TEMPORAL GEOMETRIC INTERACTIONS

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Abstract: In this work we extend our discussions on the possibility to classify geometric interactions to temporal manifolds according to the dimensions of decomposed submanifolds *n*-cells. A temporal manifold is a differentiable manifold which is accompanied a spatial manifold to form a spatiotemporal manifold which represents an elementary particle and can be assumed to have the mathematical structure of a CW complex. As in the case of spatial manifolds, a temporal differentiable manifold can also be assumed to decompose *n*-cells. The decomposed temporal *n*-cells will also be identified with force carriers for physical interactions. For the case of temporal differentiable manifolds of dimension three, there are also four different types of geometric interactions associated with 0-cells, 1-cells, 2-cells and 3-cells. We also discuss the possible dynamics from these geometric interactions in terms of Newtonian spatiotemporal mechanics. In particular we show that, unlike spatial manifolds in which the contact forces that are associated with the decomposition of 0-cells would render mass points to join to form elementary particles, the forces that are associated with the decomposition of temporal 0-cells are short-lived therefore temporal matter cannot form stable physical objects as in the case of mass points in spatial continuum. We also discuss in more details the case of geometric interactions that are associated with the decomposition of 3-cells from a spatiotemporal differentiable manifold and show that the physical interactions that are associated with the evolution of the geometric processes can be formulated in terms of general relativity.

All of the basic, seemingly obvious, physical entities that form the foundation of physics are in fact profoundly difficult to perceive and properly define. These include the so-called physical entities of matter, space and time. For matter, we still don't really know in what forms they actually exist. For space, we only have vague ideas that it is a place where physical objects occupy, if matter is not seen as geometric structures of space itself. And for time, it seems that we simply don't have the required physical ability to grasp it even though we ascertain that we are aware of its existence. If we don't even know how to properly define time then, for example, why should we insist that time must be one-dimensional. This comes from responsive recognition rather than scientific reasoning. In classical physics, even though time has had a radical change in character from Newton absolute to Einstein relative physical entity, both forms of relativistic and non-relativistic dynamics still assume one-dimensional time. On the other hand, in quantum physics, time cannot even be defined at the Planck scale. Therefore it is reasonable to say that time in physics still remains an open access. Normally we define time with respect to the motion of physical objects in the spatial continuum therefore we may suggest that if the space associated with a physical object does not change then time cannot be defined. In fact this may be seen as being implied in the standard models in the mainstream physics which assume that the observable universe was originated from a singularity with respect to the expansion of not only of space but also of time. Space is expanding with respect to a time flow and time is flowing with respect to an expansion of space. They dually exist. But if time exists dually with space and it can only be defined with respect to the expansion of space then if space can expand in three dimensions then time should also exist in three dimensions [1]. In this work we will extend our previous works to investigate time as three-dimensional continuum which may possess the geometric and topological structures of a CW complex. In spatial dynamics, we showed that spatial geometric interactions could be classified according to n-cells that are decomposed from elementary particles which are assumed to possess the geometric and topological structures of CW complexes in which the evolution of the geometric processes that involve with the intrinsic geometric structure of a manifold can be described by the Ricci flow and Einstein field equations of the gravitational field. We showed that physical interactions associated with geometric interactions can be classified according to the decomposition of submanifolds from a three-dimensional differentiable manifold [2,3]. From the results obtained for the spatial dimensions of elementary particles being considered as differentiable manifolds, we extended our discussions to temporal dynamics in which negative mass can be considered as a form of inertial reaction to the change of time, similar to the inertial reaction by inertial mass in spatial dynamics in Newtonian mechanics. We also showed that magnetic monopole can be considered as a topological structure of the temporal manifold and Dirac relationship between the electric charge and the magnetic charge $\hbar c/q_e q_m = 2$ can be derived purely in terms of topology as $k/q_e q_m = n_S n_T$ [4]. From these considerations and if we assume a complete symmetry between space and time then a temporal differentiable manifold that is associated with an elementary particle should also be a CW complex. As in the case of spatial CW complexes, in order to describe the evolution of a temporal geometric process as a physical interaction we assume that an assembly of cells of a specified dimension will give rise to a certain form of physical interactions and the intermediate particles, which are the force carriers of physical fields decomposed during a geometric evolution, may possess the geometric structures of the n-spheres and the n-tori and there should also be a classification of the geometric interactions of the temporal CW complexes. This also leads to a more profound speculation that physical properties assigned to an elementary particle, such as magnetic charge, are in fact manifestations due to the force carriers rather than physical quantities that are contained inside the elementary particle. As in the case of spatial dimensions, physical interactions in temporal dynamics will also be reduced to the analysis of the geometric processes that are related to the geometric structures of the force carriers. Therefore, for observable physical phenomena, the study of physical dynamics reduces to the study of the geometric evolution of differentiable manifolds. In particular, if an elementary particle is considered to be composed of not only a three-dimensional spatial manifold but also a three-dimensional temporal manifold then there are four different types of temporal geometric interactions that are resulted from the decomposition of 0-cells, 1-cells, 2-cells and 3-cells. We will discuss this situation further in the following using equations of motion from both the spatial and temporal Newton's second laws of motion

$$m\frac{d^2\mathbf{r}}{d\tau^2} = \mathbf{F} \tag{1}$$

$$D\frac{d^2\mathbf{t}}{ds^2} = \mathbf{F} \tag{2}$$

Depending on the topological structure of the cells it is possible to devise different forms of force. The following presentation in this work is rather suggestive therefore we will only consider radial motion in both formulations of spatial and temporal dynamics.

Forces associated with temporal 0-cells: In our previous work on spatial dimensions we suggested that, for a definite perception of physical existence, the spatial continuum is occupied with mass points which join together by contact forces to form elementary particles which have the geometric and topological structures of CW complexes. We can extend this view by stating that spatial mass is an inertial property that resists a geometric process of evolution. Now for the case of the physical existence of time, we assume that a temporal differentiable manifold is occupied by temporal mass points which interact with each other through the decomposition of temporal 0-cells. This temporal mass is also viewed as an inertial property that resists a geometric process of evolution of the temporal geometric and topological structure of the elementary particle. However, since the temporal 0-cells have temporal dimension zero therefore there are only initial forces between the temporal mass points, which can be assumed to be constant $F = h_0$. When the temporal mass points join together through the initial forces they form the temporal structures of elementary particles. The temporal 0-cells with initial forces can be arranged to form a particular topological structure. However, since $F = h_0$ at $t = 0$ and $F = 0$ for $t > 0$, the geometric structures of temporal matter of elementary particles will not be as stable as those that are formed by the inertial mass in the spatial continuum. This may be the reason why temporal physical objects cannot be observed. However, if we assume that the temporal 0-cells are continuously decomposed from a temporal manifold then the associated forces may be steady and in this case we can apply Newton's laws to determine the dynamics of a particle under their influences. Using the spatial Newton's second law, we obtain

$$m\frac{d^2r}{dt^2} = h_0 \tag{3}$$

By integration we obtain

$$r = \frac{h_0}{m}t^2 + v_0t + r_0 \tag{4}$$

Depending on the sign of h_0, with $m > 0$, we have either a repulsive or attractive gravity. On the other hand, if we apply the temporal Newton's second law then we obtain

$$D\frac{d^2t}{dr^2} = h_0 \tag{5}$$

$$t = \frac{h_0}{2D} r^2 + \frac{1}{v_0} r + t_0 \tag{6}$$

Forces associated with temporal 1-cells: For the case of 1-cells, it is anticipated that they will manifest as either a linear force $F \sim t$ or a force of inverse law $F \sim 1/t$ or a combination of the two

$$\mathbf{F} = h_1 \mathbf{t} \tag{7}$$

$$\mathbf{F} = \frac{h_2 \mathbf{t}}{t^2} \tag{8}$$

$$\mathbf{F} = \left(h_1 + \frac{h_2}{t^2}\right) \mathbf{t} \tag{9}$$

Applying the spatial Newton's second law for radial motion to Equation (7) we obtain

$$m \frac{d^2 r}{dt^2} = h_1 t \tag{10}$$

$$r = \frac{h_1}{6m} t^3 + v_0 t + r_0 \tag{11}$$

Applying the temporal Newton's second law for radial motion to Equation (7) we obtain

$$D \frac{d^2 t}{dr^2} = h_1 t \tag{12}$$

$$t = c_1 e^{\sqrt{h_1/D} r} + c_2 e^{-\sqrt{h_1/D} r} \tag{13}$$

If $D = -m$ and $h_1 > 0$, where m is the inertial mass, then we have

$$t = A \sin(\omega r) \tag{14}$$

where $\omega = \sqrt{h_1/D}$.

Applying the spatial Newton's second law to Equation (8) for radial motion we obtain

$$m \frac{d^2 r}{dt^2} = \frac{h_2}{t} \tag{15}$$

$$r = \frac{h_2}{m} t \ln(t) + c_1 t + c_2 \tag{16}$$

Applying the temporal Newton's second law to Equation (8) for radial motion we obtain

$$D \frac{d^2 t}{dr^2} = \frac{h_2}{t} \tag{17}$$

Solutions to Equation (17) can be found in terms of the inverse of the Gauss error function $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-x^2}\,dx$ as follows [5]

$$t = \exp\left(\frac{-c_1 - 2a\,\mathrm{erf}^{-1}\left(\pm i\sqrt{\frac{2}{\pi}}\sqrt{a e^{\frac{c_1}{a}}(c_2 + r)^2}\right)^2}{2a}\right) \tag{18}$$

Forces associated with temporal 2-cells: The decomposed 2-cells from a temporal manifold can manifest either as a square force $F{\sim}t^2$ or a force of inverse square law $F{\sim}\,1/t^2$ or a combination of the two

$$\mathbf{F} = h_3 t\mathbf{t} \tag{19}$$

$$\mathbf{F} = \frac{h_4\mathbf{t}}{t^3} \tag{20}$$

$$\mathbf{F} = \left(h_3 t + \frac{h_4}{t^3}\right)\mathbf{t} \tag{21}$$

Applying the spatial Newton's second law to Equation (19) for radial motion we obtain

$$m\frac{d^2 r}{dt^2} = h_3 t^2 \tag{22}$$

$$r = \frac{h_3}{12m}t^4 + v_0 t + r_0 \tag{23}$$

Applying the Newton's second law to Equation (19) for radial motion we have

$$D\frac{d^2 t}{dr^2} = h_3 t^2 \tag{24}$$

Solutions to Equation (24) can be found in terms of the Weierstrass elliptic function $\wp$ as follows [5]

$$t = \left(\frac{6D}{h_3}\right)^{\frac{1}{3}} \wp\left(\left(\frac{h_3}{6D}\right)^{\frac{1}{3}}(r + c_1); 0, c_2\right) \tag{25}$$

Applying the Newton's second law to Equation (20) for radial motion we obtain

$$m\frac{d^2 r}{dt^2} = \frac{h_4}{t^2} \tag{26}$$

$$r = -\frac{h_4}{m}\ln(t) + c_1 t + c_2 \tag{27}$$

Applying the Newton's second law to Equation (20) for radial motion we obtain

$$D\frac{d^2t}{dr^2} = \frac{h_4}{t^2} \tag{28}$$

$$r = \pm\left(\frac{1}{c_1}t\sqrt{c_1 - \frac{2h_4}{Dt}} + \frac{h_4}{Dc_1^{\frac{3}{2}}}\ln\left(\sqrt{c_1}t\sqrt{c_1 - \frac{2h_4}{Dt}} - \frac{h_4}{D} + c_1t\right)\right) + c_2 \tag{29}$$

Forces associated with temporal 3-cells: For the decomposition of 3-cells from a temporal manifold, even though it should be considered as a manifestation of either a cube force $F{\sim}t^3$ or a force of inverse cube law $F{\sim}\, 1/t^3$ or a combination of the two, this form of geometric evolution can be applied to a cosmological evolution described in terms of the temporal general relativity. The cube force and the inverse cube law are given as

$$\mathbf{F} = h_5t^2\mathbf{t} \tag{30}$$

$$\mathbf{F} = \frac{h_6\mathbf{t}}{t^4} \tag{31}$$

$$\mathbf{F} = \left(h_5t^2 + \frac{h_6}{t^4}\right)\mathbf{t} \tag{32}$$

Applying the spatial Newton's second law to Equation (30) for radial motion we obtain

$$m\frac{d^2r}{dt^2} = h_5t^3 \tag{33}$$

$$r = \frac{h_3}{20m}t^5 + v_0t + r_0 \tag{34}$$

Applying the temporal Newton's second law to Equation (30) for radial motion we obtain

$$D\frac{d^2t}{dr^2} = h_5t^3 \tag{35}$$

Applying the spatial Newton's second law to Equation (31) for radial motion we obtain

$$m\frac{d^2r}{dt^2} = \frac{h_6}{t^3} \tag{36}$$

$$r = \frac{h_6}{2mt} + c_1t + c_2 \tag{37}$$

Applying the temporal Newton's second law to Equation (31) for radial motion we obtain

$$D\frac{d^2t}{dr^2} = \frac{h_6}{t^3} \tag{38}$$

From the above considerations, in addition to geometric interactions in spatial dimensions, we can assume that a general spatiotemporal force which is a combination of the forces resulted from the decomposition of spatiotemporal *n*-cells of all dimensions to take the form

$$F = \sum_{n=-3}^{3} (k_n r^n + h_n t^n) \tag{39}$$

where k_n and h_n are constants which can be determined from physical considerations.

As in the case of spatial dimensions, we can discuss in more details the case of physical interactions that are associated with the decomposition of 3-cells from a temporal differentiable manifold and show that the physical interactions that are associated with the evolution of the temporal geometric processes can be formulated in terms of temporal general relativity [6]. Mathematically, the forming and releasing of a 3-cell from a 3-dimensional temporal manifold M can be expressed as a decomposition in the form $M = M\#S^3$ where S^3 is a three-dimensional temporal sphere. We assume that the physical interactions associated with the forming and releasing of 3-cells are temporal geometric processes that smooth out irregularities of the intrinsic geometric structure of the temporal manifold. The geometric irregularities can be viewed physically as an inhomogeneous distribution of temporal matter in the temporal manifold and the forming and releasing of the S^3 cells as a flow. With this realisation, the geometric process of decomposition of 3-cells S^3 to smooth out irregularities of the distribution of temporal matter in the observable universe can be formulated in terms of temporal general relativity in which the change of intrinsic geometric structures of the manifold is due to the change of mathematical objects that define the manifold. These mathematical objects are perceived as physical entities like the energy-momentum tensor and the equations that describe the changes can be obtained from mathematical identities, such as Bianchi identities, Ricci flow or Einstein field equations of temporal general relativity. For example, we can apply Einstein field equations of the temporal general relativity in which the roles of space and time are reversed given as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{40}$$

where $T_{\mu\nu}$ is the energy-momentum tensor, $g_{\mu\nu}$ is the metric tensor, $R_{\mu\nu}$ is the Ricci curvature tensor, R is the scalar curvature, Λ is the cosmological constant. As in the case of spatial dimensions, the decomposition of a three-dimensional temporal sphere S^3 can be described in terms of the pseudo-Euclidean Robertson-Walker metric

$$ds^2 = c_T{}^2 dr^2 - S^2(r)\left(\frac{dt^2}{1 - kt^2} + t^2(d\theta^2 + \sin^2\theta d\phi^2)\right) \tag{41}$$

or with the Euclidean Robertson-Walker metric

$$ds^2 = c_T{}^2 dr^2 + S^2(r)\left(\frac{dt^2}{1 - kt^2} + t^2(d\theta^2 + \sin^2\theta d\phi^2)\right) \tag{42}$$

where $k = -1, 0, 1$. It should be mentioned here that, contrary to the case of the geometric evolution of a decomposed spatial sphere, the geometric process of evolution of a decomposed temporal sphere from a spacetime manifold is described with respect to the expansion of space. We now discuss in more details this kind of dual expansion and show that there are profound differences between the physical properties of physical objects that exist at the macroscopic scale and those that exist at the microscopic level. These differences have led to different but equivalent forms of mathematical formulations of quantum physics. The main reason that is behind all of these differences could be due to the fact that the spacetime structure of quantum particles is different from the spacetime structure of macroscopic objects, which is contrary to the general assumption that the formulation of the classical dynamics and the formulation of the quantum dynamics are based on the same structure of spacetime. We now show that the structure of spacetime at the microscopic scale is in fact different from that at the macroscopic scale. First we assume that the universe is a six-dimensional differentiable manifold M which is composed of a three-dimensional spatial manifold and a three-dimensional temporal manifold. Furthermore, the manifold M can be decomposed in the form $M = M\#S_S^3\#S_T^3$, where S_S^3 and S_T^3 are spatial and temporal spheres, respectively. With the decomposition of the form $M = M\#S_S^3\#S_T^3$, the Robertson-Walker metric can be extended to take the form

$$ds^2 = S^2(r)\left(\frac{dt^2}{1-k_T t^2} + t^2\left(d\theta_T{}^2 + \sin^2\theta_T d\phi_T{}^2\right)\right) - S^2(t)\left(\frac{dr^2}{1-k_S r^2} + r^2\left(d\theta_S{}^2 + \sin^2\theta_S d\phi_S{}^2\right)\right) \tag{43}$$

If we arrange the (θ, ϕ) directions of both the spatial manifold and the temporal manifold so that

$$\theta_S = \theta_T = \theta, \quad \text{and} \quad \phi_S = \phi_T = \phi \tag{44}$$

then the general space-time metric given in Equation (43) becomes

$$ds^2 = \frac{S^2(r)dt^2}{1-k_T t^2} - \frac{S^2(t)dr^2}{1-k_S r^2} - \left(r^2 S^2(t) - t^2 S^2(r)\right)(d\theta^2 + \sin^2\theta d\phi^2) \tag{45}$$

However, as an illustration, in the following we will discuss a similar but simpler problem using the case when both spatial and temporal fields are centrally symmetric with the general spacetime metric of the six-dimensional spacetime written as

$$ds^2 = e^{\psi}c^2dt^2 + c^2t^2\left(d\theta_T{}^2 + \sin^2\theta_T d\phi_T{}^2\right) - e^{\chi}dr^2 - r^2\left(d\theta_S{}^2 + \sin^2\theta_S d\phi_S{}^2\right) \tag{46}$$

If we also arrange the (θ, ϕ) directions of both the spatial manifold and the temporal manifold satisfy the condition given in Equation (44) then we

$$ds^2 = e^{\psi}c^2dt^2 - e^{\chi}dr^2 - (r^2 - c^2t^2)(d\theta^2 + \sin^2\theta d\phi^2) \tag{47}$$

We now show that there are profound differences in the structure of space-time that arise from the line element given in Equation (47). First, we show that the line element given in Equation (47) can lead to the conventional structure of space-time in which, effectively, space has three dimensions and time has one dimension. The line element in Equation (47) can be re-written in the form

$$ds^2 = e^{\psi}c^2dt^2 - e^{\chi}dr^2 - r^2\left(1 - \frac{c^2}{v^2}\right)(d\theta^2 + \sin^2\theta d\phi^2) \tag{48}$$

where we have defined the new quantity that has the dimension of speed as $v = r/t$. For $v \to \infty$, the only observable structures of space-time are those that are determined by the metric of the form $ds^2 = e^{\psi}c^2dt^2 - e^{\chi}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$. This line element is the usual Schwarzschild metric in Einstein's theory of general relativity. Instead of the metric form given in Equation (48), the line element given in Equation (47) can also be re-written in a different form as follows

$$ds^2 = e^{\psi}c^2dt^2 + c^2t^2\left(1 - \frac{v^2}{c^2}\right)(d\theta^2 + \sin^2\theta d\phi^2) - e^{\chi}dr^2 \tag{49}$$

It is seen that when $v \to 0$, the line element given in Equation (49) is reduced to the line element for a spacetime manifold in which, effectively, time has 3 dimensions and space has one dimension, namely, $ds^2 = e^{\psi}c^2dt^2 + c^2t^2(d\theta^2 + \sin^2\theta d\phi^2) - e^{\chi}dr^2$. As has been discussed in our work on the temporal relativity that for the case $r^2 - c^2t^2 \neq 0$ the line element given in Equation (47) can be determined by applying Einstein's general relativity [6] as

$$ds^2 = \left(1 - \frac{2GM}{c^2r}\right)c^2dt^2 - \left(1 - \frac{2GM}{c^2r}\right)^{-1}dr^2 - (r^2 - c^2t^2)(d\theta^2 + \sin^2\theta d\phi^2) \tag{50}$$

Now for the case when $r^2 - c^2t^2 \to 0$, the line element given in Equation (47) is reduced to

$$ds^2 = e^{\psi}c^2dt^2 - e^{\chi}dr^2 \tag{51}$$

This line element is similar to the line element of the 2-dimensional space-time manifold (t, r) that we considered in our previous work on the wave-particle duality in quantum physics that can be formulated by applying the principle of least action [7]. The field equations of general relativity given in Equation (40) can be derived using the principle of least action $\delta S = 0$, where the action S is defined as

$$S = \int\left(\frac{1}{2\kappa}(R - 2\Lambda) + \mathcal{L}_M\right)\sqrt{-g}\, d^4x \tag{52}$$

where $\mathcal{L}_M$ characterises matter fields. Since $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \equiv 0$ for $n = 2$, the field equations given by Equation (40) reduces to

$$T_{\mu\nu} = \left(\frac{\Lambda}{\kappa}\right)g_{\mu\nu} \tag{53}$$

Equation (53) indicates that a physical entity can be directly identified with a mathematical object. This interesting feature can be seen as an underlying principle for quantum physics.

As long as the energy-momentum tensor $T_{\mu\nu}$ is directly identified with the metric tensor $g_{\mu\nu}$ through the relationship given by Equation (53) then when $n = 2$ we have $R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R$, therefore the resulting equations given in Equation (40) from the principle of least action are satisfied by any metric tensor $g_{\mu\nu}$. If the energy-momentum tensor is defined through the relation given in Equation (53) then it can be verified that the following local conservation of energy-momentum is satisfied

$$\nabla_\beta T^{\alpha\beta} = 0 \tag{54}$$

Since the functions $\psi = \psi(t, r)$ and $\chi = \chi(t, r)$ are arbitrary functions of space and time, in order to formulate a classical dynamics for the gravitational field we need to establish a dynamical equation. As mentioned above, due to the identity $R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R$ it is not possible to obtain the required dynamical equation for the gravitational field by applying Equation (40). In this case, however, a dynamical equation can be obtained if we assume that the energy-momentum tensor also satisfies the following equation of continuity

$$\frac{\partial T_{tt}}{c\partial t} + \frac{\partial T_{rr}}{\partial r} = 0 \tag{55}$$

Using Equation (53) we obtain

$$e^{\psi} \frac{\partial \psi}{c\partial t} - e^{\chi} \frac{\partial \chi}{\partial r} = 0 \tag{56}$$

If we assume further that $\psi \equiv \chi$, then we have

$$\frac{\partial \psi}{c\partial t} - \frac{\partial \psi}{\partial r} = 0 \tag{57}$$

This equation has the general solution of the form

$$\psi(t, r) = f(r + ct) \tag{58}$$

where $f(t, r)$ is an arbitrary function of t and r. Equation (57) is the transport equation, or the equation of continuity, in classical dynamics which describes a flow or movement of mass, charge, energy or momentum at a constant rate c in the negative direction of r. Even though Equation (57) does not have the status of Newton's dynamical equation to describe the dynamics of a particle, it represents a particle motion in a deterministic manner. On the other hand, the general solution given in Equation (58) is also a general solution of the following wave equation

$$\frac{\partial^2 \psi}{c^2 \partial t^2} - \frac{\partial^2 \psi}{\partial r^2} = 0 \tag{59}$$

From these results it can be concluded that the function $\psi(t, r) = f(r + ct)$ can be used to describe the dynamics of either a particle or a wave.

As a further remark, we would like to mention here that if time has three dimensions then any dynamics that is associated with it will require a far more complex formulation than those that assume a one-dimensional time. For example, consider the equation of motion of a free particle. The equation can be written in a vector form as follows

$$\frac{d^2\mathbf{r}}{dt^2} = 0 \tag{60}$$

In Newtonian classical mechanics with one-dimensional time, the position vector which satisfies Equation (60) takes the simple form

$$\mathbf{r} = \mathbf{v}t + \mathbf{r}_0 \tag{61}$$

where $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is the constant velocity of the particle and $\mathbf{r}_0$ is an initial position. However, with the three-dimensional temporal manifold, if we apply the temporal rate with explicit partial derivatives to Equation (60) then we obtain

$$\sum_{i,j=1}^{3} c_i c_j \frac{\partial^2\mathbf{r}}{\partial t_i \partial t_j} + \sum_{i=1}^{3} c_i \frac{\partial^2\mathbf{r}}{\partial t_i \partial t} + \frac{\partial^2\mathbf{r}}{\partial t^2} = 0 \tag{62}$$

where we have assumed $dt_i/dt = c_i$, with c_i's are constants. It is seen that the vector $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ plays the role of the velocity $\mathbf{v}$ in spatial dynamics. We now show that besides the velocity $\mathbf{v}$, there exists a physical quantity that can be interpreted as spin in quantum mechanics. The position vector that satisfies Equation (62) is

$$\begin{aligned}\mathbf{r} = (a_{11}t_1 + a_{12}t_1 + a_{13}t_1 + v_1 t)\mathbf{i} + (a_{21}t_1 + a_{22}t_1 + a_{23}t_1 + v_2 t)\mathbf{j} \\ + (a_{31}t_1 + a_{32}t_1 + a_{33}t_1 + v_3 t)\mathbf{k}\end{aligned} \tag{63}$$

where a's and v's are arbitrary constants. Using the conditions $dt_i/dt = c_i$, this position vector can be re-written as

$$\mathbf{r} = (\mathbf{i} \quad \mathbf{j} \quad \mathbf{k})\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} t + \mathbf{v}t + \mathbf{r}_0 \tag{64}$$

It is seen from this form that if a 3-dimensional temporal manifold is introduced along side with the 3-dimensional spatial manifold then, in addition to the velocity $\mathbf{v}$, the motion of a free particle is also described by a matrix, which, even though the overall effect being a linear motion in space, can be represented as rotation in the temporal manifold if the matrix satisfies the orthogonality condition $\sum_{i=1}^{3} a_{ij}a_{ik} = \delta_{jk},\ j,k = 1,2,3$. It should be mentioned here that there is an isomorphism between the set of 3-dimensional orthogonal matrices and the set of pair of matrices (Q,-Q), where Q is a transformation matrix which represents the Cayley-Klein parameters $(\alpha, \beta, \gamma, \delta)$, which can be written in terms of the Euler angles (ϕ, θ, ψ) as follows [8]

$$Q = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} e^{\frac{i(\psi+\phi)}{2}} \cos\frac{\theta}{2} & ie^{\frac{i(\psi-\phi)}{2}} \sin\frac{\theta}{2} \\ ie^{\frac{i(\psi-\phi)}{2}} \sin\frac{\theta}{2} & e^{-\frac{i(\psi+\phi)}{2}} \cos\frac{\theta}{2} \end{pmatrix} \tag{61}$$

The half angles and the double-valued property of the isomorphism are related to the fact that the value of the spin of an elementary particle such as an electron is half integral.

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