News Limit Formulas for the Exponential of $\pi/8$, Involving Pochhammer Symbols and Secant Function

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Abstract. I derive some news identities for limit of the exponential of $\pi/8$, involving Pochhammer symbols and secant function.

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1. Introduction

In present paper, I derive some news identities for limit of the exponential of $\pi/8$, involving Pochhammer symbols and secant function, such as,

$$e^{\pi/8} = \lim_{m\to\infty}\left[\frac{\sec\left(\frac{\pi}{4\sqrt{\frac{m}{m+1}}}\right)}{\sqrt{2}}\right]^m$$

and

$$e^{\pi/8} = \lim_{m\to\infty}\left\{\frac{(2m+1)_{2m}}{4^{3m}\left(\frac{1}{2}-\frac{1}{4}\sqrt{1+\frac{1}{m}}\right)_m\left(\frac{1}{2}+\frac{1}{4}\sqrt{1+\frac{1}{m}}\right)_m}\right\}^m.$$

2. The Limit Formula for $e^{\pi/8}$

Theorem 2.1. *We have*

$$e^{\pi/8} = \lim_{m\to\infty}\left[\frac{\sec\left(\frac{\pi}{4\sqrt{\frac{m}{m+1}}}\right)}{\sqrt{2}}\right]^m \tag{2.1}$$

and

$$e^{\pi/8} = \lim_{m\to\infty}\left\{\frac{(2m+1)_{2m}}{4^{3m}\left(\frac{1}{2}-\frac{1}{4}\sqrt{1+\frac{1}{m}}\right)_m\left(\frac{1}{2}+\frac{1}{4}\sqrt{1+\frac{1}{m}}\right)_m}\right\}^m \tag{2.2}$$

where e^x *denotes the expoential function and* $(x)_m$ *denotes the Pochhammer symbol.*

Proof. Leonhard Euler, see [1], give us the expansion in series

$$\begin{aligned}\frac{\pi}{4} &= \sum_{k=1}^{\infty}\left(\frac{1}{4k-3}-\frac{1}{4k-1}\right)\\ &= \sum_{k=1}^{\infty}\frac{2}{(4k-1)(4k-3)}\\ \Rightarrow \frac{\pi}{8} &= \sum_{k=1}^{\infty}\frac{1}{(4k-1)(4k-3)}\\ &= \sum_{k=1}^{\infty}\log\left[\exp\left(\frac{1}{(4k-1)(4k-3)}\right)\right]\\ \Rightarrow e^{\pi/8} &= \prod_{k=1}^{\infty}\exp\left[\frac{1}{(4k-1)(4k-3)}\right].\end{aligned} \tag{2.3}$$

On the other hand, I know [2, p. 5, Corollary 7] the limit formula for exponential function given by

$$\exp(z) = \lim_{m\to\infty}\left(1-\frac{z}{m}\right)^{-m}. \tag{2.4}$$

From (2.2) and (2.3), it follows that

$$\begin{aligned}e^{\pi/8} &= \lim_{m\to\infty}\prod_{k=1}^{\infty}\left[1-\frac{1}{m(4k-1)(4k-3)}\right]^{-m}\\ &= \lim_{m\to\infty}\left\{\prod_{k=1}^{\infty}\left[1-\frac{1}{m(4k-1)(4k-3)}\right]^{-1}\right\}^m\\ &= \lim_{m\to\infty}\left\{\prod_{k=1}^{\infty}\left[\frac{m(4k-1)(4k-3)-1}{m(4k-1)(4k-3)}\right]^{-1}\right\}^m\\ &= \lim_{m\to\infty}\left\{\prod_{k=1}^{\infty}\left[\frac{m(4k-1)(4k-3)}{m(4k-1)(4k-3)-1}\right]\right\}^m\\ &= \lim_{m\to\infty}\left\{\prod_{k=1}^{\infty}\left[\frac{1}{1-\frac{1}{m(4k-1)(4k-3)}}\right]\right\}^m\\ &= \lim_{m\to\infty}\left[\frac{\sec\left(\frac{\pi}{4\sqrt{\frac{m}{m+1}}}\right)}{\sqrt{2}}\right]^m\end{aligned}$$

or

$$
\begin{aligned}
e^{\pi/8} &= \lim_{m\to\infty}\prod_{k=1}^{m}\left[1-\frac{1}{m(4k-1)(4k-3)}\right]^{-m}\\
&= \lim_{m\to\infty}\left\{\prod_{k=1}^{m}\left[1-\frac{1}{m(4k-1)(4k-3)}\right]^{-1}\right\}^{m}\\
&= \lim_{m\to\infty}\left\{\prod_{k=1}^{m}\left[\frac{m(4k-1)(4k-3)-1}{m(4k-1)(4k-3)}\right]^{-1}\right\}^{m}\\
&= \lim_{m\to\infty}\left\{\prod_{k=1}^{m}\left[\frac{m(4k-1)(4k-3)}{m(4k-1)(4k-3)-1}\right]\right\}^{m}\\
&= \lim_{m\to\infty}\left\{\prod_{k=1}^{m}\left[\frac{1}{1-\frac{1}{m(4k-1)(4k-3)}}\right]\right\}^{m}\\
&= \lim_{m\to\infty}\left\{\frac{4^{-m}\Gamma\big(2m+\frac{1}{2}\big)}{\sqrt{\pi}\left(\frac{1}{2}-\frac{1}{4}\sqrt{1+\frac{1}{m}}\right)_m\left(\frac{1}{2}+\frac{1}{4}\sqrt{1+\frac{1}{m}}\right)_m}\right\}^{m}\\
&= \lim_{m\to\infty}\left\{\frac{4^{-m}\Gamma\big(2m+\frac{1}{2}\big)}{\sqrt{\pi}\left(\frac{1}{2}-\frac{1}{4}\sqrt{1+\frac{1}{m}}\right)_m\left(\frac{1}{2}+\frac{1}{4}\sqrt{1+\frac{1}{m}}\right)_m}\right\}^{m}.
\end{aligned}
\tag{2.5}
$$

On the other hand, in general, for non-negative value of n, I know [3] that

$$
\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n)!}{4^n n!}\sqrt{\pi}. \tag{2.6}
$$

Put $n\to 2m$ in both members of (2.5) and encounter

$$
\Gamma\left(2m+\frac{1}{2}\right)=\frac{(4m)!}{4^{2m}(2m)!}\sqrt{\pi}=4^{-2m}\frac{\Gamma(4m+1)}{\Gamma(2m+1)}\sqrt{\pi}. \tag{2.7}
$$

From (2.4) and (2.6), it follows that

$$
\begin{aligned}
e^{\pi/8} &= \lim_{m\to\infty}\left\{\frac{4^{-m}4^{-2m}\Gamma(4m+1)}{\Gamma(2m+1)\left(\frac{1}{2}-\frac{1}{4}\sqrt{1+\frac{1}{m}}\right)_m\left(\frac{1}{2}+\frac{1}{4}\sqrt{1+\frac{1}{m}}\right)_m}\right\}^{m}\\
&= \lim_{m\to\infty}\left\{\frac{4^{-3m}\Gamma(4m+1)}{\Gamma(2m+1)\left(\frac{1}{2}-\frac{1}{4}\sqrt{1+\frac{1}{m}}\right)_m\left(\frac{1}{2}+\frac{1}{4}\sqrt{1+\frac{1}{m}}\right)_m}\right\}^{m}.
\end{aligned}
\tag{2.8}
$$

The Pochhammer symbol may be defined by [4]

$$
(x)_n\equiv\frac{\Gamma(x+n)}{\Gamma(x)} \tag{2.9}
$$

for $n \geqq 0$. By (2.8), I obtain

$$\frac{\Gamma(4m+1)}{\Gamma(2m+1)} = \frac{\Gamma(2m+1+2m)}{\Gamma(2m+1)} = (2m+1)_{2m}. \tag{2.10}$$

From (2.7) and (2.9), I conclude that

$$e^{\pi/8} = \lim_{m\to\infty}\left\{\frac{(2m+1)_{2m}}{4^{3m}\left(\frac{1}{2}-\frac{1}{4}\sqrt{1+\frac{1}{m}}\right)_m\left(\frac{1}{2}+\frac{1}{4}\sqrt{1+\frac{1}{m}}\right)_m}\right\}^m,$$

which is the desired result. □

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