An operational formulation of generally covariant quantum theory.

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Abstract

Some old material of 2011 is revised in the light of the better 2016 construction.

1 Introduction

In 2011, I wrote a book about an operational approach to quantum theory with local vacua delineating a Fock-Hilbert bundle $\otimes_{x \in \mathcal{M}} \mathcal{H}_x$ over the space-time manifold $\mathcal{M}$. However, the approach was troublesome and muddled with two "fundamental errors" of mine, not due to a lack of mathematical precision, but being the consequence of a poor understanding of what curved spacetime really signifies. This error found a natural solution in a 2016 book written on general covariant quantum theory from the point of view of the Feynman series. This paper, conceived at the 2018 Marcel Grossmann conference, rectifies the latter two "mistakes".'

Concretely, we assumed $\mathcal{H}_x$ to be constructed by means of a cyclic quasi-free vacuum state $|0\rangle_x$ and multiparticle states showing Bose or Fermi statistics constructed in the Fock way. The dynamical object was a unitary bi-field $U(x, y)$ mapping $\mathcal{H}_y \to \mathcal{H}_x$ and obeying a Schroedinger like differential equation

$$\frac{d}{dt} U(t, s) = iHU(t, s)$$

but then with the times t, s replaced by x, y. The two errors in the book originated from the mathematical implementation of this idea I conceived; first of all $U(t, s) = U(t)U^\dagger(s)$ and moreover the only covariant first order differential operator *homogeneous* in the spacetime coordinates is given by the covariant Dirac operator D. The first condition is equivalent to a "cohomology" condition

$$U(x, y)U(y, z)U(z, x) = 1$$

which turns out to hold in Minkowski or any maximally symmetric spacetime only and reflects the absence of local gravitational degrees of freedom. Consequently, the only solution I was able to find of my field equations was free quantum field theory on Minkowski in a way I shall explain later. The Dirac operator gives all sorts of trouble meaning we have to replace

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the complex numbers by an appropriate Clifford algebra of signature $(1,3)$ or $(3,1)$. This gives rise to negative probabilities and huge problems with the spectral theorem even for finite dimensional Clifford bi-modules. The approach was clearly dead as it stood which I realised in 2012.

2 Taking bi-fields seriously.

As pointed out in 2016, the idea of a Hilbert bundle *is* adequate, but the correct differential equation for $U(y,x)$ needs to run over geodesics connecting x with y in a fully reparametrization invariant way. The obvious candidate is given by

$$\frac{d}{ds}U(\gamma(s),x) = i\dot{\gamma}(s)^a P_a U(\gamma(s),x)$$

where $\gamma(s)$ is the unique geodesic connecting x with y and P_a equals the free momentum generator, given by the expression

$$P_a = \sum_{\text{particles j, internal degrees } \sigma_j} \int \frac{d^3k}{\sqrt{k_0}} k_a a^{\dagger}_{k;j,\sigma_j} a_{k,j,\sigma_j}$$

at the point z with respect to the dragged vierbein in x along the geodesic. The coincidence limit is fixed by $U(x,x) = 1$; this suggests one to enlarge the notation to $U(y,x;e_a(y),e_b(x))$ as well as a unitary action $T(\Lambda, e_b(z))$ of the orthochronous Lorentz group $\Lambda \in O^+(1,3)$ on $U(z,x;{e'}_b(z),e_a(x))$ by means of conjugation $TUT^{\dagger}$. All this has been explained in the 2011 book; in order for $T(\Lambda(s), e_a(\gamma(x))$ to shift through $\frac{d}{ds}$ we need a Lorentz covariant derivative and, henceforth, an antihermitian connection $L_\mu(z, e_b(z))$ such that

$$\left(\frac{d}{ds} + \dot{\gamma}^\mu(s)L_\mu(\gamma(s),\Lambda_a^b(s)e_b(\gamma(s)))\right)T(\Lambda(s),e_b(\gamma(s)))U(\gamma(s),x;e_b(\gamma(s)),e_a(x))T^{\dagger}(\Lambda(s),e_b(\gamma(s)))$$

$$-T(\Lambda(s),e_b(\gamma(s)))U(\gamma(s),x;e_b(\gamma(s)),e_a(x))T^{\dagger}(\Lambda(s),e_b(\gamma(s)))\dot{\gamma}^\mu(s)L_\mu(\gamma(s),\Lambda_a^b(s)e_b(\gamma(s))) =$$

$$T(\Lambda(s),e_b(\gamma(s)))\left[\left(\frac{d}{ds} + \dot{\gamma}^\mu(s)L_\mu(\gamma(s),e_b(\gamma(s)))\right)U(\gamma(s),x;e_b(\gamma(s)),e_a(x))\right]T^{\dagger}(\Lambda,e_b(\gamma(s)))$$

$$-T(\Lambda,e_b(\gamma(s)))U(\gamma(s),x;e_b(\gamma(s)),e_a(x))\dot{\gamma}^\mu(s)L_\mu(\gamma(s),e_b(\gamma(s)))T^{\dagger}(\Lambda(s),e_b(\gamma(s))).$$

In case we dispose of multiple geodesics connecting x with y, we just multiply the corresponding unitary operators in the *same* vierbein at y, the order of which does not matter given that all P_a commute and because the action of the Lorentz group acts by boosting the momenta. Therefore, we can just sum up the momenta which can accomodate for topology change of Minkowski into a flat spacelike cylinder giving rise to the correct field picture.

There is however a small caveat here in case multiple geodesics connect x and y in the sense that the gauge field might acquire a nontrivial significance due to multivaluedness of $e_a(y)$, where the latter is the dragged vielbein from x to y. Hence, it is better to replace the argument y by a tangent vector V in $T\mathcal{M}_x$ and take the x perspective where $\exp_x(v) = y$. In that case, we set $L_\mu(sv, \Lambda_a^b(s)e_b(\exp_x(sv)))$ to zero in case

$$\frac{D}{ds}e_b(\exp_x(sv)) = 0$$

for $s = 0 \dots 1$. In other words, the vielbein in the warped point in that direction must be the dragged one; this makes both formalisms entirely equivalent what the free theory is concerned. Notice that by construction, $U(y,x) = U^\dagger(x,y)$ due to the minus sign caused by flipping $\dot{\gamma}^\mu(s)$. Given that the connection $L_\mu(v, e_b(\exp_x(v)))$ is a new object defined on

$$T\mathcal{M}_x(v) \times V\mathcal{M}_g(\exp_x(v))$$

where $V\mathcal{M}_g(\exp_x(v))$ is the nonlinear space of g vierbeins over $\mathcal{M}$, which is equivalent to the group manifold $O^+(1,3)$ regarded as a homogeneous space with a hyperbolic Cartan metric of signature $(3,3)$; it might be opportune to make it more dynamical and invent a new type of non abelian Yang-Mills theory over $T\mathcal{M}_x$. This author tried this also in 2011 but failed to recognize the bundle perspective as well was stuck with Clifford modules for replacements of Hilbert spaces. The easiest thing is to see L_μ as $L_{\partial_{v^j} \exp_x(v)}(v, e_b(\exp_x(v)))$ where e_b varies independently and refers to $y = \exp_x(v)$ and subsequently write out a Yang-Mills equation of the kind

$$\left(D_v L_{\partial_{v^j} \exp_x(v)}(v, e_b(\exp_x(v)))\right)_{kl} = \left(\partial_{v^{[k}} - L_{[k}\right) L_{l]} = (dL)_{kl} - (L \wedge L)_{kl} = 0$$

where d is the Hodge operator on flat tangent space. Life could be more exciting as to pick out the zero solution in parallel transport gauge and we leave this new piece of physics for further examination of the bored ones.

So far, we have determined only our quantum connection; now, we develop bi-fields which are nothing but the warps of coincidence fields meaning

$$\Phi(y,x) : \left(\prod_{v \in T\mathcal{M}_x : \exp_x(v) = y} U(y = \exp_x(v), x; (\exp_x(v))_\star e_a(x), e_a(x))\right) \Phi(x,x)$$
$$\left(\prod_{v \in T\mathcal{M}_x : \exp_x(v) = y} U(y = \exp_x(v), x; (\exp_x(v))_\star e_a(x), e_a(x))\right)^\dagger .$$

Here,

$$\Phi(x,x) = \sum_{\text{particles j with internal quantum numbers } \sigma_j} \int_{\mathbb{R}^3} \frac{d^3k}{k_0} (v_{\sigma_j} \sqrt{k_0} a^\dagger_{k,j,\sigma_j} + \overline{v}_{\sigma_j} \sqrt{k_0} a_{k,j,\sigma_j})$$

where v_{σ_j} is an internal field vector associated to the internal particle degrees of freedom. They are needed to obtain different physical behaviour, $\frac{d^3k}{k_0}$ is the on shell relativistic measure in Fourier space on Minkowski and finally, $\sqrt{k_0} a_{k,j,\sigma_j}$ is relativistic normalization of the creation annihilation algebra. I leave it as an elementary excercise to find out principles determining v_{σ_j}. So $\Phi(x,x)$ is the proper democratic relativistic expression taking into all matter degrees of freedom in the universe.

3 Interaction theory.

So far, we have delineated the free theory from an operational bi-field formalism which reduces in Minkowski to a single field formalism due to the remarkable "cohomology" property

$$U(x,y)U(y,z)U(z,x) = 1$$

where we have dropped the vielbeins and assumed dragging allalong which is logical given that dragging is trivial and hence consistent along closed paths due to the vanishing of the Riemann tensor. The trick now is to work directly into an interaction picture and forget about a closed bi-field equation. That is, we write down spacetime interaction densities of the kind

$$i\lambda \int_{\mathcal{M}} \sqrt{g(y)}\Phi(y,x)\Phi(y,z)\Phi(y,p)\Phi(y,q).$$

This is an obvious excercise leading to a completely equivalent formalism as in the 2016 book.