Shellability is NP-complete[1]

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Abstract

We prove that for every $d \geq 2$, deciding if a pure, d-dimensional, simplicial complex is shellable is NP-hard, hence NP-complete. This resolves a question raised, e.g., by Danaraj and Klee in 1978. Our reduction also yields that for every $d \geq 2$ and $k \geq 0$, deciding if a pure, d-dimensional, simplicial complex is k-decomposable is NP-hard. For $d \geq 3$, both problems remain NP-hard when restricted to contractible pure d-dimensional complexes. Another simple corollary of our result is that it is NP-hard to decide whether a given poset is CL-shellable.

1 Introduction

A d-dimensional simplicial complex is called *pure* if all its facets (i.e., inclusion-maximal faces) have the same dimension d. A pure d-dimensional simplicial complex is *shellable* if there exists a linear ordering $\sigma_1,\sigma_2,\ldots,\sigma_n$ of its facets such that, for every $i \geq 2$, $\sigma_i \cap (\cup_{j<i}\sigma_j)$ is a pure $(d-1)$-dimensional simplicial complex; such an ordering is called a *shelling* or *shelling order*.

For example, the boundary of a simplex is shellable (any order works), but no triangulation of the torus is (the condition fails for the first triangle σ_i that creates a non-contractible 1-cycle).

The concept of shellings originated in the theory of convex polytopes (in a more general version for polytopal complexes), as an inductive procedure to construct the boundary of a polytope by adding the facets one by one in such a way that all intermediate complexes (except the last one) are contractible. The fact that this is always possible, i.e., that convex polytopes are shellable, was initially used as an unproven assumption in early papers (see the discussion in [Gru¨03, pp. 141–142] for a more detailed account of the history), before being proved by Bruggesser and Mani [BM72].

The notion of shellability extends to more general objects (including non-pure simplicial complexes and posets [BW97]), and plays an important role in diverse areas including piecewise-linear topology[RS82, Bin83], polytope theory (e.g., McMullen's proof of the *Upper Bound Theorem* [McM70]), topological combinatorics [Bj¨o95], algebraic combinatorics and commutative algebra [Sta96, PRS98], poset theory, and group theory [Bj¨o80, Sha01]; for a more detailed introduction and further references see [Wac07, §3].

One of the reasons for its importance is that shellability—a combinatorial property—has strong topological implications: For example, if a pure d-dimensional complex K is a *pseudomanifold*[1]—which can be checked in linear time—and shellable, then K is homeomorphic to the sphere S^d (or the ball B^d, in case K has nonempty boundary) [DK74]—a property that is algorithmically undecidable for $d \geq 5$,

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by a celebrated result of Novikov [VKF74, Nab95]. More generally, every pure d-dimensional shellable complex is homotopy equivalent to a wedge of d-spheres, in particular it is $(d-1)$-connected.

1.1 Results

From a computational viewpoint, it is natural to ask if one can decide efficiently (in polynomial time) whether a given complex is shellable. This question was raised at least as early as in the 1970's [DK78a, DK78b] (see also [KP03, Problem 34]) and is of both practical and theoretical importance (besides direct consequences for the experimental exploration of simplicial complexes, the answer is also closely related to the question there are simple conditions that would characterize shellability). Danaraj and Klee proved that shellability of 2-dimensional pseudomanifolds can be tested in linear time [DK78a], whereas a number of related problems have been shown to be NP-complete [EG96, LLT03, JP06, MF08, Tan16] (see Section 1.2), but the computational complexity of the shellability problem has remained open. Here, we settle the question in the negative and show that the problem is intractable (modulo P 6= NP).[2]

Theorem 1. *Deciding if a pure* 2*-dimensional simplicial complex is shellable is* NP*-complete.*

Here, the input is given as a finite abstract simplicial complex (see Section 2).[3]

Remark 2. The problem of testing shellability is easily seen to lie in the complexity class NP (given a linear ordering of the facets of a complex, it is straightforward to check whether it is a shelling). Thus, the nontrivial part of Theorem 1 is that deciding shellability of pure 2-dimensional complexes is NP-hard.

It is easy to check that a pure simplicial complex K is shellable if and only if the *cone* $\{v\} * K$ is shellable, where v is a vertex not in K (see Section 2). Thus, the hardness of deciding shellability easily propagates to higher-dimensional complexes, even to cones.

Corollary 3. *For $d \geq 3$, deciding if a pure d-dimensional complex is shellable is* NP*-complete even when the input is assumed to be a cone (hence contractible).*

Moreover, our hardness reduction (from 3-SAT) used in the proof of Theorem 1 (see Section 3) turns out to be sufficiently robust to also imply hardness results for a number of related problems.

Hardness of k-decomposability. Let $d \geq 2$ and $k \geq 0$. A pure d-dimensional simplicial complex K is k-*decomposable* if it is a simplex or if there exists a face σ of K of dimension at most k such that (i) the link of σ in K is pure $(d-|\sigma|)$-dimensional and k-decomposable, and (ii) deleting σ and faces of K containing σ produces a d-dimensional k-decomposable complex. This notion, introduced by Provan and Billera [PB80], provides a hierarchy of properties (k-decomposability implies $(k+1)$-decomposability) interpolating between *vertex-decomposable* complexes ($k=0$) and shellable complexes (shellability is equivalent to d-decomposability [PB80]). The initial motivation for considering this hierarchy was to study the *Hirsch conjecture* on combinatiorial diameters of convex polyhedra, or in the language of simplicial complex, the diameter of the facet-adjacency graphs of pure simplicial complexes: at one end, the boundary complex of every d-dimensional simplicial polytope is

[2] For basic notions from computational complexity, such as NP-completeness or reductions, see, e.g., [AB09].

[3] There are a several different ways of encoding an abstract simplicial complex—e.g., we can list the facets, or we can list all of its simplices—, but since we work with complexes of fixed dimension, these encodings can be translated into one another in polynomial time, so the precise choice does not matter.

shellable [BM72], and at the other end, every 0-decomposable simplicial complex has small diameter (it satisfies the *Hirsch bound* [PB80]).

Theorem 4. *Let $d \geq 2$ and $k \geq 0$. Deciding if a pure d-dimensional simplicial complex is k-decomposable is* NP*-hard. For $d \geq 3$, the problem is already* NP*-hard for pure d-dimensional simplicial complexes that are cones (hence contractible).*

Hardness of CL-shellability of posets. Another notion related to shellability is the *CL-shellability* of a poset, introduced in [BW82]. The definition of CL-shellability is rather technical, so we do not reproduce it here, but simply note that a simplicial complex is shellable if and only if the dual of its face lattice is CL-shellable [BW83, Corollary 4.4]. Since for any fixed dimension d, the face lattice has height $d + 2$ and can be computed in time polynomial in the size of the d-complex we get:

Corollary 5. *For any fixed $d \geq 4$, deciding CL-shellability of posets of height at most d is* NP*-hard.*

1.2 Related Work on Collapsibility and Our Approach

Our proof of Theorem 1 builds on earlier results concerning *collapsibility*, a combinatorial analogue, introduced by Whitehead [Whi39], of the topological notion of contractibility.[4] A face σ of a simplicial complex K is *free* if there is a unique inclusion-maximal face τ of K with $\sigma \subsetneq \tau$. An *elementary collapse* is the operation of deleting a free face and all faces containing it. A simplicial complex K *collapses* to a subcomplex $L \subseteq K$ if L can be obtained from K by a finite sequence of elementary collapses; K is called *collapsible* if it collapses to a single vertex.

The problem of deciding whether a given 3-dimensional complex is collapsible is NP-complete [Tan16]; the proof builds on earlier work of Malgouyres and Francés [MF08], who showed that it is NP-complete to decide whether a given 3-dimensional complex collapses to some 1-dimensional subcomplex. By contrast, collapsibility of 2-dimensional complexes can be decided in polynomial time (by a greedy algorithm) [JP06, MF08]. It follows that for any *fixed* integer k, it can be decided in polynomial time whether a given 2-dimensional simplicial complex can be made collapsible by deleting at most k faces of dimension 2; by contrast, the latter problem is NP-complete if k is part of the input [EG96].[5]

Our reduction uses the gadgets introduced by Malgouyres and Francés [MF08] and reworked in [Tan16] to prove NP-hardness of deciding collapsibility for 3-dimensional complexes. However, these gadgets are not pure: they contain maximal simplices of two different dimensions, 2 and 3. Roughly speaking, we fix this by replacing the 3-dimensional subcomplexes by suitably triangulated 2-spheres and modifying the way in which they are glued. Interestingly, this also makes our reduction robust to subdivision and applicable to other types of decomposition.

Collapsibility and shellability. Furthermore, we will use the following connection between shellability and collapsibility, due to Hachimori [Hac08] (throughout, $\tilde{\chi}$ denotes the reduced Euler characteristic).

Theorem 6 ([Hac08, Theorem 8])**.** *Let K be a 2-dimensional simplicial complex. The second barycentric subdivision $\mathrm{sd}^2 K$ is shellable if and only if the link of each vertex of K is connected and there exists $\tilde{\chi}(K)$ triangles in K whose removal makes K collapsible.*

[4] Collapsibility implies contractibility, but the latter property is undecidable for complexes of dimension at least 4 (this follows from Novikov's result [VKF74], see [Tan16, Appendix A]), whereas the problem of deciding collapsibility lies in NP.

[5] We remark that building on [EG96], a related problem, namely computing *optimal discrete Morse matchings* in simplicial complexes (which we will not define here), was also shown to be NP-complete [LLT03, JP06].

At first glance, Hachimori's theorem might suggest to prove Theorem 1 by a direct polynomial-time reduction of collapsibility to shellability. However, for 2-dimensional complexes this would not imply hardness, since, as mentioned above, collapsibility of 2-dimensional complexes is decidable in polynomial time [JP06, MF08]. Instead, we will use the existential part of Hachimori's theorem ("there exists $\tilde{\chi}(K)$ triangles") to encode instances of the 3-SAT problem, a classical NP-complete problem.

2 Background and Terminology

We include here a brief summary of the main notions that we use (except for the notions already defined in the introduction, such as pure, shellable, and collapsible simplicial complexes and free faces and elementary collapses).

Simplicial complexes. A (finite) *abstract simplicial complex* is a collection K of subsets of a finite set V that is closed under under taking subsets, i.e., if $\sigma \in K$ and $\tau \subseteq \sigma$, then $\tau \in K$. The elements $v \in V$ are called the *vertices* of K (and often identified with the singleton sets $\{v\} \in K$), and the elements of K are called *faces* or *simplices* of K. The *dimension* of a face is its cardinality minus 1, and the *dimension* of K is the maximum dimension of any face. This is a purely combinatorial description of a simplicial complex and a natural input model for computational questions.

For the purposes of exposition, in particular for describing the gadgets used in the reduction, it will be more convenient to use an alternative, geometric description of simplicial complexes: A (finite) *geometric simplicial complex* is a finite collection K of geometric simplices (convex hulls of affinely independent points) in $\mathbb{R}^d$ (for some d) such that (i) if $\sigma \in K$ and τ is a face of σ, then τ also belongs to K, and (ii) if $\sigma_1, \sigma_2 \in K$, then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2. The *polyhedron* of a geometric simplicial complex K is defined as the union of simplices contained in K, $\bigcup_{\sigma \in K} \sigma$. We also say that K *triangulates* $X \subseteq \mathbb{R}^d$ if X is the polyhedron of K. Note that a given polyhedron usually has many different triangulations.

There is a straightforward way of translating between the two descriptions (see, e.g. [Mat07, Chapter 1]): On the one hand, for every geometric simplicial complex K there is an associated abstract simplicial complex, namely the collection of sets of vertices of the simplices of K (considered as finite sets, neglecting their geometric position). Conversely, for any given abstract simplicial complex K, there is a geometric simplicial complex whose associated abstract simplicial complex is (isomorphic to) K: For a sufficiently large d, pick affinely independent points $p_v \in \mathbb{R}^d$, one for each vertex v of K, and let the simplices of the geometric complex be the convex hulls of the point sets $\{p_v : v \in \sigma\}$, for all $\sigma \in K$.

For the rest of the article we work in the setting of geometric simplicial complexes (except for the definition of joins, see below, which is simpler for abstract simplicial complexes),with the understanding that a geometric simplicial complex is simply a convenient way to describe the associated abstract simplicial complex. (In particular, we will not care about issues such as coordinate complexity of the geometric complex.)

Links, subdivisions, and joins. Let K be a (geometric) simplical complex. The *link* of a vertex v in K is defined as $\mathrm{lk}_K v := \{\sigma \in K : v \notin \sigma \text{ and } \mathrm{conv}(\{v\} \cup \sigma) \in K\}$.

A *subdivision* of a complex K is a complex K' such that the polyhedron of K coincides with the polyhedron of K' and every simplex of K' is contained in some simplex of K.

We will use a specific class of subdivision, called barycentric subdivision. Given a nonempty simplex $\sigma \in \mathbb{R}^d$, let b_σ denote its barycenter (we have $v = b_v$ for a vertex v). The *barycentric subdivsion* of a complex K in $\mathbb{R}^d$ is the complex $\mathrm{sd}K$ with vertex set $\{b_\sigma : \sigma \in K \setminus \{\emptyset\}\}$ and whose simplices are of the form $\mathrm{conv}(b_{\sigma_1},\cdots,b_{\sigma_k})$ where $\sigma_1 \subsetneq \sigma_2 \subsetneq \ldots \subsetneq \sigma_k \in K \setminus \{\emptyset\}$. The *$\ell$th barycentric subdivision* of K, denoted $\mathrm{sd}^\ell K$, is the complex obtained by taking successively the barycentric subdivision ℓ times.

The *join* $K * L$ of two (abstract) simplicial complexes with disjoint sets of vertices is the complex $K * L = \{\sigma \cup \tau : \sigma \in K, \tau \in L\}$. In particular, note that $\{\emptyset\} * K = K$. For $\ell \geq 0$, let Δ_ℓ denote the *ℓ-dimensional simplex*;[6] we extend this to the case $\ell = -1$, by using the convention that $\Delta_{-1} = \{\emptyset\}$ is the abstract simplicial complex whose unique face is the empty face $\emptyset$. Note that, for a pure simplicial complex K, an ordering $\sigma_1,\sigma_2,\ldots,\sigma_n$ of the facets of K is shelling if and only if the ordering $\Delta_\ell \cup \sigma_1, \Delta_\ell \cup \sigma_2, \ldots, \Delta_\ell \cup \sigma_n$ is a shelling of $\Delta_\ell * K$.

Reduced Euler characteristic. The *reduced Euler characteristic* of a complex K is defined as

$$\tilde{\chi}(K) = \sum_{i=-1}^{\dim K} (-1)^i f_i(K)$$

where $f_i(K)$ is the number of i-dimensional faces of K and, by convention, $f_{-1}(K)$ is 0 if K is empty and 1 otherwise.

3-SAT problem. For our reduction, we use the 3-SAT problem (a classical NP-hard problem). The 3-SAT problem takes as input a 3-CNF formula φ, that is, a Boolean formula which is a conjunction of simpler formulas called *clauses*; each clause is a disjunction of three literals, where a *literal* is a variable or the negation of a variable. An example of 3-CNF formula is

$$\varphi^0 = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_4).$$

The *size* of a formula is the total number of literals appearing in its clauses (counting repetitions). The output of the 3-SAT problem states whether φ is satisfiable, that is, whether we can assign variables true or false so that the formula evaluates as true. The formula φ^0 given above is satisfiable, for example by setting x_1 to true, x_2 to false and x_3 and x_4 arbitrarily.

3 The Main Proposition and its Consequences

The cornerstone of our argument is the following construction:

Proposition 7. *There is an algorithm that, given a 3-CNF formula φ, produces, in time polynomial in the size of φ, a 2-dimensional simplicial complex K_φ with the following properties:*

(i) the link of every vertex of K_φ is connected,

(ii) if φ is satisfiable, then K_φ becomes collapsible after removing some $\tilde{\chi}(K_\varphi)$ triangles,

(iii) if an arbitrary subdivision of K_φ becomes collapsible after removing some $\tilde{\chi}(K_\varphi)$ triangles, then φ is satisfiable.

The rest of this section derives our main result and its variants from Proposition 7. We then describe the construction of K_φ in Section 4 and prove Proposition 7 in Sections 5 to 8.

Hardness of shellability. Proposition 7 and Hachimori's theorem readily imply our main result:

[6] Considered as a simplicial complex consisting of all faces of the simplex, including the simplex itself; as an abstract simplicial complex, Δ_ℓ consists of all the subsets of an $(\ell + 1)$-element set.

Proof of Theorem 1. Let φ be a 3-CNF formula and let K_φ denote the 2-dimensional complex built according to Proposition 7. Since the link of every vertex of K_φ is connected, Theorem 6 guarantees that $\mathrm{sd}^2 K_\varphi$ is shellable if and only if there exist $\tilde{\chi}(K_\varphi)$ triangles whose removal makes K_φ collapsible. Hence, by statements (ii) and (iii), the formula φ is satisfiable if and only if $\mathrm{sd}^2 K_\varphi$ is shellable. Taking the barycentric subdivision of a two-dimensional complex multiplies its number of simplices by at most a constant factor. The complex $\mathrm{sd}^2 K_\varphi$ can thus be constructed from φ in polynomial time, and 3-SAT reduces in polynomial time to deciding the shellability of 2-dimensional pure complexes. □

Hardness of *k*-decomposability. Note that statement (iii) in Proposition 7 deals with arbitrary subdivisions whereas Theorem 6 only mentions the second barycentric subdivision. This extra elbow room comes at no cost in our proof, and yields the NP-hardness of k-decomposability.

Proof of Theorem 4. Assume without loss of generality that $k \le d$. Let φ be a 3-CNF formula and K_φ the complex produced by Proposition 7. We have the following implications:[7]

φ is satisfiable	$\Rightarrow$	K_φ is collapsible after removal of some $\tilde{\chi}(K_\varphi)$ triangles
	$\Rightarrow$	$\mathrm{sd}^2 K_\varphi$ is shellable
	$\Rightarrow(b)$	$\mathrm{sd}^3 K_\varphi$ is vertex-decomposable
	$\Rightarrow(c)$	$\Delta_{d-3} * {}^{\mathrm{sd3}} K_\varphi$ is vertex-decomposable
	$\Rightarrow(a)$	$\Delta_{d-3} * {}^{\mathrm{sd3}} K_\varphi$ is k-decomposable
	$\Rightarrow(a)$	$\Delta_{d-3} * \mathrm{sd}^3 K_\varphi$ is shellable
	$\Rightarrow(d)$	$\mathrm{sd}^3 K_\varphi$ is shellable
	$\Rightarrow$	$\mathrm{sd} K_\varphi$ is collapsible after removal of some $\tilde{\chi}(K_\varphi)$ triangles
	$\Rightarrow$	φ is satisfiable

The first and last implications are by construction of K_φ (Proposition 7). The second and second to last follow from Theorem 6, given that Proposition 7 ensures that links of vertices in K_φ are connected. The remaining implications follow from the following known facts (where $\Rightarrow_{(x)}$ to mean that the implication follows from observation (x)):

(a) if K is k-decomposable, then K is k^0-decomposable for $k^0 \ge k$,

(b) if K is shellable, then $\mathrm{sd}K$ is vertex-decomposable [BW97],

(c) K is vertex-decomposable if and only if $\Delta_\ell * K$ is vertex decomposable [PB80, Prop. 2.4],

(d) K is shellable if and only if $\Delta_\ell * K$ is shellable (*c.f.* Section 2).

Since the first and last statement are identical, these are all equivalences. In particular, φ is satisfiable if and only if $\Delta_{d-3} * \mathrm{sd}^3 K_\varphi$ is k-decomposable. Since this complex can be computed in time polynomial in the size of K_φ, which is polynomial in the size of φ, the first statement follows. For $d \ge 3$, $\Delta_{d-3} * \mathrm{sd}^3 K_\varphi$ is contractible so the second statement follows. □

[7] In the case $d = 2$, we use the convention that $\Delta_{-1} * L = L$ for any simplicial complex L.

4 Construction

We now define the complex K_φ mentioned in Proposition 7. This complex consists of several building blocks, called *gadgets*. We first give a "functional" outline of the gadgets (in Section 4.1), insisting on the properties that guided their design, before moving on to the details of their construction and gluing (Sections 4.2 and 4.3).

We use the notational convention that complexes that depend on a variable u are denoted with round brackets, *e.g.* $f(u)$, whereas complexes that depend on a literal are denoted with square brackets, *e.g.* $f[u]$ or $f[\neg u]$.

4.1 Outline of the construction

The gadgets forming K_φ are designed with two ideas in mind. First, every gadget, when considered separately, can only be collapsed starting in a few *special edges*. Next, the special edges of each gadgets are intended to be glued to other gadgets, so as to create dependencies in the flow of collapses: if an edge f of a gadget $\mathbf{G}$ is attached to a triangle of another gadget $\mathbf{G}^0$, then $\mathbf{G}$ cannot be collapsed starting by f before some part of $\mathbf{G}^0$ has been collapsed.

Variable gadgets. For every variable u we create a gadget $\mathbf{V}(u)$. This gadget has three special edges; two are associated, respectively, with true and false; we call the third one "unlocking". Overall, the construction ensures that any removal of $\tilde\chi(K_\varphi)$ triangles from K_φ either frees exactly one of the edges associated with true or false in every variable gadget, or makes K_φ obviously non-collapsible. This relates the removal of triangles in K_φ to the assignment of variables in φ. We also ensure that part of each variable gadget remains uncollapsible until the special unlocking edge is freed.

Clause gadgets. For every clause $c = \ell_1 \vee \ell_2 \vee \ell_3$ we create a gadget $\mathbf{C}(c)$. This gadget has three special edges, one per literal ℓ_i. Assume that $\ell_i \in \{u, \neg u\}$. Then the special edge associated with ℓ_i is attached to $\mathbf{V}(u)$ so that it can be freed if and only if the triangle removal phase freed the special edge of $\mathbf{V}(u)$ associated with true (if $\ell_i = u$) or with false (if $\ell_i = \neg u$). This ensures that the gadget $\mathbf{C}(c)$ can be collapsed if and only if one of its literals was "selected" at the triangle removal phase.

Conjunction gadget. We add a gadget $\mathbf{A}$ with a single special edge, that is attached to every clause gadget. This gadget can be collapsed only after the collapse of every clause gadget has started (hence, if every clause contains a literal selected at the triangle removal phase). In turn, the collapse of $\mathbf{A}$ will free the unlocking special edge of every variable gadget, allowing to complete the collapse.

Notations. For any variable u, we denote the special edges of $\mathbf{V}(u)$ associated with true and false by, respectively, $f[u]$ and $f[\neg u]$; we denote the unlocking edge by $f(u)$. For every clause $c = \ell_1 \vee \ell_2 \vee \ell_3$, we denote by $f[\ell_i, c]$ the special edge of $\mathbf{C}(c)$ associated with ℓ_i. We denote by f_{and} the special edge of the conjunction gadget $\mathbf{A}$. The attachment of these edges are summarized in Table 1.

gadget	special edges	attached to	freed by
$\mathbf{V}(u)$	$f[u]$	-	triangle deletion
	$f[\neg u]$	-	triangle deletion
	$f(u)$	$\mathbf{A}$	freeing f_{and}

C$(u_2 \vee \neg u_4 \vee u_9)$	$f[u_2,c]$ $f[\neg u_4,c]$ $f[u_9,c]$	**V**(u_2) **V**(u_4) **V**(u_9)	freeing $f[u_2]$ freeing $f[\neg u_4]$ freeing $f[u_9]$
A	$f$and	every clause gadget	collapsing all clause gadgets

Table 1: Summary of the gadgets' special edges and their attachments.

Flow of collapses. Let us summarize the mechanism sketched above. Assume that φ is satisfiable, and consider a satisfying assignment. Remove the triangles from each **V**(u) so that the edge that becomes free is $f[u]$ if u was assigned true, and $f[\neg u]$ otherwise. This will allow to collapse each clause gadget in order to make f_{and} free. Consequently, we will be able to collapse **A** and make all unlocking edges $f(u)$ free. This allows finishing the collapses on all **V**(u).

On the other hand, to collapse K_φ we must collapse f_{and} at some point. Before this can happen, we have to collapse in each clause $c = \ell_1 \vee \ell_2 \vee \ell_3$ one of the edges $f[\ell_i,c]$. This, in turn, requires that $f[\ell_i]$ has been made free. If we can ensure that $f[\neg \ell_i]$ cannot also be free, then we can read off from the collapse an assignment of the variables that must satisfy every clause, and therefore φ. (If $\ell_i = u$, then we set u to true, if $\ell_i = \neg u$, then we set u to false. If there are unassigned variables after considering all clause, we assign them arbitrarily.)

4.2 Preparation: modified Bing's houses

Our gadgets rely on two modifications of Bing's house, a classical example of 2-dimensional simplicial complex that is contractible but not collapsible. Bing's house consists of a box split into two parts (or *rooms*); each room is connected to the outside by a tunnel through the other room; each tunnel is attached to the room that it traverses by a rectangle (or *wall*). The modifications that we use here make the complex collapsible, but restricts its set of free faces to exactly one or exactly three edges.

One free edge. We use here a modification due to Malgouyres and Franc´es [MF08]. In one of the rooms (say the top one), the wall has been thickened and hollowed out, see Figure 1. We call the resulting polyhedron a Bing's house with a single free edge, or a 1-*house* for short. Two special elements of a 1-house are its *free edge* (denoted f and in thick stroke in Figure 1) and its *lower wall* rectangle (denoted L and colored in light blue in Figure 1). We only consider triangulations of 1-house that subdivide the edge f and the lower wall L. We use 1-houses for the following properties:

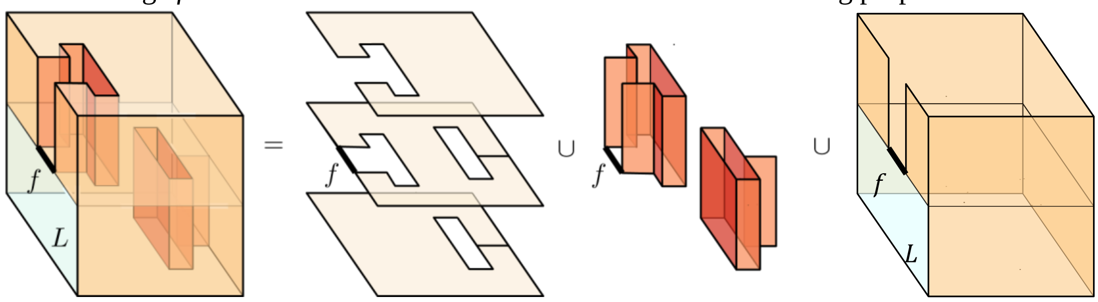


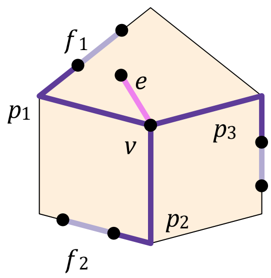
Figure 1: Bing's house modified to be collapsible with exactly one free edge f.

Lemma 8. *Let B be a* 1*-house, f its free edge and L its lower wall. In any triangulation of B, the free faces are exactly the edges that subdivide f. Moreover, B collapses to any subtree of the* 1*-skeleton of B that is contained in L and shares with the boundary of L a single endpoint of f.*

The first statement follows from the fact that the edges that subdivide f are the only ones that are not part of two triangles; see [MF08, Remark 1]. The second statement was proven in [Tan16, Lemma 7] for certain trees, but the argument holds for arbitrary trees; we spell them out in Appendix A. When working with 1-houses, we will usually only describe the lower wall to clarify which subtree we intend to collapse to.

Remark 9. We note that there exist smaller simplicial complexes that have properties analogous to those of the 1-house. The smallest such example is obtained by a slight modification of the dunce hat; has seven vertices and thirteen facets and is described as the first example in [Hac00, Section 5.3].[8] For the purposes of exposition, however, we prefer to work with 1-houses, which allows us to use some of their properties proved in [MF08, Tan16]. Moreover, the construction of the 1-house is similar to the construction of another gadget, the 3-house discussed below, and we currently do not know how to replace the latter by a smaller complex.

Three free edges. We also use the Bing's houses with three collapsed walls introduced in [Tan16]; we call them 3-*houses* for short. These are 2-dimensional complexes whose construction is more involved; we thus state its main properties, so that we can use it as a black box, and



refer the reader interested in its precise definition to [Tan16, §4]. Refer to the figure on the right (which corresponds to Figure 9 in [Tan16]). The 3house has exactly three free edges f_1,f_2,f_3, and has three distinguished paths p_1,p_2,p_3 sharing a common vertex v and such that each p_i shares exactly one vertex with f_i and no vertex with f_j for j 6= i. In addition, it$f$3 contains an edge e incident to v so that the union of p_1,p_2,p_3,f_1,f_2,f_3 and e forms a subdivided star with four rays.

Let C denote the 3-house as described above. In [Tan16], the polyhedron of C is described in detail but no triangulation is specified. We are happy with any concrete triangulation for which Lemma 10 below holds; we can in addition require that the paths p_1, p_2 and p_3 each consist of two edges.[9]

Lemma 10 ([Tan16, Lemma 8])**.** *In any subdivision of C, the free faces are exactly the edges that subdivide* f_1, f_2 *and* f_3. *Moreover, C collapses to the* 1*-complex spanned by* e,p_1,p_2,p_3 *and any two of* $\{f_1,f_2,f_3\}$.

4.3 Detailed construction

Section 4.1 gave a quick description of the intended functions of the various gadgets. We now flesh them out and describe how they are glued together.

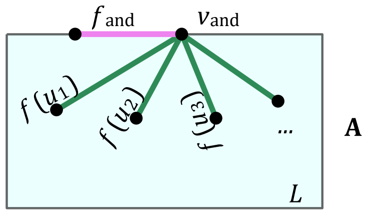
Triangulations. For some parts of the complex, it will be convenient to first describe the polyhedron, then discuss its triangulation. Our description of the triangulation may vary in precision: it may be

[8] More precisely, the complex has vertices {1,2,...,7}, facets {125,126,127,134,145,167,234,235,236,247,356,457,567}, and its only free edge is 13, contained in a unique facet 134. A computer search confirmed that this (along with with four other triangulations of the modified dunce hat) is indeed a minimal example.

[9] The value two is not important here; what matters is to fix some value that can be used throughout the construction.

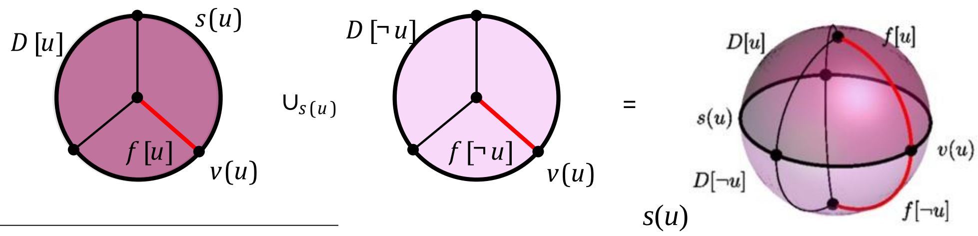
omitted (if any reasonable triangulation works), given indirectly by the properties it should satisfy, or given explicitly (for instance to make it clear that we can glue the gadgets as announced).

Conjunction gadget. The conjunction gadget **A** is a 1-house. We let f_{and} denote its free edge and v_{and} one of the endpoints of f_{and}. We further triangulate the lower wall so that v_{and} has sufficiently high degree, allowing to assign every variable u to an internal edge $f(u)$ of the lower wall incident to v_{and}. See the lower left wall on the right picture. Any triangulation satisfying these prescriptions and computable in time polynomial in the size of φ suits our purpose.

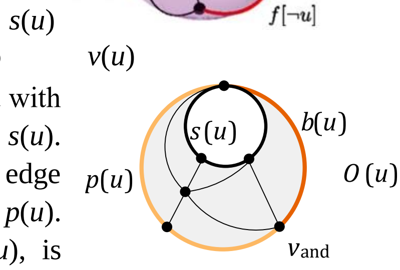


Variable gadget. The variable gadget **V**(u) associated with the variable u has four parts.

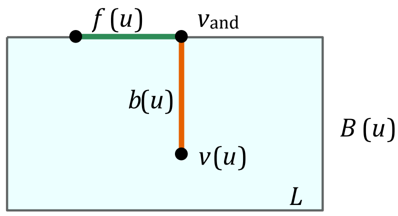
1. The first part is a triangulated 2-sphere $S(u)$ that consists of two disks $D[u]$ and $D[\neg u]$ sharing a common boundary circle $s(u)$. The circle $s(u)$ contains a distinguished vertex $v(u)$. The disk $D[u]$ (resp. $D[\neg u]$) has a distinguished edge $f[u]$ (resp. $f[\neg u]$) that joins $v(u)$ to its center.



2. The second part is a 2-complex $O(u)$ that consists of two "boundary" circles sharing a vertex. The vertex is identified with the vertex $v(u)$ of $S(u)$. One of the circles is identified with $s(u)$. The other circle is decomposed into two arcs: one is a single edge named $b(u)$, the other is a path with two edges which we call $p(u)$. The vertex common to $b(u)$ and $p(u)$, distinct from $v(u)$, is identified with the vertex v_{and} of the conjunction gadget.



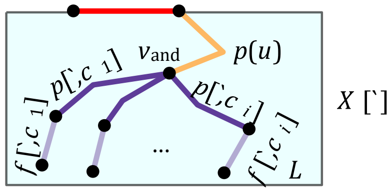
3. The third part is a 1-house $B(u)$ intended to block the edge $b(u) \in O(u)$ from being free as long as the conjunction gadget has not been collapsed. The free edge of $B(u)$ is identified with the edge $f(u)$ in the conjunction gadget **A** and the edge $b(u) \in O(u)$ is identified with an edge of the lower wall of $B(u)$ that shares the vertex v_{and} with $f(u)$.



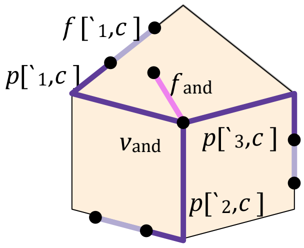
4. The fourth part consists of two complexes, $X[u]$ and $X[\neg u]$. Let ` ∈ {$u$,¬$u$} and refer to the figure on the right. The complex $X$[`] f[`] $v(u)$ is a 1-house whose free edge is identified with the edge $f$[`] from D[`], and whose lower wall contains a path identified with $p(u)$. Hence, $p(u)$ is common to $X[u]$, $X[\neg u]$ and the second part $O(u)$. For every clause $c_i$ containing the literal `, we add in the lower wall a two-edge path p[`,$c_i$] extended by an edge $f$[`,c_i]; the path p[`,c_i] intersects $p(u)$ in exactly v_{and} (in particular, these paths and edges form a subdivided star centered at v_{and}).

Clause gadget. The clause gadget **C**(c) associated with the clause

C(c) c = $`_1 \vee `_2 \vee `_3$ is a 3-house where:



- the edges f_i of C are identified with the edges $f[`_i,c]$ in $X[`_i]$;
- the paths p_i of C are identified with the paths $p[`_i,c]$ in $X[`_i]$; $f[`_3,c]$
- the vertex v of C is identified with the vertex v_{and}; and
- the edge e of C is identified with the edge f_{and}. $f[`_2,c]$



Putting it all together. Let φ be a 3-CNF formula with variables $u_1,u_2,\ldots,u_n$ and clauses $c_1,c_2,\ldots,c_m$. The complex K_φ is defined as

$$K_\varphi = \mathbf{A} \cup \bigcup_{i=1}^{n} \Big(\underbrace{S(u_i) \cup O(u_i) \cup B(u_i) \cup X[u_i] \cup X[\neg u_i]}_{v(z_i)} \Big) \cup \bigcup_{j=1}^{m} \mathbf{C}(c_j).$$

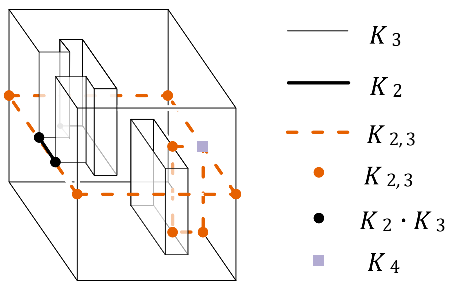
To verify the proofs in Sections 5 to 8, it may be useful to be able to quickly identify for a given vertex, edge or path which 2-complexes contain it. We provide this in Table 2.

5 Connectedness of links

In this section, we prove Proposition 7(i), *i.e.* that the link of every vertex in the complex K_φ of Section 4 is connected. By construction, the complex K_φ is covered by the following subcomplexes: **A**, $S(u)\cup O(u)$, $B(u)$, $X[`]$ and **C**($c$), where $u$ ranges over all variables, ` ranges over all literals and c ranges over all clauses. We first argue that in each subcomplex, the link of every vertex is connected. We then ensure that these subcomplexes are glued into K_φ in a way that preserves the connectedness of the links.

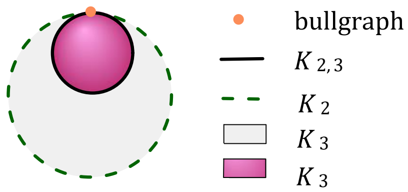
object	quantifier	in complexes
vand	1 occurence	**A**, $O(u)$, $B(u)$, $X[`]$, **C**(c)
$v(u)$	every variable u	$D[u]$, $D[\neg u]$, $O(u)$, $B(u)$, $X[u]$, $X[\neg u]$
fand	1 occurence	**A**, **C**(c)
$f(u)$	every variable u	**A**, $B(u)$
$f[`]$	every literal `	$D[`]$, $X[`]$
$b(u)$	every variable u	$O(u)$, $B(u)$
$p(u)$	every variable u	$O(u)$, $X[u]$, $X[\neg u]$
$s(u)$	every variable u	$O(u)$, $D[u]$, $D[\neg u]$
$f[`,c]$	pairs ($`,c$), $` \in c$	$X[`]$, **C**(c)
$p[`,c]$	pairs ($`,c$), $` \in c$	$X[`]$, **C**(c)

Table 2: Containments of vertices, edges and paths in 2-complexes.



Links within the subcomplexes. The proof is straightforward but rather pedestrian. Let us start with the 1-house B. Most points have a link homeomorphic to K_3, so connectedness is immediate. For the remaining points, depicted on the left, the links are homeomorphic to one of K_2, K_4, $K_{2,3}$ and $K_2 \cdot K_3$, the graph obtained by gluing K_2 and K_3 at a vertex. This settles the cases of the subcomplexes **A**, $B(u)$ and X[`].

A similar analysis for the subcomplex $S(u) \cup O(u)$ reveals that the links are homeomorphic to one of K_2, K_3, $K_{2,3}$ and the 'bull graph', that is the graph on 5 vertices formed from the triangle and two edges attached to it at different vertices.



Last, a careful inspection of the construction in [Tan16] of the 3-house yields that the link of every vertex is homeomorphic to one of K_2, K_3, $K_{2,3}$, $K_2 \cdot K_3$, $K_{2,4}$ or the graph obtained from a triangle by gluing three other triangles to it, one along each edge. This covers the subcomplexes $\mathbf{C}(c)$.

Links after gluing. We now argue that if v is a vertex shared by two of our subcomplexes C and C^0, then there is an edge incident to v and common to C and C^0. This ensures that the links of v in C and C^0 share at least a vertex, so the connectedness of $\mathrm{lk}_{C\cup C}0\ v$ follows from that of $\mathrm{lk}_C\, v$ and $\mathrm{lk}_C0\ v$. If v is shared by subcomplexes $C_1,C_2,...C_k$, we can apply this idea iteratively by finding a sequence of edges $e_1,e_2,...,e_{k-1}$ where e_i is common to C_i and *at least one of* $C_1,C_2,...,C_{i-1}$.

Let us first examine $v(u)$ for some variable u. This vertex is common to $S(u) \cup O(u)$, $B(u)$, $X[u]$ and $X[\neg u]$. The connectedness of $\mathrm{lk}_{K_\varphi}\, v(u)$ follows from the existence of the following edges incident to $v(u)$:

- $b(u)$, common to $S(u) \cup O(u)$ and $B(u)$,
- $p(u)$, common to $S(u) \cup O(u)$, $X[u]$ and $X[\neg u]$.

Let us next examine v_{and}. This vertex is common to all our subcomplexes, that is to **A**, $B(u)$, $S(u) \cup O(u)$, X[`] and $\mathbf{C}(c)$ for all variable $u$, literal `, and clause c. The connectedness of $\mathrm{lk}_{K_\varphi}\, v_{\mathrm{and}}$ follows from the existence of the following edges incident to v_{and}:

- the edges $f(u)$, common to **A** and $B(u)$,
- the edges $b(u)$, common to $B(u)$ and $S(u) \cup O(u)$, • the edges $p(u)$, common to $S(u) \cup O(u)$, $X[u]$ and $X[\neg u]$
- f_{and}, common to **A** and every $\mathbf{C}(c)$.

The remaining vertices shared by two or more of our subcomplexes are defined as part of an edge common to these subcomplexes. The connectedness of the links in K_φ of these vertices is thus immediate. This completes the proof of Proposition 7(i).

6 Reduced Euler characteristic of K_φ

In this section, we compute the reduced Euler characteristic of K_φ, preparing the proofs of Proposition 7(ii)–(iii) in the following sections. By inclusion-exclusion, for any simplicial complexes K_1 and K_2 we have:

$$\tilde{\chi}(K_1 \cup K_2) = \tilde{\chi}(K_1) + \tilde{\chi}(K_2) - \tilde{\chi}(K_1 \cap K_2). \tag{6.1}$$

In particular, if both K_2 and $K_1 \cap K_2$ are contractible, then $\tilde{\chi}(K_1 \cup K_2) = \tilde{\chi}(K_1)$.

Proposition 11. *$\tilde{\chi}(K_\varphi)$ equals the number of variables of φ.*

Proof. First, let us observe that the subcomplexes $\mathbf{A}$, $B[u]$, $X[\ell]$ and $\mathbf{C}(c)$ (for all variables u, literals ℓ and clauses c) are contractible. Indeed each of them is either a 1-house or 3-house which are collapsible by Lemmas 8 and 10, thereby contractible. In addition, each of the aforementioned subcomplexes is attached to the rest of the complex in contractible subcomplexes (trees).

Therefore, by the claim following Equation (6.1), we may replace each of these gadgets (in any order) with the shared trees without affecting the reduced Euler characteristic. That is, $\tilde{\chi}(K_\varphi) = \tilde{\chi}(K')$ where

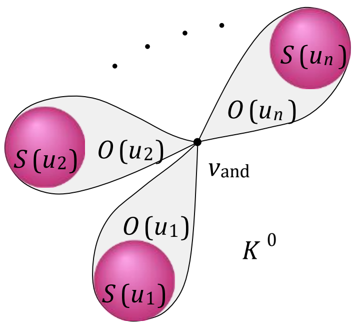
$$K' := f_{\mathrm{and}} \cup \bigcup_{u} (f(u) \cup S(u) \cup O(u)) \cup \bigcup_{(\ell,c):\ell\in c} (f[\ell,c] \cup p[\ell,c])$$

where the first (big) union is over all variables u, and the second is over all pairs (ℓ,c) where a literal ℓ belongs to a clause c.

By collapsing the pendent edges and paths, we get $\tilde{\chi}(K') = \tilde{\chi}(K'')$ where

$$K'' := \bigcup_{u} (S(u) \cup O(u)).$$

Finally, for every variable u we have $\tilde{\chi}(O(u) \cup S(u)) = 1$ as $O(u) \cup S(u)$ is homotopy equivalent to the 2-sphere. For any distinct variables u, u', the complexes $O(u) \cup S(u)$ and $O(u') \cup S(u')$ share only a vertex, namely v_{and}. Equation (6.1) then yields that $\tilde{\chi}(K_\varphi) = \tilde{\chi}(K'')$ is the number of variables.



□

Remark 12. It is possible, with slightly more effort, to show that K_φ is homotopy equivalent to K'', hence to a wedge of spheres, one for each variable. This also implies Proposition 11 but for our purpose, computing the reduced Euler characteristic suffices.

7 Satisfiability implies collapsibility

In this section we prove Proposition 7(ii), *i.e.* that if φ is satisfiable, then there exists a choice of $\tilde{\chi}(K_\varphi)$ triangles of K_φ whose removal makes the complex collapsible.

Literal $\ell(u)$. Let us fix a satisfying assignment for φ. For every variable u, we set $\ell(u)$ to u if u is true in our assignment, and to $\neg u$ otherwise.

Triangle removal. For every variable u, we remove a triangle from the region $D[\ell(u)]$ of the sphere $S(u)$. Proposition 11 ensures that this removes precisely $\tilde{\chi}(K_\varphi)$ triangles, as announced.

Constrain complex. It will be convenient to analyze collapses of K_φ locally within a subcomplex, typically a gadget. To do so formally, we use constrain complexes following [Tan16]. Given a simplicial complex K and a subcomplex M of K, we define the *constrain complex* of (K,M), denoted $\Gamma(K,M)$, as follows:

$$\Gamma(K,M) := \{\vartheta \in M : \exists \eta \in K \setminus M \text{ s.t. } \vartheta \subset \eta\}.$$

Lemma 13 ([Tan16, Lemma 4])**.** *Let K be a simplicial complex and M a subcomplex of K. If M collapses to M^0 and $\Gamma(K,M) \subseteq M^0$ then K collapses to $(K \setminus M) \cup M^0$.*

Collapses. We now describe a sequence of collapses enabled by the removal of the triangles. Recall that we started from the complex

$$K_\phi = \mathbf{A} \cup \left(\bigcup_{i=1}^{n} O(u_i) \cup S(u_i) \cup B(u_i) \cup X[u_i] \cup X[\neg u_i] \right) \cup \left(\bigcup_{j=1}^{m} \mathbf{C}(c_j) \right)$$

where $u_1,u_2,\ldots,u_n$ and $c_1,c_2,\ldots,c_m$ are, respectively, the variables and the clauses of φ. We then removed a triangle from each $D[\ell(u)]$.

(a) The removal of a triangle of $D[\ell(u)]$ allows to collapse that subcomplex to $s(u) \cup f[\ell(u)]$. This frees $f[\ell(u)]$. The complex becomes:

$$K_a = \mathbf{A} \cup \left(\bigcup_{i=1}^{n} O(u_i) \cup D[\neg\ell(u_i)] \cup B(u_i) \cup X[u_i] \cup X[\neg u_i] \right) \cup \left(\bigcup_{j=1}^{m} \mathbf{C}(c_j) \right).$$

(b) We can then start to collapse the subcomplexes $X[\ell(u)]$. We proceed one variable at a time. Assume that we are about to proceed with the collapse of $X[\ell(u)]$ and let K denote the current complex. Locally, $X[\ell(u)]$ is a 1-house with free edge $f[\ell(u)]$. Moreover, $\Gamma(K,X[\ell(u)])$ is the tree $T(u)$ formed by the path $p(u)$ and the union of the paths $p[\ell(u),c] \cup f[\ell(u),c]$ for every clause c using the literal $\ell(u)$. Lemma 8 ensures that $X[\ell(u)]$ can be locally collapsed to $T(u)$, and Lemma 13 ensures that K can be globally collapsed to $(K \setminus X[\ell(u)]) \cup T(u)$. We proceed in this way for every complex $X[\ell(u)]$. The complex becomes:

$$K_b = \mathbf{A} \cup \left(\bigcup_{i=1}^{n} O(u_i) \cup D[\neg\ell(u_i)] \cup B(u_i) \cup X[\neg\ell(u_i)] \right) \cup \left(\bigcup_{j=1}^{m} \mathbf{C}(c_j) \right).$$

(c) The collapses so far have freed every edge of $f[\ell(u_i),c]$. We now consider every clause c_j in turn. Put $c_j = (\ell_1 \vee \ell_2 \vee \ell_3)$ and let K denote the current complex. The assignment that we chose is satisfying, so at least one of ℓ_1, ℓ_2 or ℓ_3 coincides with $\ell(u_i)$ for some i; let us assume without loss of generality that $\ell_1 = \ell(u_i)$. The edge $f[\ell_1,c]$ is therefore free and Lemma 10 yields that locally, $\mathbf{C}(c_j)$ collapses to the tree $T(c_j) = f_{\text{and}} \cup p[\ell_1,c] \cup p[\ell_2,c] \cup p[\ell_2,c] \cup f[\ell_2,c] \cup f[\ell_3,c]$. Moreover, $\Gamma(K,\mathbf{C}(c_j)) = T(c_j)$ so Lemma 13 ensure that K can be globally collapsed to $(K \setminus \mathbf{C}(c_j)) \cup T(c_j)$. After proceeding in this way for every complex $\mathbf{C}(c_j)$, the complex becomes:

$$K_c = \mathbf{A} \cup \left(\bigcup_{i=1}^{n} O(u_i) \cup D[\neg\ell(u_i)] \cup B(u_i) \cup X[\neg\ell(u_i)] \right) \cup \left(\bigcup_{j=1}^{m} T(c_j) \right).$$

(d) The collapses so far have freed the edge f_{and}. We can then proceed to collapse $\mathbf{A}$. Locally, Lemma 8 allows to collapse $\mathbf{A}$ to the tree $T = f(u_1) \cup f(u_2) \cup \ldots \cup f(u_n)$. (From this point, we expect the reader to be able to check by her/himself that Lemma 13 allows to perform globally the collapse described locally.) The complex becomes:

$$K_d = \left(\bigcup_{i=1}^{n} O(u_i) \cup D[\neg\ell(u_i)] \cup B(u_i) \cup X[\neg\ell(u_i)] \right) \cup \left(\bigcup_{j=1}^{m} T(c_j) \right).$$

(e) The collapses so far have freed every edge $f(u_i)$. Thus, Lemma 8 allows to collapse each complex $B(u_i)$ to its edge $b(u_i)$. This frees the edge $b(u_i)$, so the complex $O(u_i)$ can in turn be collapsed to $s(u_i) \cup p(u_i)$. At this point, the complex is:

$$K_e = \left(\bigcup_{i=1}^{n} s(u_i) \cup p(u_i) \cup D[\neg\ell(u_i)] \cup X[\neg\ell(u_i)] \right) \cup \left(\bigcup_{j=1}^{m} T(c_j) \right).$$

(f) The collapses so far have freed every edge $s(u_i)$. We can thus collapse each $D[\neg\ell(u_i)]$ to $f[\neg\ell(u_i)]$. This frees every edge $f[\neg\ell(u_i)]$, allowing to collapse every subcomplex $X[\neg\ell(u_i)]$, again by Lemma 8, to the tree formed by the path $p(u_i)$ and the union of the paths $p[\neg\ell(u_i),c] \cup f[\neg\ell(u_i),c]$ for every clause c using the literal $\neg\ell(u_i)$.

At this point, we are left with a 1-dimensional complex. This complex is a tree (more precisely a subdivided star centered in v_{and} and consisting of the paths $p(u_i)$, the paths $p[\ell,c]$ and some of the edges $f[\ell,c]$). As any tree is collapsible, this completes the proof of Proposition 7(ii).

8 Collapsibility implies satisfiability

In this section we prove Proposition 7(iii), *i.e.* we consider some arbitrary subdivision K'_φ of K_φ, and prove that if K'_φ becomes collapsible after removing some $\tilde\chi(K_\varphi)$ triangles, then φ is satisfiable. We thus consider a collapsible subcomplex $\widehat{K}$ of K'_ϕ obtained by removing $\tilde\chi(K_\varphi)$ triangles from K'_ϕ.

Notations. Throughout this section, we use the following conventions. In general, we use hats (for example $\widehat{K}$) to denote subcomplexes of K'_φ. Given a subcomplex M of K_φ, we also write M' for the subcomplex of K'_φ that subdivides M.

Variable assignment from triangle removal. We first read our candidate assignment from the triangle removal following the same idea as in Section 7. This relies on two observations:

- The set of triangles removed in $\widehat{K}$ contains exactly one triangle from each sphere $S'(u)$. Indeed, since $\widehat{K}$ is collapsible and 2-dimensional, it cannot contain a 2-dimensional sphere. Hence, every sphere $S'(u)$ had at least one of its triangles removed. By Proposition 11, $\chi(K_\varphi) = \chi(K'_\varphi)$ equals the number of variables of φ, so this accounts for all removed triangles.
- For any variable u, any removed triangle in $S'(u)$ is either in $D'[u]$ or in $D'[\neg u]$. We give u the true assignment in the former case and the false assignment in the latter case.

The remainder of this section is devoted to prove that this assignment satisfies φ. It will again be convenient to denote by $\ell(u)$ the literal corresponding to this assignment, that is, $\ell(u) = u$ if u was assigned true and $\ell(u) = \neg u$ otherwise.

Analyzing the collapse. Let us fix some collapse of $\widehat{K}$. We argue that our assignment satisfies φ by showing that these collapses must essentially follow the logical order of the collapse constructed in

Section 7. To analyze the dependencies in the collapse, it is convenient to consider the partial order that it induces on the simplices of K_b: $\sigma \prec \tau$ if and only if in our collapse, σ is deleted before τ. We also write $\sigma \prec M_c$ for a subcomplex M_c of K_b if σ was removed before removing any simplex of M_c.

The key observation is the following dependency:

Lemma 14. *There exists an edge e of* $\mathbf{A}^0$ *such that* $e \prec D^0[\neg`(u)]$ *for every variable u.* b
b

Proof. We first argue that for every variable u, there exists an edge $\widehat{e}_1(u) \in b'(u) \cup p'(u)$ such that $\widehat{e}_1(u) \prec D'[\neg\ell(u)]$. To see this, remark that the complex $D^0[\neg`(u)] \cup O^0(u)$ is fully contained in $K_b$ since the triangle removed from $S^0(u)$ belongs to $D^0[`(u)]$. It thus has to be collapsed. Since this complex is a disk, the first elementary collapse in $D^0[\neg`(u)] \cup O^0(u)$ has to involve some edge $\widehat{e}_1(u)$ of its boundary. This boundary is $b^0(u) \cup p^0(u)$, so it contains no edge of $D^0[\neg`(u)]$. It follows that $\widehat{e}_1(u) \prec D'[\neg\ell(u)]$. We next claim that $\widehat{e}_1(u) \in b'(u)$. Indeed, remark that every edge in $p^0(u)$ belongs to two triangles of $X^0[\neg`(u)]$. By Lemma 8, any collapse of $X^0[\neg`(u)]$ must start by an elementary collapse using an edge of $f^0[\neg`(u)]$ as a free face. Any edge of $f^0[\neg`(u)]$ is, however, contained in two triangles of $D^0[\neg`(u)]$ and thus cannot precede $D^0[\neg`(u)]$ in $\prec$. It follows that $\widehat{e}_1(u) \in b'(u)$. We can now identify e. Observe that $b^0(u) \subset B^0(u)$. As $B^0(u)$ is a 1-house, Lemma 8 ensures that b

the first edge removed from $B^0(u)$ must subdivide $f^0(u)$. Hence, there is an edge $eb_2(u) \subset f^0(u)$ such that $eb_2(u) \prec e_b1(u)$. Since $f^0(u) \subset \mathbf{A}^0$, another 1-house, the same reasoning yields an edge $e_b3(u)$ in $f_{and}{}^0$

such that $eb_3(u) \prec e_b2(u)$. Let e_b denote the first edge removed from $\mathbf{A}^0$ among all edges $e_b3(u)$. At this point, we have for every variable u $eb \prec eb_2(u) \prec e_b1(u) \prec D^0[\neg`(u)]$, as announced. □

Let e denote the edge of $\mathbf{A}^0$ provided by Lemma 14, *i.e.* satisfying $e \prec D^0[\neg`(u)]$ for every variable u. b
b

We can now check that the variable assignment does satisfy the formula:

- Consider a clause $c = (`_1 \vee `_2 \vee `_3)$ in $\varphi$. The complex $\mathbf{C}^0(c)$ is a 3-house, so Lemma 10 restricts its set of free edges to the $f^0[`_i,c]$. Hence, there is $i \in \{1,2,3\}$ and an edge $\widehat{e}_4(c)$ in $f'[\ell_i,c]$ such that $\widehat{e}_4(c) \preceq \mathbf{C}'(c)$. Note that, in particular, $\widehat{e}_4(c) \prec \widehat{e}$ as the edge f_{and} also belongs to $\mathbf{C}(c)$ and must be freed before collapsing $\mathbf{A}^0$ (by Lemma 8).

- The subcomplex $f^0[`_i,c]$ is contained not only in $\mathbf{C}^0(c)$, but also in $X[`_i]$ which is a 1-house with free edge $f[`_i]$. By Lemma 8, the first elementary collapse of $X[`_i]$ uses as free face an edge $\widehat{e}_5(c)$ that subdivides $f^0[`_i]$. In particular, $\widehat{e}_5(c) \prec f'[\ell_i,c]$ and $\widehat{e}_5(c) \prec \widehat{e}_4(c)$.

- Let u be the variable of the literal $`_i$, that is, $`_i = u$, or $`_i = \neg u$; in particular $`_i \in \{`(u), \neg`(u)\}$. From $\widehat{e}_5(c) \prec \widehat{e}_4(c) \prec \widehat{e} \prec D'[\neg\ell(u)]$ it comes that $\widehat{e}_5(c)$ cannot belong to $D^0[\neg`(u)]$. Yet, $\widehat{e}_5(c)$ belongs to $f^0[`_i]$. It follows that $`_i 6= \neg`(u)$ and we must have $`_i = `(u)$. The definition of $`(u)$ thus ensures that our assignment satisfies the clause c.

Since our assignment satisfies every clause, it satisfies φ.

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References

[AB09] S. Arora and B. Barak. *Complexity Theory: A Modern Approach*. Cambridge University Press, Cambridge, 2009. http://www.cs.princeton.edu/theory/complexity/.

[Bin83] R. H. Bing. *The geometric topology of 3-manifolds*, volume 40 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1983.

[Bjö80] A. Björner. Shellable and Cohen-Macaulay partially ordered sets. *Trans. Amer. Math. Soc.*, 260(1):159–183, 1980.

[Bjö95] A. Björner. Topological methods. In *Handbook of combinatorics, Vol. 1, 2*, pages 1819–1872. Elsevier Sci. B. V., Amsterdam, 1995.

[BM72] H. Bruggesser and P. Mani. Shellable decompositions of cells and spheres. *Mathematica Scandinavica*, 29(2):197–205, 1972.

[BW82] A. Björner and M. Wachs. Bruhat order of Coxeter groups and shellability. *Advances in Mathematics*, 43(1):87 – 100, 1982.

[BW83] A. Björner and M. Wachs. On lexicographically shellable posets. *Trans. Amer. Math. Soc.*, 277(1):323–341, 1983.

[BW97] A. Björner and M. L. Wachs. Shellable nonpure complexes and posets. II. *Trans. Amer. Math. Soc.*, 349(10):3945–3975, 1997.

[DK74] G. Danaraj and V. Klee. Shellings of spheres and polytopes. *Duke Mathematical Journal*, 41(2):443–451, 1974.

[DK78a] G. Danaraj and V. Klee. A representation of 2-dimensional pseudomanifolds and its use in the design of a linear-time shelling algorithm. *Ann. Discrete Math.*, 2:53–63, 1978. Algorithmic aspects of combinatorics (Conf., Vancouver Island, B.C., 1976).

[DK78b] G. Danaraj and V. Klee. Which spheres are shellable? *Ann. Discrete Math.*, 2:33–52, 1978. Algorithmic aspects of combinatorics (Conf., Vancouver Island, B.C., 1976).

[EG96] O. Eğecioğlu and T. F. Gonzalez. A computationally intractable problem on simplicial complexes. *Comput. Geom.*, 6(2):85–98, 1996.

[Grü03] B. Grünbaum. *Convex polytopes*, volume 221 of *Graduate Texts in Mathematics*. SpringerVerlag, New York, second edition, 2003. Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler.

[Hac00] M. Hachimori. *Combinatorics of Constructible Complexes*. PhD thesis, Graduate School of Arts and Sciences, the University of Tokyo, 2000.

[Hac08] M. Hachimori. Decompositions of two-dimensional simplicial complexes. *Discrete Math.*, 308(11):2307–2312, 2008.

[JP06] M. Joswig and M. E. Pfetsch. Computing optimal Morse matchings. *SIAM Journal on Discrete Mathematics*, 20(1):11–25, 2006.

[KP03] V. Kaibel and M. E. Pfetsch. Some algorithmic problems in polytope theory. In *Algebra, geometry, and software systems*, pages 23–47. Springer, Berlin, 2003.

[LLT03] T. Lewiner, H. Lopes, and G. Tavares. Optimal discrete Morse functions for 2-manifolds. *Comput. Geom.*, 26(3):221–233, 2003.

[Mat07] J. Matoušek. *Using the Borsuk-Ulam theorem*. Universitext. Springer-Verlag, Berlin, 2007.

[McM70] P. McMullen. The maximum numbers of faces of a convex polytope. *Mathematika*, 17(2):179– 184, 1970.

[MF08] R. Malgouyres and A. R. Franc´es. Determining whether a simplicial 3-complex collapses to a 1-complex is NP-complete. *DGCI*, pages 177–188, 2008.

[Nab95] A. Nabutovsky. Einstein structures: Existence versus uniqueness. *Geom. Funct. Anal.*, 5(1):76– 91, 1995.

[PB80] J. S. Provan and L. J. Billera. Decompositions of simplicial complexes related to diameters of convex polyhedra. *Math. Oper. Res.*, 5(4):576–594, 1980.

[PRS98] I. Peeva, V. Reiner, and B. Sturmfels. How to shell a monoid. *Math. Ann.*, 310(2):379–393, 1998.

[RS82] C. P. Rourke and B. J. Sanderson. *Introduction to piecewise-linear topology*. Springer Study Edition. Springer-Verlag, Berlin-New York, 1982. Reprint.

[Sha01] J. Shareshian. On the shellability of the order complex of the subgroup lattice of a finite group. *Trans. Amer. Math. Soc.*, 353(7):2689–2703, 2001.

[Sta96] R. P. Stanley. *Combinatorics and commutative algebra*, volume 41 of *Progress in Mathematics*. Birkh¨auser Boston, Inc., Boston, MA, second edition, 1996.

[Tan16] M. Tancer. Recognition of collapsible complexes is NP-complete. *Discrete Comput. Geom.*, 55(1):21–38, 2016.

[VKF74] I.A. Volodin, V.E. Kuznetsov, and A.T. Fomenko. The problem of discriminating algorithmically the standard three-dimensional sphere. *Usp. Mat. Nauk*, 29(5):71–168, 1974. In Russian. English translation: Russ. Math. Surv. 29,5:71–172 (1974).

[Wac07] M. L. Wachs. Poset topology: Tools and applications. In *Geometric combinatorics*, volume 13 of *IAS/Park City Math. Ser.*, pages 497–615. American Mathematical Soc., 2007.

[Whi39] J. H. C. Whitehead. Simplicial spaces, nuclei and m-groups. *Proc. London Math. Soc. (2)*, 45(1):243–327, 1939.

A Collapsing a 1-house

In this section we prove the second statement of Lemma 8 (recalled below). We use an auxiliary lemma:

Lemma 15. *A triangulation of a topological disk collapses to any tree contained in its* 1*-skeleton.*

Proof. Let K be a triangulation of a topological disk (that is, the polyhedron of K is homeomorphic to a 2-dimensional disk) and T a tree contained in the 1-skeleton of K. While there is an edge of K that is free and not in T, we collapse such an edge. Let K^0 denote the resulting complex.

Let us first argue that K^0 contains no triangle. Let c denote the (possibly empty) Z2-chain obtained by summing the triangles of K^0. The 1-chain ∂c is a 1-cycle by definition and it is supported on T. Indeed, every edge in K^0 is contained in zero, one or two triangles and any edge contained in exactly one triangle and not in T could be used as a free face to further collapse K^0. Since a tree does not contain any nontrivial 1-cycle, it follows that ∂c is empty. In K, any nonempty 2-chain has a non-empty boundary. It follows that c is empty and K^0 is indeed 1-dimensional.

Since K^0 is a collapse of K, it must be contractible. Hence, K^0 is a tree and, by construction, it contains T. The statement follows since a tree always collapses to any of its sub-trees. □

We can now prove the announced statement: that a 1-house with free edge f and lower wall L collapses to any subtree t of the 1-skeleton of B that is contained in L and shares with the boundary of L a single endpoint of f.

Proof of Lemma 8. We apply Lemma 15 repeatedly. First, we collapse the lower wall L to the tree formed by t and the subcomplex of B triangulating $(\partial L) \setminus f$. Next, we collapse the lowest floor, except for the edges that belong to walls that are still present. We proceed to collapse every wall that used to be attached to the lowest floor. The resulting complex is already a disk with t attached to it. We collapse the disk to the attachment point and are done. □