ $\{(1+\cos\theta)-i\sin\theta\}^n$

$$= 1+\binom{n}{1}\cos\theta+\binom{n}{2}\cos 2\theta+\cdots+\binom{n}{n}\cos n\theta-i\binom{n}{1}\sin\theta-i\binom{n}{2}\sin 2\theta-\cdots-i\binom{n}{n}\sin n\theta.\ \cdots(2)$$

$after\ adding$ (1) and (2) $we\ get,$ $\{(1+\cos\theta)+i\sin\theta\}^n+\{(1+\cos\theta)-i\sin\theta\}^n = 2\left\{1+\binom{n}{1}\cos\theta+\binom{n}{2}\cos 2\theta+\cdots+\binom{n}{n}\cos n\theta\,.\right\}$

$or, \left\{2cos^2\frac{\theta}{2}+i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right\}^n+\left\{2cos^2\frac{\theta}{2}-i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right\}^n = 2\left\{1+\binom{n}{1}\cos\theta+\binom{n}{2}\cos 2\theta+\cdots+\binom{n}{n}\cos n\theta\right\}.$

$or, \left(\cos\frac{\theta}{2}\right)^n\left\{2\cos\frac{\theta}{2}+i\,2\sin\frac{\theta}{2}\right\}^n+\left(\cos\frac{\theta}{2}\right)^n\left\{2\cos\frac{\theta}{2}-i\,2\sin\frac{\theta}{2}\right\}^n = 2\left\{1+\binom{n}{1}\cos\theta+\binom{n}{2}\cos 2\theta+\cdots+\binom{n}{n}\cos n\theta\right\}.$

$or, (2\cos\frac{\theta}{2})^n\cos\frac{n\theta}{2} = \left\{1+\binom{n}{1}\cos\theta+\binom{n}{2}\cos 2\theta+\cdots+\binom{n}{n}\cos n\theta\right\}.$

$Q.5.$ $Find\ the\ value\ of$ $\sum_{r=1}^{10}(r^2+1)r!\,.$

$Answer: we\ know\ that$ $1\cdot 1!+2\cdot 2!+3\cdot 3!+4\cdot 4!+\cdots+n\cdot n! = (n+1)!-1.\ \cdots(1)$

$so, \sum_{r=1}^{10}(r^2+1)r! = \sum_{r=1}^{10}(r+1)^2\cdot r! - 2\sum_{r=1}^{10} r\cdot r!$

$$= \sum_{r=1}^{10}(r+1)\cdot(r+1)! - 2\sum_{r=1}^{10} r\cdot r! = 12!-1-2(11!-1)-1 \quad [using\ the\ property\ (1)].$$

$= 12! - 2 \cdot 11! = 10 \cdot 11!$

$Q.6. If\ S_n = \sum_{r=0}^{n} \frac{1}{\binom{n}{r}}$ $and\ t_n = \sum_{r=0}^{n} \frac{r}{\binom{n}{r}}$ $, then\ find\ the\ value\ of\ \frac{t_n}{S_n}.$

$Answer: so, t_n = \sum_{r=0}^{n} \frac{r}{\binom{n}{r}}.$

$or,\ t_n = -\sum_{r=0}^{n} \frac{-r}{\binom{n}{r}}.$

$= -\sum_{r=0}^{n} \frac{n-r-n}{\binom{n}{r}}.$

$= -\sum_{r=0}^{n} \frac{n-r}{\binom{n}{n-r}} + \sum_{r=0}^{n} \frac{n}{\binom{n}{r}}.$

$= -t_n + nS_n.$

$or, t_n = -t_n + nS_n.$

$or, \frac{t_n}{S_n} = \frac{n}{2}.$

$Q.7. Compute\ the\ sum : S_n = \sum_{k=0}^{n} \frac{1}{(k+1)(k+2)} \binom{n}{k}.$

$Answer: we\ know\ that\ (1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n.$

$after\ integrating\ both\ side\ with\ respect\ to\ x\ we\ get, \int (1+x)^n\,dx = \int \{1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n.\}$

$or, \frac{(1+x)^{n+1}}{n+1} = x + \binom{n}{1}\frac{x^2}{2} + \binom{n}{3}\frac{x^3}{3} + \cdots + \binom{n}{n}\frac{x^{n+1}}{n+1} + c_1,$ $where\ c_1\ is\ integrating\ constant.$

$To\ find\ the\ value\ of\ c_1\ we\ have\ to\ put\ x = 0\ and\ we\ get, c_1 = \frac{1}{n+1}.$

$Now, \frac{(1+x)^{n+1}}{n+1} - \frac{1}{n+1} = x + \binom{n}{1}\frac{x^2}{2} + \binom{n}{3}\frac{x^3}{3} + \cdots + \binom{n}{n}\frac{x^{n+1}}{n+1}. \cdots (1)$

$Now, after\ integrating\ both\ side\ again\ we\ get, \frac{(1+x)^{n+2}}{(n+1)(n+2)} - \frac{x}{n+1} = \frac{x^2}{1\cdot 2}\binom{n}{0} + \binom{n}{1}\frac{x^3}{2\cdot 3} + \binom{n}{3}\frac{x^4}{3\cdot 4} + \cdots + \binom{n}{n}\frac{x^{n+2}}{(n+1)\cdot(n+2)} + c_2. \cdots (2)$

$putting\ x = 0\ we\ get, c_2 = \frac{1}{(n+1)(n+2)}.$

$putting\ the\ value\ of\ c_2\ in\ the\ equation\ (2),\ \sum_{k=0}^{n} x^{k+2}\binom{n}{k} \cdot \frac{1}{(k+1)(k+2)} = \frac{(1+x)^{n+2}}{(n+1)(n+2)} - \frac{x}{n+1} - \frac{1}{(n+1)(n+2)}. \cdots (3)$

$putting\ x = 1\ in\ equation\ (3) we\ get, \sum_{k=0}^{n} \binom{n}{k} \cdot \frac{1}{(k+1)(k+2)} = \frac{2^{n+2}}{(n+1)(n+2)} - \frac{1}{n+1} - \frac{1}{(n+1)(n+2)}.$

$$= \frac{2^{n+2}-(n+3)}{(n+1)(n+2)}.$$

: **Reference:**

- $Test\ of\ Mathematics\ at\ the\ 10+2\ Level - INDIAN\ STATISTICAL\ INSTITUTE$