An Upper Bound on the number of distinct Composition Series in a finite group*

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Abstract

In this paper we prove that among all finite groups of order $\leq n$ (where $n \geq 4$ be a natural number) the number of distinct composition series is bounded above by $\prod_{i=1}^{[\log_2 n]} (2^i - 1)$ and it is attained if and only if G is the elementary abelian 2-group of order 2^α, where $\alpha = [\log_2 n]$. This bound is the a non trivial upper bound of composition series and so far best possible.

Keywords: Finite group, composition series, Sylow subgroup, elementary abelian group
Mathematics Subject Classifications(2010): 20D30, 20D15

1 Introduction

A composition series, that is a series of subgroups each normal in the previous such that corresponding factor groups are simple. Any finite group has a composition series.The concept of a composition series in a group is due to Evariste Galois(1831)[Theorem 5.9, [1]]. The famous Jordan-Holder theorem which says that, the composition factors in a composition series are unique up to isomorphism was proved in nineteenth century [5, 6]. Sometimes a group of small order has a huge number of distinct composition series. For example an elementary abelian group of order 64 has 615195 distinct composition series.To find an upper bound for composition series in a finite group is a natural question. In [7], there is an algorithm in GAP to find the distinct composition series of any group of finite order.The aim of this paper is to provide an upper bound of the number of distinct composition series of any group of finite order. The approach of this is combinatorial and the method is elementary. All the groups considered in this paper are of finite order.

There are three sections in the paper namely section 2, section 3 and section 4. In section 2 we prove Theorem 2.1. which is the main theorem of section 2. All the theorems and results in section 2 are well known. In section 3 we prove Theorem 3.7. which is a new result and it is the main theorem of section 3. In section 4 we prove Theorem 4.3. by proving an inequality in number theory. This theorem is the main theorem of this paper.

2 Preliminary Results

Theorem 2.1. *Among all finite groups of order n, the abelian group with elementary abelian Sylow subgroups, has the highest number of distinct composition series.*

Before proving theorem 2.1 we recall some useful theorems in group theory at first.

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Theorem 2.2 (Theorem 2.28, [1]). *(Correspondence Theorem) Given $G \triangleright N$, denote the the family of subgroups of G that contain N by $sub(G;N)$ and the family of subgroups of G/N by $sub(G/N)$. Then there is a bijection*

$$\phi : sub(G;N) \longrightarrow sub(G/N),$$

$$H \mapsto H/N$$

which preserves all subgroup lattice and normality relationships.

Theorem 2.3 (Exercise 14, Section 5.2, [8]). *A finite abelian group is self dual.*

Lemma 2.4 (Lemma 4.2.7, [9]). *Suppose G be a finite group and let $\prod(G) = \{p_1, p_2, \ldots, p_r\}$ where $\prod(G)$ denotes the distinct prime factors in the order of G. Let $P_i \in Syl_{p_i}(G)$. If $G = P_1 \times P_2 \times \cdots \times P_r$ then*

$$\mid m(G) \mid = \sum_{i=1}^{r} \mid m(P_i) \mid,$$

where $m(G)$ denotes the distinct maximal subgroups in G.

Corollary 2.5. *If G be the abelian group with elementary abelian Sylow subgroups of order n where $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ then $m(G) = \sum_{i=1}^{r} \left(\frac{p_i^{\alpha_i} - 1}{p_i - 1} \right)$.*

Proof. Since G is a finite abelian group then it is self dual and direct products of its Sylow subgroups. Therefore number of maximal subgroups in G is equal to the sum of all the maximal subgroups in it's Sylow subgroups. Now elementary abelian group is self dual. So number of maximal subgroups in an elementary abelian group is exactly same with the number of minimal subgroups.Therefore number of subgroups of index p_i is number of distinct subgroups of order $p_i$in an elementary abelian $p_i$group of order $p_i^{\alpha_i}$ which is $\frac{p_i^{\alpha_i} - 1}{p_i - 1}$ and the result follows immediately. □

Example 2.6. *If $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5$ then $\mid G \mid = 3600$ and*

$$m(G) = \frac{2^4 - 1}{2 - 1} + \frac{3^2 - 1}{3 - 1} + \frac{5^2 - 1}{5 - 1} = 15 + 4 + 6 = 25.$$

Lemma 2.7. *If $G = S_1 \times S_2$, where S_1, S_2 are two finite simple groups. If G_1 is a proper non-trivial normal subgroup of G then G_1 is isomorphic to either S_1or S_2.*

Proof. We consider the projection map $p : G \to S_1$ and the projection map is onto. Therefore the image of a normal subgroup under the projection map p will be a normal subgroup in S_1. Since S_1 is a simple group we have the following two choices :

Case 1 : If $p(G_1) = \{e\}$. Then G_1 is a subgroup of S_2 and it is a non trivial normal subgroup of the simple group group S_2 then $G_1 = S_2$ i.e. $G_1 \backsimeq S_2$.

Case 2 : If $p(G_1) = S_1$. Now we will take the kernel $K = S_2 \cap G_1$of the projection map. Here K will become a normal subgroup of the simple group S_2. So either $K = \{e\}$or S_2. If $K = \{e\}$ then $G_1 \backsimeq S_1$. If $K = S_2$ then $G_1 \backsimeq S_1 \times S_2$, which is a contradiction since $G_1$is a proper subgroup of G. □

Corollary 2.8. *If $G = S_1 \times S_2 \times \cdots \times S_k$ be the direct product of k non-abelian simple groups then G has 2^k normal subgroups namely $N_1 \times N_2 \times \cdots \times N_k$ where either $N_i = \{e\}$ or $N_i = S_i$ for $1 \leq i \leq k$. Also G has only k maximal normal subgroups namely $M_1, M_2, \ldots, M_k$ where $M_i = S_1 \times S_2 \times \cdots \times S_{i-1} \times e \times S_{i+1} \times \cdots \times S_k$ and $G/M_i \backsimeq S_i$ for $1 \leq i \leq k$.*

Example 2.9. *Let $G = A_5 \times A_5$ then $\mid G \mid = 3600$ and G has only $2^2 = 4$ normal subgroups namely $e \times e$, $e \times A_5$, $A_5 \times e$, G and it has only 2 maximal normal subgroups namely $e \times A_5$ and $A_5 \times e$.*

Theorem 2.10 (Theorem 8.6, [4]). *(Birkhoff) Every algebra A is isomorphic to subdirect product of subdirectly irreducible algebras.*

Now we present the proof of theorem 2.1.

Proof. Let G be the abelian group with elementary abelian Sylow subgroups such that $\mid G \mid = n$. Now by lemma 2.4, number of maximal subgroups in a finite nil-potent group of order n is exactly same of the sum of distinct maximal subgroups in each Sylow subgroup. But we know that among all finite p groups of same order the elementary abelian group has highest number of maximal subgroups. This proves that among all nil-potent groups of order n, G has the highest number of maximal normal subgroups and hence the highest number of composition series. Let K be a solvable group of order n with m maximal normal subgroups namely $M_1, M_2, \ldots, M_m$. Then index of M_i in K is a prime and $K/M_i \backsimeq \mathbb{Z}_p$ for some prime p. Let C' be the commutator subgroup of K. Then $C' \trianglelefteq M_i$. Let $M = \cap_{i=1}^{m} M_i$ be the intersection of all the maximal normal subgroups of K. M is a normal subgroup of K such that $C' \trianglelefteq M$ and therefore K/M is abelian. Now by Correspondence Theorem there exists a bijection

$$\phi : sub\,(K; M) \longrightarrow sub\,(K/M)\,,$$

which preserves subgroup lattice and normality relationship. Therefore K/M is an abelian group with exactly m maximal normal subgroups. But it is already proved that among all abelian groups of same order the abelian group with elementary abelian Sylow subgroups, has the highest number of maximal normal subgroups.For non solvable non simple group we take the Jacobson radical $J(G)$ which is the intersection of all the maximal normal subgroups of G and we will use the subdirect irreducible theorem in universal algebra which says any algebra can be expressed as a subdirect product of subdirectly irreducible algebras . Then $J(G)$ is a characteristic subgroup of G and also $J(G/J(G)) = 1$ (By [3], page - 4). So $J(G)$ is a normal subgroup of G such the the quotient $G/J(G)$ is the direct product of simple groups.(By [2], Lemma 4.1). Now if $J(G)$ is trivial then G is direct product of simple groups. Otherwise, let $J(G)$ be a non trivial then we can get a minimal normal subgroup P such that P is a subgroup of $J(G)$. Then we take the quotient G/P and get a normal covering. Proceeding this way it reduces to the case when G is a direct products of simple groups as G embeds in $G/N_1 \times G/N_2 \times \cdots \times G/N_k$ such that $1 = \cap_{i=1}^{k} N_i$ where each N_i is a maximal normal subgroup and where k is a natural number. Now if H is a non-solvable non-simple group of order n then H is direct product of isomorphic non-abelian simple groups by Corollary 2.8, direct product of k copies of non-abelian simple groups has only k maximal normal subgroups, which proves the theorem. □

3 Some New Results

In this section we prove Theorem 3.7, which is the main result in this section.

Definition 3.1. *We define C_G as the set of all distinct composition series of the group G.*

Theorem 3.2. *Let $n \geq 2$ be a positive integer where $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$. Then*

$$\mid C_{\mathbb{Z}_n} \mid = \frac{(\sum_{i=1}^{r} \alpha_i)!}{\prod_{i=1}^{r} \alpha_i!}.$$

Proof. We know that for each positive divisor d of n there exists a unique subgroup of $\mathbb{Z}_n$ of order n. Let S be the set of sequence of primes from the set $\{p_1, \ldots, p_r\}$ of length $\alpha_1 + \alpha_2 + \cdots + \alpha_r$, where p_i occurs α_i times. Then

$$\mid S \mid = \frac{(\sum_{i=1}^{r} \alpha_i)!}{\prod_{i=1}^{r} \alpha_i!}.$$

Now $C_{\mathbb{Z}_n}$ is the set of distinct composition series of $\mathbb{Z}_n$. Now define a function $f : S \to C_{\mathbb{Z}_n}$ by

$$f\left(\beta_1\beta_2 \ldots \beta_{\alpha_1+\alpha_2+\cdots+\alpha_{r-1}}\beta_{\alpha_1+\alpha_2+\cdots+\alpha_r}\right) = \{e\} \trianglelefteq \mathbb{Z}_{\beta_1} \trianglelefteq \mathbb{Z}_{\beta_1\beta_2} \trianglelefteq \cdots \trianglelefteq \mathbb{Z}_{\beta_1\beta_2\ldots\beta_{\alpha_1+\alpha_2+\cdots+\alpha_r}} = \mathbb{Z}_n,$$

where $\beta_1, \beta_2, \ldots, \beta_{\alpha_1+\alpha_2+\cdots+\alpha_r}$ are primes such that $\beta_i = p_i$ has exactly α_i solutions for $1 \leq i \leq \alpha_1 + \alpha_2 + \cdots + \alpha_r$. So $\beta_1\beta_2 \ldots \beta_{\alpha_1+\alpha_2+\cdots+\alpha_{r-1}}\beta_{\alpha_1+\alpha_2+\cdots+\alpha_r} \in S$. Now $\{e\} \trianglelefteq \mathbb{Z}_{\beta_1} \trianglelefteq \mathbb{Z}_{\beta_1\beta_2} \trianglelefteq \cdots \trianglelefteq \mathbb{Z}_{\beta_1\beta_2\ldots\beta_{\alpha_1+\alpha_2+\cdots+\alpha_r}} = \mathbb{Z}_n$ i.e.$\{e\} \trianglelefteq \mathbb{Z}_{\theta_1} \trianglelefteq \mathbb{Z}_{\theta_2} \trianglelefteq \cdots \trianglelefteq \mathbb{Z}_{\theta_{\alpha_1+\alpha_2+\cdots+\alpha_r}} = \mathbb{Z}_n$ is a composition series of $\mathbb{Z}_n$ where $\theta_i = \prod_{j=1}^{i} \beta_j$, $1 \leq i \leq \alpha_1 + \alpha_2 + \cdots + \alpha_r$. Then $f : S \to C_{\mathbb{Z}_n}$ is an injective mapping by its construction. Now we will prove that it is surjective also. Let $\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{\alpha_1+\alpha_2+\cdots+\alpha_r} = \mathbb{Z}_n$ be a composition series of $\mathbb{Z}_n$. Since for each divisor d of n, there exists a unique subgroup of order d of $\mathbb{Z}_n$ and any subgroup of $\mathbb{Z}_n$ is cyclic then $\dfrac{\mid G_i \mid}{\mid G_{i-1} \mid}$ is a prime number.

Define $g : C_{\mathbb{Z}_n} \longrightarrow S$ by

$$g\left(C_{\mathbb{Z}_n}\right) = q_1 q_2 \dots q_{\alpha_1+\alpha_2+\cdots+\alpha_r},$$

where $q_i = \dfrac{\mid G_i \mid}{\mid G_{i-1} \mid}$, for $1 \leq i \leq \alpha_1+\alpha_2+\cdots+\alpha_r$. Then each q_i is a prime for $1 \leq i \leq \alpha+\alpha_2+\cdots+\alpha_r$. So $q_1 q_2 \dots q_{\alpha_1+\alpha_2+\cdots+\alpha_r}$ is a sequence of primes such that $q_i = p_i$ has exactly α_i solutions for $1 \leq i \leq \alpha_1+\alpha_2+\cdots+\alpha_r$. So, $q_1 q_2 \dots q_{\alpha_1+\alpha_2+\cdots+\alpha_r} \in S$ and therefore $f^{-1} = g$ and hence f is surjective and hence bijective also. Therefore,

$$\mid S \mid = \mid C_{\mathbb{Z}_n} \mid = \frac{(\sum_{i=1}^{r} \alpha_i)!}{\prod_{i=1}^{r} \alpha_i!}$$

i.e. $\mathbb{Z}_n$ has $\dfrac{(\sum_{i=1}^{r} \alpha_i)!}{\prod_{i=1}^{r} \alpha_i!}$ distinct composition series. □

Example 3.3. Let us try to understand the above theorem for $\mathbb{Z}_{360}$. Note that, $360 = 2^3 3^2 5^1$. Now $\frac{(3+2+1)!}{3!2!1!} = 60$ distinct sequence of primes can be formed of length 6 such that each sequence contains 3 times prime 2,2 times prime 3 and 1 time prime 5. Some of them are $232253, 323522, 523322$. Now we will construct a composition series corresponding to these numbers.

1. The composition series corresponding to 232253 will be
$$\{e\} \trianglelefteq \mathbb{Z}_2 \trianglelefteq \mathbb{Z}_6 \trianglelefteq \mathbb{Z}_{12} \trianglelefteq \mathbb{Z}_{24} \trianglelefteq \mathbb{Z}_{120} \trianglelefteq \mathbb{Z}_{360}.$$
2. The composition series corresponding to 323522 will be
$$\{e\} \trianglelefteq \mathbb{Z}_3 \trianglelefteq \mathbb{Z}_6 \trianglelefteq \mathbb{Z}_{18} \trianglelefteq \mathbb{Z}_{90} \trianglelefteq \mathbb{Z}_{180} \trianglelefteq \mathbb{Z}_{360}.$$
3. The composition series corresponding to 523322 will be
$$\{e\} \trianglelefteq \mathbb{Z}_5 \trianglelefteq \mathbb{Z}_{10} \trianglelefteq \mathbb{Z}_{30} \trianglelefteq \mathbb{Z}_{90} \trianglelefteq \mathbb{Z}_{180} \trianglelefteq \mathbb{Z}_{360}.$$

Theorem 3.4. *Let G be an abelian group of order n where $n \geq 2$ be a natural number such that $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$. Then $\mid C_G \mid = \prod_{i=1}^{r} t_i \frac{(\sum_{i=1}^{r} \alpha_i)!}{\prod_{i=1}^{r} \alpha_i!}$ where t_i are the numbers of distinct composition series of the Sylow p_i- subgroups of G.*

Proof. Note that in finite abelian group there exists at least one subgroup of order h for each divisor h of the order of the group n and also each composition factor of an abelian group is group of prime order. The order of composition factors of an finite abelian group is a prime number and using them we can make a sequence of primes which belongs to S (which is described in the Theorem 3.2).

Therefore the number of distinct composition series of G is a multiple of $\frac{(\sum_{i=1}^{r} \alpha_i)!}{\prod_{i=1}^{r} \alpha_i!}$. Now let $\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_m \trianglelefteq G_{m+1} \trianglelefteq \cdots \trianglelefteq G_{\alpha_1+\alpha_2+\cdots+\alpha_r} = G$ be a composition series of G where j be the least positive integer such that $p_1^{s_1}$ divide $\mid G_j \mid$and k be the least positive integer such that $p_1^{s_1+1}$ divide $\mid G_k \mid$ when

$$1 \leq j < k \leq \alpha_1+\alpha_2+\cdots+\alpha_r,\ 1 \leq s < s+1 \leq \alpha_1.$$

Therefore $\mid G_j \mid = p_1^{s_1} q$ where $p_1^{s_1}, q \in \mathbb{N}$ with $\gcd\left(p_1^{s_1}, q\right) = 1$. $\mid G_k \mid = p_1^{s_1+1} u$, where $p_1^{s_1+1}, u \in \mathbb{N}$ with $\gcd\left(p_1^{s_1+1}, u\right) = 1$.

Let $G_j = A_j B_j$ where A_j and B_j are two subgroups of G of order p_1^s and q respectively. Let $G_k = C_k D_k$ where C_k and D_k are two subgroups of G of order $p_1^{s_1+1}$ and u respectively. We will prove that $A_j \subseteq C_k$; if not then, $\exists$ some $x \in A_j$ such that $x \notin C_k$. $x \in A_j \Rightarrow x = p_1^{v_1}$. Now $x \in A_j \implies x \in G_j$, $x \in G_j \implies x \in G_{j+1}$ as G_j is a normal subgroup of G_{j+1}. $x \in G_{j+1} \implies x \in G_{j+2}$ as G_{j+1} is a normal subgroup of G_{j+2}. So proceeding in this way we can prove that $x \in G_k$. Now $G_k = C_k D_k$ where $C_k \cap D_k = \{e\}$. $x \in G_k \implies x \in C_k D_k$, i.e. $x = cd$ where $c \in C_k, d \in D_k$. Therefore $x = p_1^{c_1} d, p_1^{c_1} \in C_k$. So $p_1^{v_1} = p_1^{c_1} d$ i.e. $p_1^{v_1 - c_1} = d$, As $\gcd\left(p_1^{\alpha_1}, u\right) = 1$ so $P_1 \cap D_k = \{e\}$ but $p_1^{v_1-c_1} \in P_1 \cap D_k$ i.e. $p_1^{v_1-c_1} = e$,$p_1^{v_1} = p_1^{c_1}$ hence $x\left(= p_1^{c_1}\right) \in C_k$, a contradiction.

Therefore in the composition series of Sylow p_1-group, A_j is a subgroup of C_k. So if the Sylow p_1-group has t_1 distinct composition series then we have $t_1 \frac{(\sum_{i=1}^{r} \alpha_i)!}{\prod_{i=1}^{r} \alpha_i!}$ such possible composition series due to Sylow p_1-subgroup in G. In a similar way for the other Sylow p_2-group, ..., Sylow p_r- group we have $\prod_{i=1}^{r} t_i \frac{(\sum_{i=1}^{r} \alpha_i)!}{\prod_{i=1}^{r} \alpha_i!}$ distinct composition series for G. □

Theorem 3.5. *In an elementary abelian group of order p^n, there are $\prod_{k=1}^{n}\left(\frac{p^k-1}{p-1}\right) = \frac{\prod_{k=1}^{n}(p^k-1)}{(p-1)^n}$ distinct composition series.*

Proof. Let $\{e\} \trianglelefteq p \trianglelefteq p^2 \trianglelefteq \cdots \trianglelefteq p^{k-1} \trianglelefteq p^k \trianglelefteq \cdots \trianglelefteq p^n$ be a composition series for the elementary abelian group of order p^n. To find the number of distinct composition series, we will find the possible choice from p^{k-1} to p^k at first.

Let there be C_{k-1} choice available for p^{k-1} to p^k. Now in an elementary abelian group of order p^n there are

$$\frac{\prod_{i=0}^{k-1}\left((p^n-1)-(p^i-1)\right)}{\prod_{i=0}^{k-1}(p^k-p^i)} = A \text{ (say)}$$

distinct subgroups of order p^k each of which is an elementary abelian (By [10], page - 65). In an elementary abelian group of order p^n, also there are

$$\frac{\prod_{i=0}^{k-2}\left((p^n-1)-(p^i-1)\right)}{\prod_{i=0}^{k-2}(p^k-p^i)} = B \text{ (say)}$$

distinct subgroups of order p^{k-1} each of which is an elementary abelian(By [10], page-65). Again in an elementary abelian group of order p^k there are

$$\frac{\prod_{i=0}^{k-2}\left((p^k-1)-(p^i-1)\right)}{\prod_{i=0}^{k-2}(p^k-p^i)} = D \text{ (say)}$$

distinct subgroups of order p^{k-1} each of which is elementary abelian (By [10], page-65). Now there is an identical algebraic structure of any two subgroups of same order in an elementary abelian group i.e. in elementary abelian group of order p^n , any two distinct subgroups of order p^k have the same number of subgroup order p^{k-1}. Then

$$C_{k-1} = \frac{A.B}{D} = \frac{p^{n+1-k}-1}{p-1}.$$

As a result we have $\prod_{k=1}^{n-1}\left(\frac{p^{n+1-k}-1}{p-1}\right)$distinct possible choice for the composition series of an elementary abelian group of order p^n (as $k \geq 1$), which is same as

$$\prod_{k=1}^{n}\left(\frac{p^{n+1-k}-1}{p-1}\right) = \prod_{k=1}^{n}\left(\frac{p^k-1}{p-1}\right) = \frac{\prod_{k=1}^{n}(p^k-1)}{(p-1)^n}.$$

□

Theorem 3.6. *Let G be an abelian group with elementary abelian Sylow subgroups of order $n \geq 2$ where $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_r^{\alpha_r}$, then*

$$| C_G |= \prod_{i=1}^{r}\left(\prod_{j=1}^{\alpha_i}\frac{p_i^j-1}{p_i-1}\right)\frac{(\sum_{i=1}^{r}\alpha_i)!}{\prod_{i=1}^{r}\alpha_i!}.$$

Proof. This followed from theorem 3.4 and theorem 3.5. □

Theorem 3.7. *Let G be a finite group of order n where $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_r^{\alpha_r}$, then*

$$| C_G |\leq \prod_{i=1}^{r}\left(\prod_{j=1}^{\alpha_i}\frac{p_i^j-1}{p_i-1}\right)\frac{(\sum_{i=1}^{r}\alpha_i)!}{\prod_{i=1}^{r}\alpha_i!}$$

and the equality is hold if and only if G is the abelian group with elementary abelian sylow subgroups.

Proof. This followed from theorem 2.1 and theorem 3.6. □

4 Main Result

In this section we prove theorem 4.3 which is the main theorem of the paper. This section is more elementary and computational.

Lemma 4.1. *Let* $X = \prod_{i=\alpha_1+1}^{\alpha_1+k}(2^i-1)\alpha_1!\alpha_r!$ *and* $Y = \prod_{j=1}^{\alpha_r}(\frac{p^j-1}{p-1})(\alpha_1+\alpha_r)!$*, where* $p \geq 5$ *is a prime and* $\alpha_1 \geq 0$ *is an integer,* $\alpha_r \in \mathbb{N}$*, such that* $t = p^{\alpha_r}$ *and* $[\log_2 t] = k$*. Also* $\alpha_r \geq 2$ *whenever* $p = 3$*. Then* $\frac{X}{Y}$ *is a monotone increasing function of* α_1*.*

Proof. Let

$$f(\alpha_1) = \frac{\prod_{i=1}^{\alpha_1+k}(2^i-1)}{\prod_{i=1}^{\alpha_1}(2^i-1)\prod_{j=1}^{\alpha_r}(\frac{p^j-1}{p-1})\frac{(\alpha_1+\alpha_r)!}{\alpha_1!\alpha_r!}}.$$

Then

$$f(\alpha_1+1) = \frac{\prod_{i=1}^{\alpha_1+k+1}(2^i-1)}{\prod_{i=1}^{\alpha_1+1}(2^i-1)\prod_{j=1}^{\alpha_r}(\frac{p^j-1}{p-1})\frac{(\alpha_1+\alpha_r+1)!}{(\alpha_1+1)!\alpha_r!}}.$$

Therefore

$$\frac{f(\alpha_1+1)}{f(\alpha_1)} = \frac{2^{\alpha_1+k+1}-1}{2^{\alpha_1+1}-1}\frac{\alpha_1+1}{\alpha_1+\alpha_r+1}.$$

In order to prove $f(\alpha_1+1) > f(\alpha_1)$ we have to prove that

$$\frac{2^{\alpha_1+k+1}-1}{2^{\alpha_1+1}-1} > \frac{\alpha_1+\alpha_r+1}{\alpha_1+1}.$$

As $\frac{2^{\alpha_1+1+k}-1}{2^{\alpha_1+1}-1} > 2^k$ and $\alpha_r+1 \geq \frac{\alpha_1+1+\alpha_r}{\alpha_1+1}$. So it is sufficient enough to prove

$$2^k > \alpha_r+1 \iff k > \log_2(\alpha_r+1).$$

Now $\log_2 t = k+f$, where $0 < f < 1$. So $k = \log_2 t - f$. Therefore, we have to prove that

$$\log_2 t - f > \log_2(\alpha_r+1) \iff \log_2(\frac{t}{\alpha_r+1}) > f.$$

As $0 < f < 1$ it is sufficient enough to prove $\log_2(\frac{t}{\alpha_r+1}) > 1$ since $\log x$ is a monotone increasing function for $x > 0$, i.e.

$$\frac{t}{\alpha_r+1} > 2 \iff p^{\alpha_r} > 2\alpha_r+2$$

for all prime $p \geq 5$ with $\alpha_r \geq 1$ and $p = 3$ with $\alpha_r \geq 2$, which can easily be proved using mathematical induction. Hence $\frac{X}{Y}$ is a monotone increasing function of α_1. □

Theorem 4.2. *Among all p groups of order* $\leq n$*, where* $n \geq 4$ *is a positive integer; the elementary abelian group of order* 2^α *where* $\alpha = [\log_2 n]$*, has the highest number of composition series.*

Proof. We know that among all finite p-groups of equal order the elementary abelian group has the highest number of composition series. Let G_2 be an elementary abelian group of order 2^α, where $\alpha = [\log_2 n]$ and let G_p be an elementary abelian p group of order p^β, where $\beta = \left[\log_p n\right]$ with $p \geq 3$ be a prime and $n \geq 4$.

Now we know that if $\left[\log_q r\right] = t$ then $\left[\log_q qr\right] = t+1$ for all prime $q \geq 2$ and $r, t \in \mathbb{N}$ with $q \leq r$. Obviously 2 divides $p-1$. Therefore we get

$$2^{[\log_2 n]} - 1 > \frac{p^{\left[\log_p n\right]}-1}{p-1}, \text{ whenever } n \geq 4. \tag{1}$$

$$| C_{G_2} | = \prod_{i=1}^{[\log_2 n]} (2^i-1),$$

$$| C_{G_p} | = \prod_{i=1}^{\left[\log_p n\right]} \left(\frac{p^i-1}{p-1}\right).$$

From (1) it implies that

$$\prod_{i=1}^{[\log_2 n]} \left(2^i - 1\right) > \prod_{i=1}^{[\log_p n]} \left(\frac{p^i - 1}{p - 1}\right)$$

i.e.$\mid C_{G_2} \mid > \mid C_{G_p} \mid$. Therefore among all p groups of order $\leq n$, the elementary abelian group of order 2^α has the highest number of composition series where $\alpha = [\log_2 n]$. □

Theorem 4.3. *Let $n \geq 4$ be a positive integer. Among all finite groups of order $\leq n$, the elementary abelian group of order 2^α has the highest number of composition series where $\alpha = [\log_2 n]$.*

Proof. $n \geq 4$ is a natural number and $\alpha = [\log_2 n]$ and let G be the elementary abelian group of order 2^α then

$$\mid C_G \mid = \prod_{i=1}^{[\log_2 n]} \left(2^i - 1\right).$$

As we know that among all finite groups of order $n \geq 2$, the abelian group with elementary abelian Sylow subgroups has the highest number of composition series. So in order to prove our claim we have to prove that

$$\prod_{i=1}^{[\log_2 n]} \left(2^i - 1\right) \geq \prod_{i=1}^{q} \left(\prod_{j=1}^{\beta_i} \frac{p_i^j - 1}{p_i - 1}\right) \frac{\left(\sum_{i=1}^q \beta_i\right)!}{\prod_{i=1}^q \beta_i!}$$

where each p_i is a prime and $\beta_i, q \in \mathbb{N}$ such that $\prod_{i=1}^q p_i^{\beta_i} \leq n$ and the equality is hold if and only if $q = 1, p_1 = 2$ and $\beta_1 = [\log_2 n]$. The inequality will be proved in the following steps.

Step 1 : Let H be an abelian group of order m with elementary abelian Sylow subgroups such that $4 \leq m \leq n$ where $m = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ also $\mid C_H \mid$denotes the number of distnct composition series of H. Let p_r be a prime such that $p_r \neq 2$ and $p_r^{\alpha_r} = t, [\log_2 t] = k$. Then $2^k < p_r^{\alpha_r}$. Let H' be the abelian group of order $\frac{|H|.2^k}{p_r^{\alpha_r}}$ with elementary abelian Sylow subgroups. Then $\mid H' \mid < \mid H \mid$. We will prove that $\mid C_{H'} \mid > \mid C_H \mid$.

Step 2 : If $\mid H' \mid = 2^k p_1^{\alpha_1} \ldots p_{r-1}^{\alpha_{r-1}}$ i.e. 2 is not a divisor of m then we have $k > \alpha_r$ and

$$\prod_{i=1}^{k} \left(2^k - 1\right) > \prod_{i=1}^{\alpha_r} \left(\frac{p_r^i - 1}{p_r - 1}\right).$$

Therefore, obviously $\mid C_{H'} \mid > \mid C_H \mid$.

Step 3 : If $\mid H \mid = 2^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ then $\mid H' \mid = 2^{\alpha_1 + k} p_2^{\alpha_2} \ldots p_{r-1}^{\alpha_{r-1}}$. In this case we will prove that $\mid C_{H'} \mid > \mid C_H \mid$.

$$\mid C_H \mid = \prod_{i=1}^{\alpha_1} \left(2^i - 1\right) \prod_{i=1}^{\alpha_r} \left(\frac{p_r^{\alpha_r} - 1}{p_r - 1}\right) \prod_{i=2}^{r-1} \left(\prod_{j=1}^{\alpha_i} \frac{p_i^j - 1}{p_i - 1}\right) \frac{\left(\sum_{i=1}^r \alpha_i\right)!}{\prod_{i=1}^r \alpha_i!},$$

$$\mid C_{H'} \mid = \prod_{i=1}^{\alpha_1 + k} \left(2^i - 1\right) \prod_{i=2}^{r-1} \left(\prod_{j=1}^{\alpha_i} \frac{p_i^j - 1}{p_i - 1}\right) \frac{\left(\sum_{i=1}^{r-1} \alpha_i + k\right)!}{\prod_{i=2}^{r-1} \alpha_i! \left(\alpha_1 + k\right)!}.$$

Therefore,

$$\frac{\mid C_{H'} \mid}{\mid C_H \mid} = \frac{\prod_{i=\alpha_1+1}^{\alpha_1+k} \left(2^i - 1\right) (k+s)! \alpha_1! \alpha_r!}{\prod_{j=1}^{\alpha_r} \left(\frac{p_r^j - 1}{p_r - 1}\right) \left(\alpha_1 + k\right)! \left(\alpha_r + s\right)!},$$

where $s = \sum_{i=1}^{r-1} \alpha_i$. So in order to prove our claim we have to prove that

$$\prod_{i=\alpha_1+1}^{\alpha_1+k} \left(2^i - 1\right) (k+s)! \alpha_1! \alpha_r! > \prod_{j=1}^{\alpha_r} \left(\frac{p_r^j - 1}{p_r - 1}\right) \left(\alpha_1 + k\right)! \left(\alpha_r + s\right)! \tag{2}$$

Note that $s = \sum_{i=1}^{r-1} \alpha_i \geq \alpha_1, \Longrightarrow s = \alpha_1 + a$, where $a \geq 0,\ k > \alpha_r \Longrightarrow k = \alpha_r + b,\ b > 0;\ a, b \in \mathbb{Z}$. Then inequality (2) reduces to

$$\prod_{i=\alpha_1+1}^{\alpha_1+k} \left(2^i - 1\right) (\alpha_1 + \alpha_r + a + b)!\alpha_1!\alpha_r! > \prod_{j=1}^{\alpha_r} \left(\frac{p_r^j - 1}{p_r - 1}\right) (\alpha_1 + \alpha_r + a)!\,(\alpha_1 + \alpha_r + b)! \tag{3}$$

[Here we are ignoring the case $p_r = 3$ with $\alpha_r = 1$, as $m \geq 4$ at least one other case must exist.] Note

$$\frac{(\alpha_1 + \alpha_r + a + b)!\alpha_1!\alpha_r!}{(\alpha_1 + \alpha_r + a)!\,(\alpha_1 + \alpha_r + b)!}$$

is a monotone increasing function of a for fixed α_1, α_r, b. So it is sufficient enough to prove the inequality for $a = 0$ i.e.

$$\prod_{i=\alpha_1+1}^{\alpha_1+k} \left(2^i - 1\right) \alpha_1!\alpha_r! > \prod_{j=1}^{\alpha_r} \left(\frac{p_r^j - 1}{p_r - 1}\right) (\alpha_1 + \alpha_r)! \tag{4}$$

Let $X = \prod_{i=\alpha_1+1}^{\alpha_1+k}(2^i - 1)\alpha_1!\alpha_r!$ and $Y = \prod_{j=1}^{\alpha_r}(\frac{p_r^j-1}{p_r-1})(\alpha_1 + \alpha_r)!$. Then according to lemma 4.1 $\frac{X}{Y}$ is a monotone increasing function of α_1 whenever p_r and α_r are fixed. So it is sufficiently enough to prove the inequality (4) for $\alpha_1 = 0$ i.e.

$$\prod_{i=1}^{k} \left(2^i - 1\right) > \prod_{j=1}^{\alpha_r} \left(\frac{p_r^j - 1}{p_r - 1}\right),$$

which is following from theorem 4.2. As a result we prove that $\mid C_{H'} \mid > \mid C_H \mid$. Now we will repeat the same process in the group H' and get a group H'' such that $\mid C_{H''} \mid > \mid C_{H'} \mid$. Since $\mid H \mid$is finite then $\mid H \mid$has only finitely many prime divisors. So after a finite number of steps we get an elementary abelian group of order 2^η, where $\eta \leq [\log_2 n]$.

Step 4 : Let $\mid H \mid = 2^{\alpha_1}3^1$ for any $\alpha_1 \in \mathbb{N}$. Then $\mid C_H \mid = \prod_{i=1}^{\alpha_1}\left(2^i - 1\right)(\alpha_1 + 1)$. Then we can find an elementary abelian group H' such that$\mid H' \mid = 2^{\alpha_1+1}$ and $\mid C_{H'} \mid = \prod_{i=1}^{\alpha_1+1}\left(2^i - 1\right)$. Obviously $\mid C_{H'} \mid > \mid C_H \mid$ as $2^{\alpha_1+1} > \alpha_1 + 2$ for each $\alpha_1 \in \mathbb{N}$. Among all 2 groups of order $\leq n$, G has the highest number of composition series which proves the theorem. □

References

[1] J.J. Rotman, An Introduction to the Theory of Groups, fourth ed., Springer Verlag, 1995.

[2] A.V.Borovik, A.Lubotzky, and A.G.Myasnikov. The finitary Andrews-Curtis conjecture. In Infinite groups: geometric, combinatorial and dynamic aspects, volume 248 of progr. Math., pages 15-30. Birkhauser, Basel,2005.

[3] L. Guyot. On Andrews-Curtis conjecture for the soluble groups. Int. J. Algebra Comput., 28(01), 97-113, 2018.

[4] S.Burris and H.P. Sankappanavar, A Course in Universal Algebra, The Millennium edition, Springer Verlag, 2012.

[5] C.Jordan, Traite des substitutions et des equations algebraique, Gauthier-Villars,1870.

[6] O.Holder, Zuruckfuhrung einer beliebigen algebraischen Gleichung auf eine kette von Gleichungen, Math. Ann.34 (1889) 26-56.

[7] Alexander Hulpke, Constructing All Composition Series of a Finite Group, 2015 ACM 978-1-4503-8/15/07.

[8] D.S.Dummit and R.M.Foote, Abstract algebra, 3rd ed., John Whily and Sons, Inc., Hoboken, NJ, 2004.

[9] Lindsey Kay Lauderdale, Maximal Subgroups of Finite Groups; a dissertation presented to graduate school, University of Florida, 2014.

[10] Gustav Saeden Stahl, Johan Laine, Gustav Behm ; On p-groups of low power order, Department of Mathematics, KTH 2010.