# The Net Positive Charges And Additional Pressure of The White Dwarf Star

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**Abstract** The upper mass limit of the white dwarf star is predicted 1.44 $M_\odot$ based on the ideally degenerate Fermi electron gas at the temperature of absolute zero. However, more conditions should be considered like temperature and charges. In this research, first we use the grand partition function in statistical mechanics to build the expressions of the electron gas pressure and the particle number depending on temperature. At $1.16\text{x}10^7$ K, there is about $1.50\text{x}10^{-4}$ of total electrons exceeding the Fermi energy, and about $1.50\text{x}10^{-7}$ at $1.16\text{x}10^4$ K. Because some of this Fermi electron gas are the extremely relativistic electrons, some of them can escape the gravity and some return the star after leaving. These two mechanisms result in a positively charged star and the net positive charges produce the repulsive force and the pressure to against gravity. The increases pressure is comparable to that of the Fermi electron gas at $T$=0 when the star is charged at $10^{20}$ $C$.

**Keywords:** white dwarf star, degenerate Fermi electron gas, pressure, upper mass limit, Coulomb's interaction

## I. Introduction

The white dwarf star is thought to be the type of the low to medium mass stars in the final evolution stage and it was named first in 1922 [1]. Its density is usually very high with the mass similar to our sun but the size is as small as the Earth. The nucleus of the white dwarf star stops the nuclear-fusion reaction and cools down gradually. It is believed that the pressure of the degenerate Fermi electron gas mainly support the balance with the gravitation so as to the mass upper limit exist [2-7]. The calculation is based on the assumption that all electrons like free particles occupy all energy levels until to Fermi energy as they are at temperature $T$ of absolute zero. Due to this, such stars has been called the "zero-temperature stars" [8]. The early highest reported temperature for the white dwarf star was the HD 62166 at the center of the planetary nebula NGC 2440 [9,10]. Its effectively surface temperature is about 200,000 K. The new record is about 230,000 K – 300,000 K investigated at the white dwarf star RX J0439.8-6809 [11-13]. This star is even speculated to reach 400,000 K, the maximum temperature, a thousand years ago [13]. It seems that the ideal Fermi electron gas at $T$= 0 K still works in the high-temperature and high-pressure situation even some electrons have kinetic energy higher than the Fermi energy.

From astronomical observations, most stars are rotating and might be easily charged because the relativistically massive particles can escape the gravitational attraction. According to the calculations of statistical mechanics [2-6], the relativistic electrons have more possibility to escape gravity than helium nuclei at the same temperature. Because of this factor, we consider the positively charged star and the Coulomb's interaction existing between the net positive charges, and further calculate the pressure produced by these net positive charges. The Coulomb's force is also an important one against gravity so we discuss its possibly effect on the inner pressure.

## II. The Degenerate Fermi Electron Gas For The White Dwarf Star

The calculation of the upper mass limit for the white dwarf star adopts the ideally degenerate Fermi electron gas at and considers the relativistic kinetic energy [3,4]. Each energy state permits two electrons occupied because of the electron spin s$=\pm\frac{1}{2}$. Each electron has the rest mass $m_{e,}$ and its relativistic kinetic energy $E$ at momentum $p$ is

$$E_k = m_e c^2 \left\{ \left[ 1 + \left( \frac{p}{m_e c} \right)^2 \right]^{1/2} - 1 \right\}. \tag{1}$$

The Fermi electron gas has total kinetic energy

$$\begin{aligned} E_0 &= 2m_e c^2 \sum_{|\vec{p}|<p_F} \left\{ \left[ 1 + \left( \frac{\vec{p}}{m_e c} \right)^2 \right]^{1/2} - 1 \right\} \\ &= \frac{2Vm_e c^2}{h^3} \int_0^{p_F} dp 4\pi p^2 \left\{ \left[ 1 + \left( \frac{\vec{p}}{m_e c} \right)^2 \right]^{1/2} - 1 \right\}, \end{aligned} \tag{2}$$

where $h$ is the Planck's constant and Fermi momentum is defined

$$p_F = h \left( \frac{3N}{8\pi V} \right)^{1/3}. \tag{3}$$

The pressure produced by the ideal Fermi electron gas at $T$=0 K is [4]

$$P_0 = -\frac{\partial E_0}{\partial V} = \frac{8\pi m_e^4 c^5}{h^3} \left[ \frac{1}{3} x_F^3 \sqrt{1 + x_F^2} - f(x_F) \right], \tag{4}$$

where

$$E_0 = \frac{8\pi m_e^4 c^5 V}{h^3} \left[ f(x_F) - \frac{1}{3} x_F^3 \right], \tag{5}$$

$$f(x_F) = \int_0^{x_F} dx x^2 \left[ (1 + x^2)^{1/2} \right], \tag{6}$$

and

$$x_F \equiv \frac{p_F}{m_e c} = \frac{h}{2m_e c}\left(\frac{3N}{8\pi V}\right)^{1/3}, \tag{7}$$

A white dwarf star mainly consists of helium nuclei so the total mass $M$ is

$$M = (m_e + m_p + m_n)N \approx 2m_p N \approx 2m_n N, \tag{8}$$

where $m_p$ is the mass of a proton and $m_n$ is the mass of a neutron. Then the relationship between the radius $R$ and mass $M$ of the star is

$$\bar{R} = \bar{M}^{2/3}\left[1 - \left(\frac{\bar{M}}{\bar{M}_0}\right)^{2/3}\right]^{1/2}, \tag{9}$$

where

$$\bar{M}_0 = \left(\frac{27\pi}{64\delta}\right)^{3/2}\left(\frac{hc}{2\pi G m_n^2}\right)^{3/2}, \tag{10}$$

$$\bar{R} = \left(\frac{2\pi m_e c}{h}\right) R, \tag{11}$$

and

$$\bar{M} = \frac{9\pi}{8}\frac{M}{m_n}. \tag{12}$$

In Eq. (10), $\delta$ is a parameter of pure number and $G$ is the gravitational constant. Further calculations [4] the upper mass limit $M_0$ is given by

$$M_0 \approx 1.44 M_\odot. \tag{13}$$

## III. The Temperature Effect On The Pressure of The Ideal Fermi Electron Gas

The inner temperature of a star is usually about $10^7$ K, and the upper mass limit in Eq. (13) is calculated at $T$=0 which seems to be unreasonable and doubtful. Then we consider the grand partition function for $T$>>0 in statistical mechanics [4]

$$q(T,V,z) = \ln Z = \textstyle\sum_k \ln[1 + z \cdot exp(-\beta E_k)], \tag{14}$$

where $E_k$ is the kinetic energy, $\beta=1/k_B T$, $z=\exp(\mu\beta)$, and $\mu$ the chemical potential of the Fermi electron gas. The chemical potential depends on $T$ weakly. The grand partition function can change to the integral from

$$\ln Z = \int_0^\infty dE[D(E_k)] \ln[1 + zexp(-\beta E_k)], \tag{15}$$

where $D(E_k)$ is the density of state. When integrating it by parts, then it gives [4]

$$\ln Z = g\frac{4\pi V}{h^3}\frac{\beta}{3}\int_0^\infty p^3 dp\frac{dE_k}{dp}\frac{1}{z^{-1}exp(\beta E_k)+1}, \tag{16}$$

where

$$p^2c^2 = E_k^2 + 2m_ec^2E_k \tag{17}$$

and $g$=2$s$+1 is the degeneracy factor. Substituting Eq. (17) into Eq. (16), it gives

$$\ln Z = g\frac{4\pi V\beta}{3h^3c^3}\int_0^\infty dE_k\frac{E_k^3\left[1+\frac{2m_ec^2}{E_k}\right]^{3/2}}{z^{-1}exp(\beta E_k)+1}. \tag{18}$$

Using the Taylor series expansion to the first-order term, then we have

$$\ln Z \approx g\frac{4\pi V}{3h^3c^3\beta^3}\int_0^\infty d(\beta E_k)\frac{(\beta E_k)^3\left[1+3\left(\frac{m_ec^2\beta}{\beta E_k}\right)\right]}{z^{-1}exp(\beta E_k)+1}. \tag{19}$$

Similarly, we can obtain the expression of the total number $N(T, V, z)$ is

$$\begin{aligned} N(T,V,z) &= \mathrm{g}\frac{4\pi V}{h^3}\int_0^\infty p^2dp\frac{1}{z^{-1}exp(\beta E_k)+1} \\ &= \mathrm{g}\frac{4\pi V}{h^3c^3}\int_0^\infty dE_k\frac{E_k^2\left(1+\frac{2m_ec^2}{E_k}\right)^{1/2}\left(1+\frac{m_ec^2}{E_k}\right)}{z^{-1}exp(\beta E_k)+1}. \end{aligned} \tag{20}$$

A useful function in both above integrals is [4]

$$f_n(z) = \frac{1}{\Gamma(n)}\int_0^\infty dx\frac{x^{n-1}}{z^{-1}e^x+1}, \tag{21}$$

where $x = \beta E_F$. When $z$>>1, Eq. (21) approximates [4]

$$f_n(z) \approx \frac{(\ln z)^n}{n!}. \tag{22}$$

For example, the calculation of $\ln Z$ can be written as

$$\ln Z \approx g\frac{4\pi V}{3h^3c^3\beta^3}\left[\Gamma(4)f_4(z)+3\left(\frac{m_ec^2}{k_BT}\right)\Gamma(3)f_3(z)\right], \tag{23}$$

where $\Gamma(n)$ is the gamma function. Further calculations in Eq. (20) gives

$$N(T,V,z) \approx \frac{8\pi V(k_BT)^3}{h^3c^3} \times \left[\Gamma(3)f_3(z) + 2\left(\frac{m_ec^2}{k_BT}\right)\Gamma(2)f_2(z) + \left(\frac{m_ec^2}{k_BT}\right)^2 f_1(z)\right]. \tag{24}$$

The results in Eqs. (23) and (24) can also be applied to neutron when the electron mass is changed to the neutron mass, so we can also use the similar way to discuss the neutron star.

## IV. The Additional Pressure Considering The Net Charges

However, above discussions are based on the neutral condition for the white dwarf star that the negative charges balance the positive charges. The relativistic electrons have possibility to escape the gravity of a star much higher than the heavier nuclei, so the star would very be the positively charged one. In the classical statistical mechanics, the Maxwell-Planck velocity distribution tells us the most probable, the mean, and the root mean square root absolute velocities $v^*$, $\langle v\rangle$, and $\sqrt{\langle v^2\rangle}$ in the ideal gas are

$$v^* = \sqrt{\frac{2k_BT}{m}}, \tag{25}$$

$$\langle v\rangle = \sqrt{\frac{8k_BT}{m\pi}}, \tag{26}$$

and

$$\sqrt{\langle v^2\rangle} = \sqrt{\frac{3k_BT}{m}}. \tag{27}$$

Although we deal with the indistinguishable quantum particles, in the ultrahigh-temperature case the classical results are still good references. All those three velocities for the electron gas are very close to $c$, but they are only 7.3x10$^{-4}c$, 8.2x10$^{-4}c$, and 8.9x10$^{-4}c$ for helium nucleus at the same temperature. Even for the hydrogen nucleus, its average velocity is still much less than electron. According to these, some electrons escape the gravitation of the star and the star is reasonably positive-charged. A similar phenomenon is the well-known solar wind raising from the surface of the star and moving outward to the space. The solar wind includes a lot of charged particles so each time it happens, the sun also changes its net charges. Even the sun is charged, it still has ability to emit more charged particles as long as the providing energy is great enough.

Then considering the total negative and positive charges are -Q and Q+ΔQ in the star. Supposing the net positive charges ΔQ distribute homogeneously in the star, then the density $\rho_{\Delta\mathrm{Q}}$ of the net positive charges is

$$\rho_{\Delta\mathrm{Q}} = \frac{\Delta\mathrm{Q}}{\frac{4}{3}\pi R^3}. \tag{28}$$

The electrostatic self-energy $E_{self}$ of this charged sphere [14,15] is

$$E_{self} = \frac{3K_e(\Delta\mathrm{Q})^2}{5R} = \frac{3K_e(\Delta\mathrm{Q})^2}{5}\left(\frac{3V}{4\pi}\right)^{-\frac{1}{3}}. \tag{29}$$

The Coulomb's law still works in the length scale of $10^{-17}$ *m* [16] to $5\text{x}10^{-17}$ *m* [17]. The pressure $P_{\Delta\mathrm{Q}}$ produced by the rest positive charges is

$$P_{\Delta Q} = -\frac{\partial E_{self}}{\partial V} = \frac{K_e(\Delta\mathrm{Q})^2}{5}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}} V^{-4/3}. \tag{30}$$

Using Eqs. (8), (11), and (12), then we have

$$P_{\Delta Q} = \frac{K_e(\Delta\mathrm{Q})^2}{5}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}\left(\frac{3M}{8\pi m_n N R^3}\right)^{\frac{4}{3}} = \frac{3K_e(\Delta\mathrm{Q})^2}{5\pi^2 N}\frac{1}{9}\left(\frac{4}{9\pi N}\right)^{\frac{1}{3}}\left(\frac{2\pi m_e c}{h}\right)^4 \frac{\bar{M}^{4/3}}{\bar{R}^4}. \tag{31}$$

This net-positive-charges pressure should be also considered into the contribution of the total pressure. When we consider the Fermi electron gas in metal, the net charges stay in the surface because of the zero electric field inside the perfect metal. However, the solar system is consisting of high-temperature and viscous plasma, and the heat convection continuously happens. The homogeneous distribution is theoretically reasonable assumption as long as the evolutional time is enough to reach this situation.

Next, $P_{\Delta\mathrm{Q}}$ can be further arranged as follows

$$P_{\Delta Q} = \frac{2\pi m_e^4 c^5}{3h^3}\left[\frac{8K_e(\Delta\mathrm{Q})^2\pi}{5Nhc}\left(\frac{4}{9\pi N}\right)^{\frac{1}{3}}\frac{\bar{M}^{4/3}}{\bar{R}^4}\right]. \tag{32}$$

Using these data [18], $K_e$=8.987x10$^9$ $N{\cdot}m^2/C^2$, $h$=6.626x10$^{-34}$ $J{\cdot}m$, $c$=2.998x10$^8$ $m/s$, and $N$~9x10$^{56}$ for our sun [3], the coefficient can be simplified to

$$\frac{8K_e(\Delta\mathrm{Q})^2\pi}{5Nhc}\left(\frac{4}{9\pi N}\right)^{\frac{1}{3}} \approx 1.364\times10^{-41}(\Delta Q)^2. \tag{33}$$

When ΔQ=2.708x10$^{20}$ *C*, this term is 1.0. This effect is caused by

$(2.708\text{x}10^{20})/(1.602\text{x}10^{-19})=1.67\text{x}10^{39}$ electrons leaving the star which only occupies about $10^{-17}$ of the total Fermi electron gas in the white dwarf star.

## V. The Number of Electrons Above Fermi Energy From About $10^4$ To $10^7$ K

As mentioned previously, the high-energy electrons have ability to leave the star far away. Before discussing how many high-energy electrons possibly leave the star, it is a point to understand the occupation of electrons on high-energy level as it is in metal or semiconductor. First of all, we estimate the ratio of the particle number above $E_F$ by using Eq. (24). The three parts in Eq. (24) are divided into two sub-integrals for each part, and these sub-integrals are

$$\Gamma(3)f_3(z)=\int_0^{\beta E_F}\frac{x^2}{z^{-1}e^x+1}dx+\int_{\beta E_F}^{\infty}\frac{x^2}{z^{-1}e^x+1}dx, \qquad (34a)$$

$$\Gamma(2)f_2(z)=\int_0^{\beta E_F}\frac{x}{z^{-1}e^x+1}dx+\int_{\beta E_F}^{\infty}\frac{x}{z^{-1}e^x+1}dx, \qquad (34b)$$

and

$$\Gamma(1)f_1(z)=\int_0^{\beta E_F}\frac{1}{z^{-1}e^x+1}dx+\int_{\beta E_F}^{\infty}\frac{1}{z^{-1}e^x+1}dx. \qquad (34c)$$

All the second integrals in Eqs. (34a) - (34c) can be estimated by changing the variable $x$ to $y=x-\beta E_F$, that is,

$$\int_{\beta E_F}^{\infty}\frac{x^2}{z^{-1}e^x+1}dx=\int_0^{\infty}\frac{(y+\beta E_F)^2}{e^y+1}dy$$

$$\approx(\beta E_F)^2\int_0^{\infty}\left(1+2\frac{y}{\beta E_F}+\left(\frac{y}{\beta E_F}\right)^2\right)e^{-y}dy$$

$$=(\beta E_F)^2\left[\Gamma(1)+2\frac{\Gamma(2)}{\beta E_F}+\frac{\Gamma(3)}{(\beta E_F)^2}\right], \qquad (35a)$$

$$\int_{\beta E_F}^{\infty}\frac{x}{z^{-1}e^x+1}dx=\int_0^{\infty}\frac{y+\beta E_F}{e^y+1}dy$$

$$\approx\beta E_F\int_0^{\infty}\left(1+\frac{y}{\beta E_F}\right)e^{-y}dy$$

$$=\beta E_F\left[\Gamma(1)+\frac{\Gamma(2)}{\beta E_F}\right], \qquad (35b)$$

and

$$\frac{1}{\Gamma(1)}\int_{\beta E_F}^{\infty}\frac{1}{z^{-1}e^{x}+1}dx=\frac{1}{\Gamma(1)}\int_{0}^{\infty}\frac{1}{e^{y}+1}dy\approx\frac{1}{\Gamma(1)}\int_{0}^{\infty}e^{-y}dy=1. \qquad (35c)$$

Using $N(T, V, z)$ in Eq. (24) as the total number of electron in a star, the ratio of the whole second integrals to the whole integrals in Eqs. (35a) - (35c) approximates

$$\frac{\left[(\beta E_F)^2+\beta(2+2\beta m_e c^2)E_F+4+2(\beta m_e c^2)+(\beta m_e c^2)^2\right]}{\left[\frac{(\beta E_F)^3}{3}+\beta^3(m_e c^2)E_F^2+\beta^3(m_e c^2)^2 E_F\right]}$$
$$\approx\frac{3k_B T}{E_F}\approx 1.50\times10^{-4}, \qquad (36)$$

where $\beta E_F$~20000.0. It means that there are 1.50x$10^{-4}$ of the total electrons above Fermi energy at $T$=1.16x$10^{7}$ K or $k_BT$~1000 eV. When the total number of electrons is $N$~9x$10^{56}$, there are 1.35x$10^{53}$ electrons above Fermi energy. The temperature of 1.16x$10^{7}$ K is the core temperature of the white dwarf star. The highest reported effectively surface temperature for the white dwarf star until now is 300,000 K investigated at the white dwarf star RX J0439.8-6809 [11]. When we use this temperature in Eq. (36), the ratio is 7.2x$10^{-6}$. If we treat the surface temperature at 10,000 K for the general white dwarf stars and use it in Eq. (36), the ratio is about 1.50x$10^{-7}$. According to these above discussions, the ratio is reasonable between 1.50x$10^{-7}$ and 1.50x$10^{-4}$, or the number of the electrons above Fermi energy is between 1.35x$10^{50}$ and 1.35x$10^{53}$. The Fermi-Dirac distribution function for $T$>0 is schematically shown in Fig. 1.

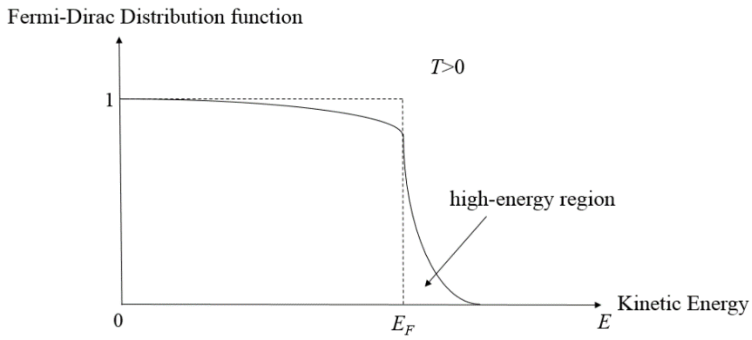


Figure 1. The Fermi-Dirac distribution function for electrons at $T$>0 in the white dwarf star.

## VI. The Charged Star Due To The Escaping High-Energy Electrons In The Electrostatic Consideration

After obtaining the number of electrons above Fermi energy for a white dwarf star with mass $M_\odot$, it allows us to estimate the number of the possibly escaping electrons. Using the conservation of energy between the kinetic and electric potential energy, we can estimate the maximal number of electrons $(\Delta Q)_{max}$ escaping the star to infinity in the electrostatic consideration. As we know, the Coulomb's interaction is much larger than the gravitational interaction for two protons or two electrons at the same distance, so we only consider the Coulomb's interaction here. The electric potential energy at infinity is zero as a reference. Supposing the minimum kinetic energy for escaping the Coulomb's interaction is $E_{min}$, then we have

$$(\gamma-1)m_e c^2 = E_{min} \geq \frac{K_e(\Delta\mathrm{Q})_{max} e}{R}. \tag{37}$$

where $\gamma$ is the Lorentz factor for the massive particle with the velocity $v$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{38}$$

Let $(\Delta\mathrm{Q})_{max}$=$(\Delta N)_{max}e$, using the Fermi-Dirac distribution gives

$$\begin{aligned}
&\frac{E_{min}R}{K_e e^2} \geq (\Delta N)_{max} \\
&\quad = g\frac{4\pi V}{h^3c^3\beta^3}\int_{E_{min}}^{\infty} d(\beta E_k)\frac{(\beta E_k)^2\left(1+\frac{2m_e c^2}{E_k}\right)^{1/2}\left(1+\frac{m_e c^2}{E_k}\right)}{z^{-1}exp(\beta E_k)+1} \\
&\quad \approx \mathrm{g}\frac{4\pi V(k_B T)^3}{h^3c^3}\int_{E_{min}}^{\infty} dx\frac{x^2\left[1+2\left(\frac{\beta m_e c^2}{x}\right)+\left(\frac{\beta m_e c^2}{x}\right)^2\right]}{z^{-1}e^x+1},
\end{aligned} \tag{39}$$

where $x$=$\beta E_k$, $z$=$exp(\beta\mu)$ and $\mu$ is the chemical potential very close to $E_F$. Choosing the lowest limit is zero, the right-hand side of Eq. (39) is equal to the total electron number $N$.

Next, we want to obtain the number of electrons above the minimum energy $E_{min}$ in Eq. (39). The integral in Eq. (39) can be divided into two parts which gives

$$\begin{aligned}
&\frac{(\Delta N)_{max}}{N}\left[\Gamma(3)f_3(z)+2\left(\frac{m_e c^2}{k_B T}\right)\Gamma(2)f_2(z)+\left(\frac{m_e c^2}{k_B T}\right)^2 f_1(z)\right] \\
&\approx (\beta E_F)^2\int_{\beta(E_{min}-E_F)}^{\infty}\left(1+2\frac{y}{\beta E_F}+\left(\frac{y}{\beta E_F}\right)^2\right)e^{-y}\,dy \\
&\quad +2\beta^2(m_e c^2)E_F\int_{\beta(E_{min}-E_F)}^{\infty}\left(1+\frac{y}{\beta E_F}\right)e^{-y}dy \\
&\quad +\beta^2(m_e c^2)^2\int_{\beta(E_{min}-E_F)}^{\infty} e^{-y}dy
\end{aligned}$$

$$= \beta^2 E_F^2 \begin{bmatrix} \Gamma[1, \beta(E_{min} - E_F)] \left(1 + 2\left(\frac{m_e c^2}{E_F}\right) + \left(\frac{m_e c^2}{E_F}\right)^2\right) \\ +2\Gamma[2, \beta(E_{min} - E_F)] \left(1 + \frac{m_e c^2}{E_F}\right)\left(\frac{k_B T}{E_F}\right) \\ +\Gamma\big(3, \beta(E_{min} - E_F)\big)\left(\frac{k_B T}{E_F}\right)^2 \end{bmatrix}, \tag{40}$$

where $\Gamma(n,y)$ is the incomplete Gamma function [19]. Substituting Eq. (40) into Eq. (39), it becomes

$$\frac{E_{min} R}{K_e e^2} \frac{\left[\Gamma(3) f_3(z) + 2\left(\frac{m_e c^2}{k_B T}\right)\Gamma(2) f_2(z) + \left(\frac{m_e c^2}{k_B T}\right)^2 f_1(z)\right]}{N}$$
$$\gtrsim \beta^2 E_F^2 \begin{bmatrix} \Gamma[1, \beta(E_{min} - E_F)] \left(1 + 2\left(\frac{m_e c^2}{E_F}\right) + \left(\frac{m_e c^2}{E_F}\right)^2\right) \\ +2\Gamma[2, \beta(E_{min} - E_F)] \left(1 + \frac{m_e c^2}{E_F}\right)\left(\frac{k_B T}{E_F}\right) \\ +\Gamma\big(3, \beta(E_{min} - E_F)\big)\left(\frac{k_B T}{E_F}\right)^2 \end{bmatrix}. \tag{41}$$

Using Eq. (37) in Eq. (41), it is simplified to

$$\frac{E_{min} R}{K_e e^2} \gtrsim (1.50 \times 10^{-4} N)$$
$$\times \left\{ \frac{\begin{bmatrix} \Gamma[1, \beta(E_{min} - E_F)] \left(1 + 2\left(\frac{m_e c^2}{E_F}\right) + \left(\frac{m_e c^2}{E_F}\right)^2\right) \\ +2\Gamma[2, \beta(E_{min} - E_F)] \left(1 + \frac{m_e c^2}{E_F}\right)\left(\frac{k_B T}{E_F}\right) \\ +\Gamma\big(3, \beta(E_{min} - E_F)\big)\left(\frac{k_B T}{E_F}\right)^2 \end{bmatrix}}{\left[1 + \left(\frac{2 + 2\beta m_e c^2}{\beta E_F}\right) + \frac{4 + 2(\beta m_e c^2) + (\beta m_e c^2)^2}{(\beta E_F)^2}\right]} \right\}. \tag{42}$$

The incomplete Gamma function has another expression [19]

$$\Gamma(n, y) = (n-1)!\, e^{-y} \sum_{m=0}^{n-1} \frac{y^m}{m!}. \tag{43}$$

Using Eq. (43) into Eq. (42) and considering $E_F$>>$m_ec^2$, we obtain

$$\frac{E_{min}R}{K_e e^2} \gtrsim (1.50\times 10^{-4}N)e^{-\beta(E_{min}-E_F)} \times \left\{ \begin{array}{c} \left[1+2\left(\frac{m_ec^2}{E_F}\right)+\left(\frac{m_ec^2}{E_F}\right)^2\right] \\ +2\left(1+\frac{E_{min}-E_F}{k_BT}\right)\left(1+\frac{m_ec^2}{E_F}\right)\left(\frac{k_BT}{E_F}\right) \\ +2\left[1+\left(\frac{E_{min}-E_F}{k_BT}\right)+\frac{1}{2}\left(\frac{E_{min}-E_F}{k_BT}\right)^2\right]\left(\frac{k_BT}{E_F}\right)^2 \end{array} \right\}. \tag{44}$$

Then we substitute some constants [18] into Eq. (44) to obtain $E_{min}$ and $(\Delta Q)_{max}$, they are $K_e$=8.987x10$^9$ $nt\cdot m^2\cdot C^{-2}$, $e$=1.6022x10$^{-19}$ $C$, $k_B$=1.38066x10$^{-23}$ $J\cdot K^{-1}$, $T$=1.16x10$^7$ K, $E_F$=20 MeV [3], and $z$=exp(20000.0). The radius is $R$=6.378x10$^6$ $m$ of a typical white dwarf star similar to the earth in size. After substituting those data into Eq. (45), it gives

$$\frac{E_{min}(eV)}{2.2601\times 10^{-16}(\mathrm{eV})} \gtrsim (1.35\times 10^{53})e^{-\beta(E_{min}-E_F)}. \tag{45}$$

According to Eq. (45), the lowest limit of $E_{min}$ is

$$E_{min} \gtrsim E_F + 6.946\times 10^4\ (\mathrm{eV}). \tag{46}$$

It means that the electron at least with kinetic energy 6.946x10$^4$ eV more than $E_F$ can escape the Coulomb's attraction to infinity in the electrostatic case. Furthermore, the maximally positive charges are

$$(\Delta \mathrm{Q})_{max} \lesssim 1.476\times 10^4\ C. \tag{47}$$

When we adopt $k_BT$=1.0 eV or $T$=1.16x10$^4$ K, the lowest limit of $E_{min}$ is

$$E_{min} \gtrsim E_F + 62.59\ (\mathrm{eV}). \tag{48}$$

The maximally positive charges are

$$(\Delta \mathrm{Q})_{max} \lesssim 1.418\times 10^4\ C. \tag{49}$$

By comparing Eq. (49) with Eq. (47), temperature only has 4.1% increase on the maximally positive charges from 1.16x10$^4$ K to 1.16x10$^7$ K in the electrostatic case. The lowest limit of $E_{min}$ is larger when temperature is higher. In conclusion, when we adopt the core temperature to calculate the maximally positive charges of the white dwarf star, it is a good approximation in the electrostatic consideration. Then substituting Eq. (49) into Eq. (33), it gives the increase of the pressure is

$$P_{\Delta Q} \approx 1.364 \times 10^{-41}(1.418 \times 10^4)^2 = 2.74 \times 10^{-33}\left(\frac{2\pi m_e^4 c^5}{3h^3}\right)\left(\frac{\bar{M}^{4/3}}{\bar{R}^4}\right). \quad (50)$$

This is the inner increasing pressure in the white dwarf star for the electrostatic consideration.

## VII. The Charged Star Due To The Leaving High-Energy Electrons In The Dynamical Consideration

In the previous electrostatic consideration, we can calculate the maximal number of electrons escaping the white dwarf star to infinity. However, electrons don't have to really reach the infinity or the edge of the observable universe but are just far away from the star. In fact, even electrons don't have enough energy to freely leave the star, they still possible leave the star at a distance and then return during some time. After those high-energy electrons leave the surface, the net charges of the star are positive and the Coulomb's field establishes soon. However, this field produced by the total net charges is not established immediately in the space. The photon is responsible for this establishment and it needs some time to reach the field point at the propagation speed of light. Some parts of the field induced by the charges from the other side of the star as shown in Fig. 2(a). Photons have to cross the inner of the star to establish the Coulomb's field in the space. The propagation speed of photon in the star is much slower than it in the free space. The random walk method has been applied to estimate the time scale of a photon diffusing from the center to its surface for the sun [20]. This time scale was reported as $1.7\text{x}10^5$ yr [20]. It is much longer time to build the electric field from the net charges on the other side of the star. Except for this much slow propagation process, the electric field is actually established in a short time by the net charges on the near side of the star as shown in Fig. 2(b). The photon responsible for establishing the electric field just propagates in nearly free space and then catch up with the early leaving electrons later. It also means that there is a little time delay for the electric field affecting the early leaving electrons.

According to Fig. 2(b), the propagation distance can be estimated. In the beginning, the high-energy electron leaves the star at speed very close to $c$. The ratio of the light speed $c$ to the electron speed $v$ is

$$\frac{c}{v} = \frac{E_k + m_e c^2}{(E_k^2 + 2m_e c^2 E_k)^{1/2}} \approx \frac{1 + \dfrac{m_e c^2}{E_k}}{1 + \dfrac{m_e c^2}{E_k} - \dfrac{1}{2}\left(\dfrac{m_e c^2}{E_k}\right)^2}$$

$$\approx 1 + \frac{1}{2}\left(\frac{m_e c^2}{E_k}\right)^2. \qquad (E_k \ll m_e c^2) \quad (51)$$

Here the effect of the gravitation on the speed of light near the star is also included in the estimation. The photon responsible for establishing the electric field is behind the early leaving electron some time $\Delta t$. When the photon catches up with the leaving electron, it has propagated a distance $\Delta d$ so we have

$$\frac{\Delta d}{c} + \Delta t = \frac{\Delta d}{v}. \tag{52}$$

Substituting Eq. (51) into Eq. (52), it gives

$$\Delta d \approx \ 2c\Delta t \left(\frac{E_k}{m_e c^2}\right)^2. \tag{53}$$

When $E_k$~20 MeV~$40m_ec^2$ and the time delay $\Delta t$=0.03 $s$, the distance is

$$\Delta d|_{\Delta t=0.02\ s} \approx 92c. \tag{54}$$

This time delay and propagation distance are estimated for the photon from the two ports of the diameter perpendicular to the electron trajectory as shown in Fig. 2(b). After the photon catches up, the electron starts to slow down and return to the star surface, as it is a process for an electron emitted and adsorbed by the star. This emission-absorption process continuously takes place. As time goes by, the net positive electrons gradually distribute toward the inner of the star due to the heat convection. Although the nuclear-fusion reaction stops, the temperature difference between the core and surface still causes the heat convection. After the long-time charge-distribution process, the net positive charges probably distribute the whole star as shown in Fig. 2(c). The number of the net positive charges should have something to do with the core and surface temperatures. The temperature is higher, the leaving electrons are more. The distribution of the leaving electron around the star is roughly in the range of the distance in Eq. (54) denoted as the grey region in Fig. 2(c). The distribution of the electron density decreases from the surface outward to the outer space.

The leaving electrons are mainly from the surface and it is easy to estimate the thickness of the surface providing so many leaving electrons. Most of them are close to and above the Fermi energy. Supposing the homogeneous distribution of electrons in the white dwarf star and the number of the leaving electrons is $1.67\text{x}10^{39}$ as mentioned in Sec. IV. At $T$=$1.16\text{x}10^4$ K, the thickness of the surface providing $10^{-17}$ of the total electrons is $\Delta R$, then we have

$$1.50 \times 10^{-7} \frac{4\pi R^2 \Delta R}{\frac{4}{3}\pi R^3} = 1.50 \times 10^{-7} \times 3\frac{\Delta R}{R} \approx 10^{-17}. \tag{55}$$

Using the radius of the earth as an example, it gives

$$\Delta R \approx 1.80 \times 10^{-4}\ m = 180\ \mu m. \tag{56}$$

It means that so many electrons leaving the star are mainly from the surface within the thickness of 180 $\mu m$, a very thin layer on the surface.

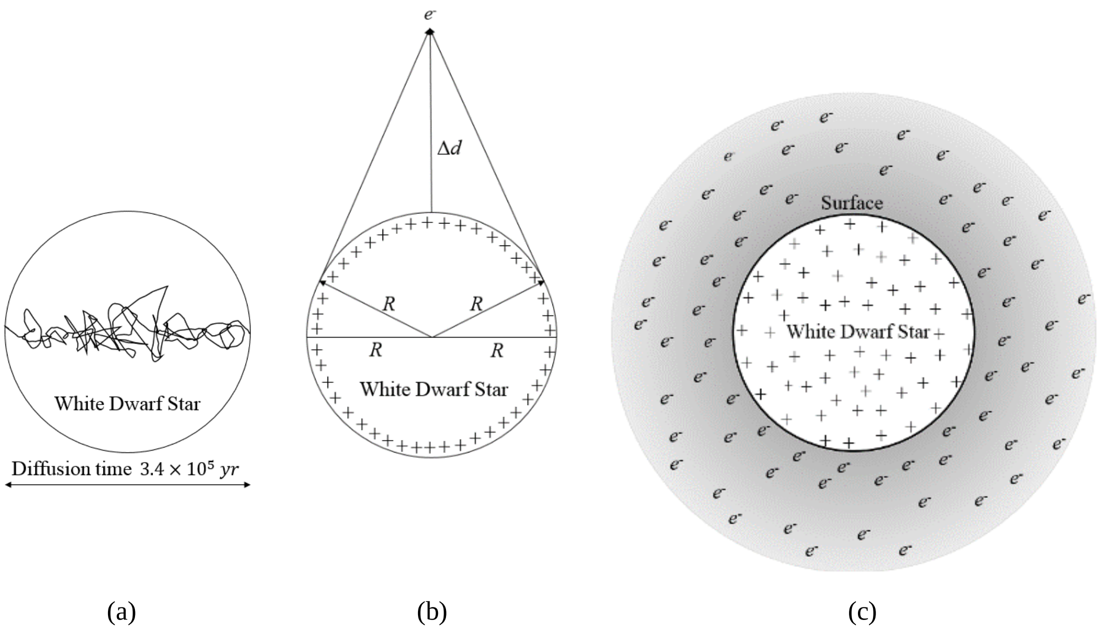


Figure 2. (a) The schematic picture for a photon diffusing from one side to the other side through a white dwarf star with mass of 1 $M_{\odot}$ and the radius $R_{\oplus}$. The diffusion time is roughly 3.4x10$^5$ yr. (b) The charges on the surface of the star and the part on the near side of the star affects the leaving electron in a very short time. (c) After the long-time revolution, the positive charges distributed in the white dwarf star and the distribution of the leaving electrons around the star is denoted as the grey region.

After many electrons leave and distribute around the star, those net charges $(\Delta Q)_{max}$ staying in the white dwarf star can increase the total inner pressure. Because the leaving electrons induce Coulomb's field, they also increase the Coulomb's potential energy of the positive charges. The main constitutions of the white dwarf stars are some elements, such as He, C, O, Ne, Mg, and Fe [13, 21, 22]. Those elements are about 7x10$^3$ to 10$^5$ times as heavy as the electron. Even they leave the star, they spend much shorter time than electrons coming back to the star. The we consider a situation that the star is charged $(\Delta Q)_{max}$=2.708x10$^{20}$ $C$. According to Eq. (31), the additional pressure due to the net positive charges inside the white dwarf star is almost the same as the pressure of the Fermi electron gas at $T$=0 K. The relationship between the net charges and the additionally induced pressure inside the white dwarf star is drawn in Fig. 3.

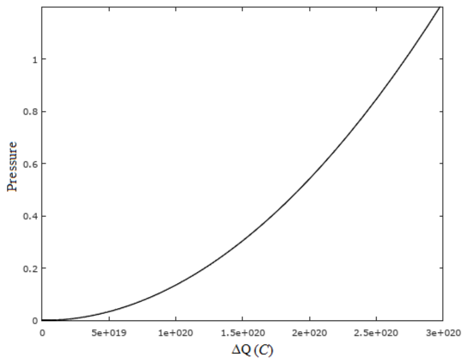


Figure 3. The relationship between the net charges and additionally induced pressure inside the white dwarf star. The unit of the pressure is in terms of $(2\pi m_e^4 c^5/3h^3)$.

Another possibility for charging so much is due to the supernova while a lot of electrons obtain enough energy and spread widely in space. After supernova, it leaves a lot of positive charges on the star and the Coulomb's fields due to these positive charges have to spend much time to catch up with the leaving electrons. So theoretically speaking, it possibly holds a positively charged star for a long time.

## VIII. Conclusion

In summary, the calculation from statistical mechanics shows that the temperature effect causes $1.5\text{x}10^{-7}$ of the total electrons exceeding the Fermi energy at about $10^4$ K, and $1.5\text{x}10^{-4}$ at about $10^7$ K. This effect makes it possible that some electrons have ability leaving the white dwarf star even escaping it to infinity. The Coulomb interaction should be considered for the contribution of the inner pressure inside the star. In the considerations of the electrostatic case, the number of electrons escaping to infinity is about $1.418\text{x}10^4$ *C* at the surface temperature of $1.16\text{x}10^4$ K. It is only slightly different when the core temperature is used. However, the charge flow is dynamical and we have to consider the case that some electrons leave the white dwarf star soon and return later due to the Coulomb's field induced by the net positive charges on the star surface. When this term is significant and comparable to the degenerate Fermi gas pressure, the number of the positive charges is about $10^{20}$ *C*. The number of electrons exceeding the Fermi energy is about $1.35\text{x}10^{50}$ at $1.16\text{x}10^4$ K, so theoretically speaking, it is possible for a star charging so much for a while and increasing its inner pressure gradually.

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