GREAT UNIFICATION THEORY : A SOLUTION

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Abstract

The paper presents a unified model representing the gravitational, electromagnetic, weak and strong fields, fermions and bosons, in the Geometry of General Relativity. It is based on a group belonging to the Clifford algebra Cl(C,4), acting on the algebra itself. It uses an original real structure on the Clifford algebra, accounting for the physical specificities of the geometry. An explicit expression of the group, its action and of the vector states and charges of the known fermions is given. Bosons are represented as discontinuities in the derivative of the potential of the force field. No additional dimension, physical object or exotic property are required. The model appears as the continuation and extension of the Spinor model of Mechanics which holds at any scale.

The purpose of this paper is to propose a solution to one of the great problems of contemporary Physics : an unification theory. It seems a bit preposterous to pretend to have a solution to a problem which has eluded so many bright scientists. However I will not follow the usual path. Most of the previous attempts, if not all, are based on the concepts and formal system of Quantum Fields Theory, with virtual particles and bosons, and are focused on improving the mathematical representation of the Standard Model, including gravitation, with often the introduction of some new physical objects. Overall the goal is to find a "Theory of Everything", following the idea that, elementary particles being the ultimate constituants of matter and fields, a theory which explains their behavior should be the keystone for all Physics, if not of all Science.

I do not believe in the existence of a Theory of Everything, Science is organized in domains, with their own Theories which address efficiently the phenomena they study. In Physics there are Mechanics, Thermodynamics, ... They propose models and well proven laws, which can address their problems and are the basis of all the technological feats of the last century. They accept shortcuts, in that they do not pretend to account for everything, and they neglect, in full knowledge, phenomena which have a small impact in the problem at hand. Relativity has not changed this picture : Special Relativity is necessary when high spatial speeds are considered, and General Relativity when gravitation changes significantly from one point to another. However these theories are based on some common Principles and the recognition of basic physical objects, such as material bodies and force fields, with specific properties, and common concepts such as momentum, energy,.. They are stated or defined in a somewhat vague way, flexible enough to be adjusted to the representations used in the different theories. So it seems appropriate to base the search of a Great Unification Theory (GUT) on this common ground.

The first section of the paper is dedicated to a review of the concepts of measure, space-time, material body, force fields, and the implications of the Principles of Relativity, Locality and Least Action. A GUT should be fully consistent with these concepts and principles. But it is focused on the phenomena which occur in a specific domain : Particles Physics, and the goal is to provide a representation which helps to understand what happens and a model which enables to do practical computations, and gives results which are in accordance with the experimental facts. In building its theories Physics has proceeded layers by layers, exploring, through increasingly complicated experiments, new phenomena which require more sophisticated representations. I do not believe in the common interpretation of Quantum Mechanics (QM) and the existence of two Physics, with concepts and principles which would be different at the macroscopic and the atomic level. And, anyway, they would be contrary to the idea of GUT. The "Axioms" of QM, like the Hilbert spaces, observables, eigen-values,... are actually theorems, which can be proven, common to all mathematical models sharing some well defined properties, and from the way variables are estimated practically. These theorems, for the most part, validate the usual computations which are done, but with 2 major additions : they provide precise conditions for their validity, and they hold whatever the scale. There is no need for a "physical interpretation" of QM, with some bizarre properties which would hold only at the atomic scale, because there is none to be found. But of course we can use these theorems in a GUT, to explore what we can expect with a model of elementary particles. The proper quantization of the properties of elementary particles, following the path of the experiments, provides a new approach to the problem : rather than postulate a mathematical model, I give the conditions which should be met by the model, and in particular by the group, starting from the characteristics of the particles as they appear in the successive layers given by the experiments with the gravitational and electromagnetic (EM) fields, the weak fields and the strong fields.

Keeping in mind that a Unified Theory should encompass all known facts, whatever the scale, one cannot focus on the discontinuous processes. There are actually 2 Physics : continuous and discontinuous processes require different mathematical tools, and our models must account for both, with some adjustments. There are no totally discontinuous processes, there are continuous processes in which a discontinuity appears. This transition (such as in change of phase) is of course significant, and is usually the focus in the study of discontinuous processes. But a physical theory must first account in a consistent way for continuous processes in order to deal efficiently with discontinuities. The key discontinuities in a GUT theory occur in the interaction particles / fields and in the annihilation / creation of particles. So we must address these discontinuities for what they are, from a continuous representation. After all even the Standard Model is based on continuous variables. And if we acknowledge that there can be discontinuities in the life of a particle, we must acknowledge that there can be discontinuities in force fields, and this is the genuine nature of bosons.

This first section is completed by a short review of the experimental facts, as they are explained in the Standard Model, which gives a basic picture of the challenges.

The second section is dedicated to the core of the problem, the search for a unified representation of particles and fields for elementary particles. The results of the first section lead naturally to a solution based on a group defined in the Clifford algebra $Cl(\mathbb{C}, 4)$ and acting on the same algebra. This path has been attempted many times. I introduce several new ingredients : a real structure on $Cl(\mathbb{C}, 4)$ based on the only structure which has a physical meaning, the Clifford algebra $Cl(3, 1)$ which is at the core of the representation of motion in General Relativity, and new mathematical theorems on Clifford algebras such as the exponential and the extension of reflections to unitary maps. As the reader may be not familiar with Clifford algebras this section is introduced by a mathematical review of the concepts. I show that a 16 real dimensional Lie group acting on $Cl(\mathbb{C}, 4)$ by the adjoint map gives a representation of the known fermions. I give the format of the state vectors of these particles, the value of their charges, the explicit matricial representation of the action and of the lagrangian, which is quite simple. The concept of momentum can then be extended to fermions, which enables to study efficiently the collision of particles in the context of General Relativity. The study of composite particles and their decay can then be done in the usual way, with the decomposition of representations. I give an explicit description of the Cartan algebra and the root vectors, which are an essential tool in these endeavours.

The third section is dedicated to the propagation of the field. The idea of a force field existing everywhere and propagating by self-interaction raises several issues, which are more acute in the context of General Relativity. The usual models based on the implementation of the Principle of Least Action provide a set of differential equations, representing the conditions of the balance of energy at equilibirum, which do not cover all the phenomena involved in propagation. I introduce a new assumption, based on considerations which go beyond particle physics : in its propagation the force field follows Killing curves. It is consistent with the known facts, and is crucial in the representation of bosons. The interactions particles / field can be at the origin of discontinuities of the derivative of the potential, these discontinuities can be represented like particles, the bosons. Their mathematical representation is given, and from there their basic properties can be explained.

Many subjects of the first section are seen in more details in my book "Theoretical Physics". The Annex gives a short reminder of essential mathematical results, which is also useful to fix the notations. More can be found in my book "Advanced Mathematics for Theoretical Physics".

1 THE MEANING OF A GREAT UNIFICATION THEORY

1.1 Measure and standards

In a Theory we assume the existence of objects, characterized by properties, that is phenomena which can be observed and measured. These properties define the objects, and their measures are the signature of the object in the theory, however they are deeply rooted in the representation which is used.

A measure is always an experiment in which a phenomenon is interpreted with respect to another, similar, phenomena. The measure of the length of an object is the comparison of some characteristics of one object, with similar characteristics of another object. An object does not "have" a length, it behaves similarly as another object in experiments which follow a precise protocol. A law is deemed scientific if it can be checked : the experiment can be reproduced and the results compared. The protocols used in the experiments must define the conditions in which the data are collected and analyzed. To do this one uses models, based on a formal representation of the phenomenon involved : to each data which is collected is associated a mathematical object, a variable, which precise the format of the data : scalars are not collected as vectors, and the data are usually related to different events, in space and time, of the experiment. Mathematically these features are defined in standards. They give the rules which apply to adjust the data from one standard to another, to compare the results between experiments, for instance in a change of units, or when the experiment is done in another context, in space and time.

The Principle of Relativity states that the laws of nature do not depend on the observer. The observer has free will, he can choose the standards inside the procedure. The principle does not give the laws of nature, it tells only that the data which are collected much follow some rules when one goes from one standard to another.

The Principle of Locality states that a phenomenon at a given point, in space and time, depends only on the value of the variables at this point. The Principle is the affirmation that there is no action at a distance. As a consequence it should be possible, at least theoretically, to use different standards at different points, or equivalently an experiment can be done by the aggregation of data collected by different observers located at any point, using their own standards, and exchanging their data at the end of the experiment. This requisite is necessary because any practical experiment covers an extended area in space and time. Mathematically it implies that the variables must be represented in fiber bundles : their value is fixed with respect to a standard, given at each point.

The choice of the standard is arbitrary, but the rules according to which they change are intimately linked to the mathematical object chosen to represent the variable, and are a fundamental part of the model. Usually it is assumed that the standards vary according to the rules given by a group, the choice of an element of the group is arbitrary, but the group itself is part of the model.

Then the unity element corresponds to a "neutral state", a state of reference of the system, in which no phenomenon occurs. A standard located at each point is represented by a principal bundle, and the quantities which are measured are represented in associated fiber bundles.

In any scientific theory it is assumed that the objects are real physical entities, in the meaning that they have an existence independent of our measures. In this picture of physical objects, with their intrinsic properties, Physics can be seen as a Natural Science : the role of the Physicist would be discover new facts, as the explorer who is looking for new worlds. Cosmology and the media lead to this perception. But Physics has a higher goal, not only to measure but to understand, and both proceed from the use of models, in which the physicist assume that a quantity can, a priory, takes any value, and compute what it would then happen, so that he can check the validity of his theories. And this gives to models and standards their full meaning : as well as the standard "1" corresponds to a state of reference, another value of the standard corresponds to a change of the value of the variable.

1.2 Geometry

We have 3 objects : the Geometry, Particles, Force Fields. The Geometry of the Universe is an object of Physics in itself. It deals with the measure of location, lengths in time and space, and how they change from one observer to the other. Material bodies have geometric properties, a position and a motion, whose measure is done in a representation of the Geometry of the Universe. They constitute the first layer of experiments which are done on material bodies.

1.2.1 Location and the manifold representation

Observational facts lead to assume that the Geometry can be represented by a 4 dimensional manifold M. Events are located in a chart $\varphi : \mathbb{R}^4 \to M$, specific to each observer, which is arbitrary : the only purpose of a chart is to provide a protocol to locate the event in space and time. The only requisite is that the charts must be compatible : there are precise rules which tell how to compute the coordinates of the same event in different charts. For instance the "cosmic ladder" provides the location of events at increasing distances in Astrophysics.

One fundamental feature of the Geometry of the Universe is the "fundamental symmetry breakdown" : time has specific properties, it is not measured by the same protocols and one cannot travel in time, it has a specific, given, orientation imposed to any physical phenomenon. A chart, by definition, defines the set of events which are simultaneous for an observer (the value of the coordinate time is the same), it defines a foliation of the manifold in 3 dimensional hypersurfaces, but this foliation is specific to each observer.

Any displacement in the universe is then represented by a 4 dimensional vector in the tangent bundle TM to the manifold. There is an euclidean metric, a 2 symmetric definite positive form, to measure the spatial distance between simultaneous events. Using this metric any observer can define at a point a basis of the 3 dimensional space and physically check that it is orthonormal. One proceeds to measures of length and rotation through orthonormal bases, which provide the standards. On goes from one orthonormal basis to another by operations depending on a group which is $SO(3)$ for 3 dimensional euclidean bases. The freedom of observers is that they can proceed to a change of orthonormal basis.

The Principle of Causality shows that the order of events itself does not depend on the observer, and from this fact it is not difficult to deduce the existence of a special metric in the universe, which is not euclidean : the time coordinate is special, and the metric must have the signature $(3,1)$ or $(1,3)$. Vectors $V \in TM$ can be distinguished according to the value of the scalar product $\langle V,V\rangle_{TM}$ and vectors which represent non simultaneous events are such that $\langle V,V\rangle_{TM} < 0$ (or > 0 with the signature 1,3). Moreover one can distinguish future oriented vectors [1]. The metric is the physical part of the Geometry. An event, by definition, is a point in the Universe, its location is absolute, and similarly the metric : it is defined everywhere and it is given.

1.2.2 Position and motion of a material body

A material body occupies a definite location at any time, which is measured by an observer through a set of coordinates in a chart. All material bodies, and it is assumed that this is also true for particles, are characterized by an arrangement, which can be represented by a 3 dimensional orthonormal basis attached to the body, and measured by its rotation $r \in SO(3)$ with respect to the orthonormal basis of an observer. Location + Arrangement, that we will call Position, are static, they are defined at any time and measured by an observer with variables depending on his time.

Material bodies are not immobile : they travel along world lines in the Universe, and the change of their position is their motion, which can be represented and measured in two different ways.

The motion can be with respect to a given observer, this is the relative motion, which is measured by the derivative, with respect to the time of the observer, of the variables used to measure the position.

But one can also define an absolute motion, without any reference to an external observer : it sums up to measure the variation of position with respect to the position of the body itself. The world line along which travels a material body is represented by a curve, the tangent to this curve is a geometric object, a 4 vector V, the velocity of the body, whose existence does not depend on any observer. The motion of the body is a map $q : \mathbb{R} \to M$ and there is a unique (up to the choice of an origin) parameter τ such that $\frac{dq}{d\tau} = V$. This parameter is the proper time of the body, and this is the time measured by an observer on his own clock. The world line has then for map : $q : \mathbb{R} \to M :: q(\tau) = \Phi_V(\tau, x)$ where Φ_V is the flow of any vector field which supports V, and x the location

[1] This is no longer possible if there are more than one time dimension.

at $\tau = 0$.

The change of arrangement can be measured with respect to an orthonormal basis attached to the body itself : this is equivalent to measure the instantaneous rotation by the quantity $r^{-1}\frac{dr}{d\tau} \in so(3)$ and one can check easily that this quantity does not depend on the choice of an orthonormal basis (it does not depend on the observer).

A basic assumption of Relativity is that the velocity of any material body is a future oriented 4 vector whose length $\langle V, V\rangle_{TM} = -c^2$ is constant (this is $+c^2$ with the signature $(1,3)$). As the proper time of an observer is his "biological time" this assumption is equivalent to say that the clocks of the observers "run" at the same speed. The universal constant c is just a number which enables to go from the measures of time to the measures of length, and is not, a priory, related to the speed of light.

From these assumptions one can easily prove the usual formulas to go from the measures done in the orthonormal basis of an observer to the measures done in the orthonormal basis of another observer located at the same point, and these formula hold without the usual requisite of "inertial frame".

With these assumptions we can build a comprehensive representation of the motion.

1.2.3 Tetrads and the principal fiber bundle P_G

An observer can choose a 3 dimensional orthonormal basis $(\varepsilon_j)_{j=1}^3$, from which can be deduced a 4 dimensional basis, orthonormal for the metric : there is no choice for the 4th vector ε_0, which must be future oriented and in the direction of his velocity. The set of 4 vectors $(\varepsilon_j)_{j=0}^3$ is a tetrad. From his chart of the manifold M the observer can deduce a local holonomic basis $(\partial\xi_\alpha)_{\alpha=0}^3$ of TM (they correspond to infinitesimal displacements along the axes, along the time at the same location for $\partial\xi_0$) and there is a relation between ε_j and $\partial\xi_\alpha$: $\varepsilon_j = [P]_j^\alpha \partial\xi_\alpha$ where $[P]$ is a 4×4 matrix which has necessarily the format :

$$[P] = \begin{bmatrix} 1 & 0 \\ 0 & [Q] \end{bmatrix}$$

where $[Q]$ gives the components of the spatial orthonormal basis in $(\partial\xi_\alpha)_{\alpha=1}^3$.

The tetrad fixes the geometric standards for the observer. His freedom of geometric gauge is expressed in the group by which one goes from one tetrad to another. Because the 4th vector is imposed one considers that a change of this vector corresponds to a change of observer (another observer, located at the same point, but with a different velocity).

We assume that there is a tetrad similarly attached to any material body, then the arrangement of the body with respect to the observer if measured by the rotation necessary to go from one orthonormal basis to another, that is by an element of the group of rotations.

1.2.4 The Clifford bundle

For a change of 3 dimensional orthonormal basis the group is $SO(3)$, but it raises an issue when considering rotational motions : the instantaneous rotational motion is measured by $r^{-1}\frac{dr}{dt}$, and we get the same result with a rotational speed ω and axis $\overrightarrow{r}$ and rotational speed $-\omega$ and axis $-\overrightarrow{r}$. This comes from the fact that $r^{-1}\frac{dr}{dt}$ belongs to the Lie algebra $so(3)$. The Spin group $Spin(3)$ has the same Lie algebra $so(3)$ and distinguishes the 2 rotations (the scalars $+1$ and -1 belong to $Spin(3)$), so we should actually consider $Spin(3)$ to measure rotations in Galilean Geometry. The same issue happens with the tetrad, the right group is $Spin(3,1)$ and not the Lorentz group $SO(3,1)$. This is at the origin of the - confusing - issue of the "spin" in Physics. It has nothing to do with a "quantic phenomenon", this is just a matter of choosing the right representation. The choice between one or the other representation is linked to the possibility to physically differentiate one or the other rotation. In the relativist picture, with the tetrad, the choice is possible : the 4th vector is imposed, and it suffices to say that it is in the direction of the axis of rotation ("spin up") then the sign of $\sigma_r \in Spin(3,1)$ identifies the rotational motion.

With the group $Spin(3,1)$, which has a physical meaning, one can define a principal bundle $P_G(M, Spin(3,1), \pi_G)$, that is the choice (arbitrary up to the 4th vector) of the tetrad of an observer at each point. The map used to go from one tetrad to another is the adjoint map, denoted Ad, defined in the Clifford algebra by $Ad_\sigma Z = \sigma \cdot Z \cdot \sigma^{-1}$, which is similar to the same operation on matrices. According to the Principle of Locality the measures must be done at the location where the phenomenon occurs, so we must assume the existence of a network of observers sharing (with a delay) their data, or equivalently of a principal bundle $P_G(M, Spin(3,1), \pi_G)$ which tells what is the tetrad chosen by the observer at each point. The tetrad are then defined in the associated vector bundle $P_G[\mathbb{R}^4, Ad]$.

The Spin group is built from a special mathematical object, the Clifford algebra $Cl(3,1)$ which can be seen as the extension of the vector space spanned by the tetrad to a 16 dimensional algebra. Using the principal bundle P_G one can define, at each point, a Clifford bundle P_{Cl} which is a vector bundle $P_G[Cl(3,1), Ad]$. There is a Clifford bundle located at each point, and a change of tetrad implies a change of basis in the Clifford algebra, given by the adjoint map with an element of $Spin(3,1)$.

The principal bundle P_G can then be used to represent the position of any material body : one assigns another tetrad to the material body and its relative position is represented by an element of $Spin(3,1)$ at its location.

The instantaneous rotational motion is measured by $\sigma_r^{-1} \cdot \frac{d\sigma_r}{dt}$ which belongs to a sub Lie algebra of $Cl(3,1)$. And we can get similar relations for the transversal motion. Let $(e_j)_{j=0}^3$ be the tetrad attached to a material body and $V = \frac{dq}{dt}$ its velocity with respect to the observer. By definition $V = c\varepsilon_0 + \overrightarrow{v}$ where $\overrightarrow{v}$ is the spatial speed as measured by the observer (located at the same point). And there is some $\sigma \in Spin(3,1)$ such that $e_j = Ad_\sigma \varepsilon_j$. Using the fact that $V = ce_0$ and denoting U the velocity V as measured by the observer in his tetrad, a computation gives :

$U = -\frac{c}{\langle Ad_\sigma \varepsilon_0, \varepsilon_0 \rangle} Ad_\sigma \varepsilon_0$

$\frac{de_j}{dt} = \left[\frac{d\sigma}{dt} \cdot \sigma^{-1}, e_j\right]$

$\frac{dU}{dt} = \frac{U}{c} \left\langle \left[\frac{d\sigma}{dt} \cdot \sigma^{-1}, U\right], \varepsilon_0 \right\rangle + \left[\frac{d\sigma}{dt} \cdot \sigma^{-1}, U\right]$

$\frac{d\sigma}{dt} \cdot \sigma^{-1}$ belongs to the Lie algebra $T_1 Spin(3,1) \subset Cl(3,1)$ and the bracket $[]$ is the bracket in the Clifford algebra.

In a discontinuous motion $\frac{d\sigma}{dt} \cdot \sigma^{-1}$ is replaced by $\delta\sigma \in T_1 Spin(3,1)$.

In all cases $\delta\sigma$ has a unique decomposition $\delta\sigma = \delta\sigma_r + \delta\sigma_w$ where the first component corresponds to a rotational motion (the usual spin) and the second to a translational motion.

So the position and motion of a material body can be fully represented in the Clifford bundle. For any motion, not necessarily continuous (for instance if the velocity is not continuous as in a collision), the representation is in the 1st jet extension $J^1 P_{Cl}$ of P_{Cl} : $j^1 P_{Cl} = (q(t), \sigma(t), \delta\sigma(t))$ where $\delta\sigma(t) \in T_1 Spin(3,1)$.

1.2.5 Motion of a deformable solid

A deformable solid (or a fluid) is represented by a set of material points moving on integral curves of a common vector field V. This picture can immediately be extended in General Relativity : the vector field defines the proper time τ of the solid, the location of a material point is fixed by this parameter τ and the location x at some initial time through the flow $\Phi_V : q(\tau) = \Phi_V(\tau, x)$. By affecting a tetrad to each material point the deformation of the body can be represented by a section $\sigma \in \mathfrak{X}(P_G)$. A rigid solid is then a deformable solid such that $\sigma = s(t) \cdot \sigma_0(x)$ where σ_0 does not depend on the time but on the initial location x.

The same representation can be used for a set of particles following similar trajectories (such as in a beam) : one defines a common section $\sigma : M \to P_G$ and then the position of a given particle is $\sigma(q(t))$. Then the velocity V of a given particle is given through the common section.

Symmetries in material bodies and their motion (such as periodic motion) can be represented using the Clifford bundle. For instance the periodic motion of a particle on a closed curve (such as the electronic shell around a nucleus) can be easily represented in GR with 2 maps $\mathbb{R} \to \mathbb{R}^3$.

A common case is the constant instantaneous rotational motion of a material body, such as a particle at the atomic level. The speed of rotation is fixed but the axis can be changed. When, in the inversion of the axis, the particle "looks the same", this can be interpreted as the existence of a spatial symmetry. Then the representation of the group involves integers and it is said that the particle has a "spin n", $n \in \mathbb{N}$, and conversely the particles without this symmetry are labeled as having a "spin$\frac{n}{2}$".

The standards belong to the principal bundle P_G. All geometric measures are done in a vector bundle associated to P_G. By itself P_G does not give the metric, only how the measures of length and time change with a change of tetrad. This model, which implements fully the Geometry of General Relativity, does not require any assumption about the speed of light or inertial frames. Special Relativity adds an assumption : the metric is assumed to be constant, so it enables to consider universal orthonormal frames and affine coordinates, however they hold only for observers moving at a constant velocity.

1.3 Mechanics

The second layer of experiments involve forces, and lead to the introduction of the concepts of momentum, kinetic energy, and additional characteristics of material bodies.

1.3.1 Momentum and kinetic energy

Material bodies show a resistance to change their motion, both translational and rotational, and this leads to the introduction in Newtonian Mechanics of the concepts of translational $\overrightarrow{p}=M_p\overrightarrow{v}$ and rotational momentum $\Gamma = R[J]R^{-1}\frac{dR}{dt}$, with 2 characteristics of the material body : its inertial mass M_p and its rotational inertial tensor $[J]$. Only the change of momentum can be measured and it is the opposite of the force or the torque necessary to change the motion. The kinetic energy, both translational and rotational are then defined by integration over a trajectory, and only their change can be measured. Formally the change of kinetic energy is related to the change $\overrightarrow{\delta p}$ of momentum (not necessarily continuous) by $\delta K = \frac{1}{M_p}\langle\overrightarrow{p},\delta\overrightarrow{p}\rangle$. The translational momentum is a vector, but the rotational momentum is not and its definition is more convoluted.

In the early days of Relativity the translational momentum has been defined as M_pV with the 4 velocity V, and the identity $\langle V,V\rangle = -c^2$ has lead to define a "mass at rest" and to the decomposition of M_pV in a 3 dimensional relativist momentum, the 4th component is assumed to represent the "energy" of a particle (without proper definition). As all simplistic ideas it stuck, but it has no physical ground. Meanwhile there is no clear definition of the rotational momentum, and anyway Relativity did not acknowledge the existence of solids.

1.3.2 Spinors

Definition

Position and motion can be represented in the 1st jet extension J^1P_{Cl}, for particles as well as deformable solids, so it is logical to look for a similar representation of the momentum, which would be a differential operator. To have a vectorial representation of the momentum we need a representation of the Clifford algebra. They have faithful representations on matrices, however the Clifford algebras $Cl(3,1), Cl(1,3)$ are not isomorphic and their representations are different. However there are morphisms $C : Cl(3,1) \to Cl(\mathbb{C},4), C' : Cl(1,3) \to Cl(\mathbb{C},4)$ such that the images of the real Clifford algebras are real subalgebras of the complex 4 dimensional Clifford algebra $Cl(\mathbb{C},4)$ (with a 2 bilinear symmetric form with signature $(++++)$). $Cl(\mathbb{C},4)$ has a faithful representation $\gamma : Cl(\mathbb{C},4) \to L(\mathbb{C},4)$ on 4×4 complex matrices, so that γC or $\gamma C'$ provide a representation of the real Clifford algebras. The matrices are built from a set of 4 generators $\gamma_j = \gamma(\varepsilon_j)$, representing the vectors of a basis $(\varepsilon_j)_{j=0}^3$ of $\mathbb{C}^4$, which must meet the identities : $\gamma_k\gamma_k + \gamma_k\gamma_j = 2\delta_{jk}I_4$. The solution is not unique, equivalent representations are deduced by conjugation with a fixed matrix. A convenient representation is with :

$\gamma_0 = \begin{bmatrix} 0 & -i\sigma_0 \\ i\sigma_0 & 0 \end{bmatrix}; j = 1,2,3 : \gamma_j = \begin{bmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{bmatrix}$

where

$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

are the Dirac's matrices, such that $\sigma_j = \sigma_j^*; \sigma_j\sigma_k + \sigma_k\sigma_j = \delta_{jk}I_2$.

Then the generators γ_j are both Hermitian and unitary : $j = 0...3 : \gamma_j = (\gamma_j)^* = (\gamma_j)^{-1}$.

Each element Z of the Clifford algebra gives, by sum or product of generators, a matrix $\gamma(Z)$ which acts on a 4 complex dimensional vector space E, whose vectors are called spinors. With the representation above, spinors read $S = \begin{bmatrix} S_R \\ S_L \end{bmatrix}$ with a right handed component S_R and a left handed component S_L, both 2 dimensional complex vectors. The decomposition comes from the existence of a specific operator in Clifford algebras. The image of the real Clifford algebras should lead to globally invariant subsets of E, which are such that $S_L = \epsilon i S_R, \epsilon = \pm 1$.

There is a Spinor bundle $P_{Cl}[E,\gamma C]$ based on the Clifford bundle P_{Cl}. The main assumption is then, for any particle, the existence of a differential operator

$J^1P_{Cl} \to J^1P_{Cl}[E,\gamma C] ::$

$j^1S = ((q(t), S(t), \delta S)) = (q(t), \gamma C(\sigma(t))S_0, \gamma C(\delta\sigma)S_0)$

The fixed vector $S_0 \in E$ characterizes the kinematic properties of the particle, $S(t) = \gamma C(\sigma(t))S_0$ its momentum, both translational and rotational, and $\delta S = \gamma C(\delta\sigma)S_0$ its change of momentum, in a continuous or discontinuous motion. In a change of gauge S_0 does not change, and for an observer attached to the particle $S(t) = S_0$.

Forces and torques are represented by vectors of E, whose basis is arbitrary and chosen in each context. They are actually identified through the motion (represented by σ and then measured in the tetrad) to which they are associated which enables to distinguish a translational and rotational momentum.

Momentum and kinetic energy

There is a Hermitian scalar product, preserved by $\gamma C(\sigma), \sigma \in Spin(3,1)$ given by the matrix γ_0 :

$\langle S, S'\rangle = \langle \gamma C(\sigma)S, \gamma C(\sigma)S'\rangle = [S]^*[\gamma_0][S]$

$\langle \gamma C(\sigma)S_0, \gamma C(\sigma)S_0\rangle = \langle S_0, S_0\rangle = 2\epsilon[S_R]^*[S_R]$

It is then logical to identify the mass to $M_p = \sqrt{2[S_R]^*[S_R]}$ then

$S_R = \frac{M_p}{\sqrt{2}} \begin{bmatrix} e^{i\alpha_1} \cos \alpha_0 \\ e^{i\alpha_2} \sin \alpha_0 \end{bmatrix}$. The kinematic characteristics of a particle are fixed by 4 real parameters.

By derivation along a trajectory :

$\frac{d}{dt} \langle \gamma C(\sigma) S_0, \gamma C(\sigma) S_0 \rangle = \left\langle \gamma C\left(\frac{d\sigma}{dt}\right) S_0, \gamma C(\sigma) S_0 \right\rangle + \left\langle \gamma C(\sigma) S_0, \gamma C\left(\frac{d\sigma}{dt}\right) S_0 \right\rangle = 0$

and, by analogy with Newtonian Mechanics, we can define the change of total kinetic energy, translational and rotational, by :

$\delta K = \frac{1}{i} \frac{1}{M_p} \langle \gamma C(\sigma) S_0, \gamma C(\delta\sigma) S_0 \rangle = \frac{1}{i} \frac{1}{M_p} \langle S_0, \gamma C(Ad_{\sigma^{-1}} \delta\sigma) S_0 \rangle$

The computation of δK shows that $\delta K = -\epsilon \frac{M_p}{2} k_S^t \operatorname{Re} Ad_{\sigma^{-1}} \delta\sigma$ where k_S is a fixed 3 dimensional real vector, which sums up the kinematic characteristics of the particle. And two particles such that $\psi_0^- = e^{i\phi}\psi_0$ share the same vector k_S.

Collisions

It is usually said that the solution of the problem of the elastic collision (that is without change of the total energy) of two material bodies is given by the conservation of momentum $M_p \overrightarrow{v}$, generalized in Special Relativity. But this is deceptive. Even in the simplest case it is easy to check that the equation in the 3 dimensional Galilean space does not provide a solution : it is necessary to involve, one way or another, the conservation of rotational momentum.

With this remark the model of elastic collisions in General Relativity comes naturally from the conservation of the sum of the spinors, which account for the rotational motion. The collision occurring, by definition, at a point, the standards are identical for the 2 bodies and the sum is well defined. We must consider separately the two components in $\delta\sigma = \delta\sigma_r + \delta\sigma_w$ and the equations read :

$\sum_{p=1,2} \gamma C(\delta\sigma_{rp}) S_{0p} = \sum_{p=1,2} \gamma C\left(\delta\sigma'_{rp}\right) S_{0p}$

$\sum_{p=1,2} \gamma C(\delta\sigma_{wp}) S_{0p} = \sum_{p=1,2} \gamma C\left(\delta\sigma'_{wp}\right) S_{0p}$

Deformable solids

The great interest of this representation is that it can be easily extended to deformable solids with a common vector field V (this can be useful in Astrophysics where General Relativity is required). A deformable solid is represented by a section of $J^1 P_{Cl}[E, \gamma C]$, the vector S_0 is associated to its elementary components. One can introduce a density function $\mu : M \to \mathbb{R}$ giving the number of components present at a point m. An elementary computation gives then the conservation law : $\frac{d\mu}{dt} + \mu div V = 0$. A spinor S_B for the solid can be computed by integration, and for a rigid solid $S_B = \gamma C(s(t)) \gamma C\left(\int_{\omega_3(0)} s_0(x) \mu(x) \omega_3(x)\right) S_0$ where the integral is computed over a 3 dimensional spatial hypersurface at $t = 0$. Which gives a generalization of common formulas of mechanics. The representation gives also the stress tensor of the solid.

Elementary particles

So far the word "particle" has been used in the usual meaning of Mechanics, as material point. But several features of this representation are of interest for elementary particles. The fact that the spinor has two components, whose definition is chiral. The relation $S_L = \epsilon i S_R$ between these components. The definition of the energy $2\epsilon [S_R]^* [S_R]$ which can give a positive or a negative result. Because the choice of the basis of E is arbitrary, in the usual cases ϵ does not matter, but it can be interpreted as differentiating particles and antiparticles.

The model has several similarities with the usual spinors of the standard model (they are basically the same objects, with a different signature and choice of γ matrices). The vector k_S corresponds to the Dirac's current.

So we have a clear representation of material bodies and of Mechanics, consistent at any scale, from electrons to galaxies, in the Geometry of General Relativity, based on standards defined through the principal bundle P_G.

1.4 Force fields

The third layer of experiments introduce the third object of Physics : the force fields, which have the properties to be defined everywhere, to interact with particles and to propagate in the vacuum by interacting with themselves. They have been introduced at the end of the XIX° century to explain electromagnetism, and the model has been extended to the gravitational field. Relativity, then the phenomenon discovered at the level of elementary particles, have lead to a profound revision of the model. Its main architecture appears to be robust enough to be used as a basis for a GUT. So I will give here a general description of what could be the model of a GUT, and a presentation of the main tools which could be used.

1.4.1 Elementary particles

The spinor model accounts for the motion and the kinematic characteristics of particles. With force fields new characteristics appear, the charges, which are constant and tell how the fields interact with them.

In a GUT there should be, for each elementary particle, a variable ψ that we can assume to belong to a vector space F, which sums up all the measures which can be done on the particle at any point $m \in M$. These measures are done with respect to some standard, given locally by a principal bundle $P_U(M, U, \pi_U)$ with a group U. A state of reference ψ_0, corresponding to $u = 1$, is defined and one goes from ψ_0 to $\psi(m)$ by an action ϑ of the group U, so there is a representation (V, ϑ) of U and ψ itself is represented by $\psi(m) = \vartheta(u(m))\psi_0$ in an associated vector bundle $P_U[V, \vartheta]$.

On its trajectory, in a continuous or discontinuous process, the state ψ of a particle and its change $\delta\psi$ are represented in the 1st jet extension : $J^1 P_U [V, \vartheta] = (m, \psi, \delta\psi)$.

In a continuous process $\psi = \vartheta(u)\psi_0 \Rightarrow \delta\psi = \left(\frac{d\vartheta}{du}\right)|_{u=u_0}(\delta u)\psi_0$ and

$\left(\frac{d\vartheta}{du}\right)|_{u=u_0}(\delta u) = \left(\frac{d\vartheta}{du}\right)|_{u=1}\left(L'_{u_0^{-1}}u_0\right)(\delta u)$ with the derivative $L'_{u_0^{-1}}u_0$ of the translation on U.

$\left(\frac{d\vartheta}{du}\right)|_{u=1} = \vartheta'(1)$ is a linear map $T_1U \to \mathcal{L}(V;V)$

δu and $\left(L'_{u_0^{-1}}u_0\right)(\delta u)$ belongs to the Lie algebra T_1U of the group U.

Any change, continuous or discontinuous (as long as the fundamental state ψ_0 stays the same) can then be written :

$\delta\psi = \vartheta'(1)(X)\psi_0$ with $X \in T_1U$

The adjoint representation (T_1U, Ad) gives the adjoint bundle $P_U[T_1U, Ad]$ and the 1st jet extension of P_U can be written :

$J_1P_U = (m, u \in U, X \in T_1U)$

The idea that the state of the particle changes according to the change of u and its own characteristics (its charges) represented in ψ_0 can be summed up in the existence of a linear differential operator :

$J_1P_U[V,\vartheta] \to J^1P_U[V,\vartheta] :: (m,\psi,\delta\psi) = (m,\vartheta(u)\psi_0,\vartheta'(1)(X)\psi_0)$

The change of the state of the particle is attributed to the action $\vartheta'(1)(X)\psi_0$ of a force field represented by $X \in T_1U$.

This is the extension of the representation by σ and $P_G[Cl(3,1), Ad]$ with the group $Spin(3,1)$, and the vector ψ_0 is the generalization of the spinor S_0.

The state ψ should account for the motion, and so encompasses σ, and, in a unified theory, accounts also for the charges and the specific features of elementary particles. Charges are measured by comparing the behavior of particles under the action of a known force field, so the standard given by U are closely related to the definition of the fields. The value $u = 1$ corresponds to the absence of field and gives the value $\psi = \vartheta(1)\psi_0 = \psi_0$.

1.4.2 Interactions particles / fields

A particle is never immobile in the 4 dimensional Universe, the value of the field changes along its trajectory, and this entails a change of its state ψ, notably of its motion, depending both of the characteristics of the particle and of the change $\delta\grave{A}$ in the value of the field. And the simplest assumption is that the infinitesimal change $\delta\psi$ is linearly linked to the change $\delta\grave{A}$. This is the basic idea of gauge fields theories.

The value of the fields is measured in experiments where the field interacts with known particles. So the standards come from the common structure P_U. The value unity of U corresponds to a state of reference, which is, by definition, the absence of field.

Let us consider a particle moving on its trajectory $q : \mathbb{R} \to M$ the state of a particle which is measured is given by : $(\mathbf{p}(q(t)), \psi(t))$ where $\mathbf{p}(q(t)) = \varphi_U(q(t), u) \in P_U$ is the gauge used by the observer. It is arbitrary, and its physical meaning is only given by the protocol for the measure. However one can see the picture the other way around : by changing the standard one can come back to a state where there is no field.

In the absence of field, assimilated to $u = 1$, with the standard gauge $\mathbf{p} = \varphi_U(m, 1)$, by definition $\psi(t) = \psi_0$. Because

$(\mathbf{p},\psi_0) \sim (\mathbf{p} \cdot u^{-1}, \vartheta(u)\psi_0) = (\varphi_U(q(t), u^{-1}), \vartheta(u)\psi_0) = (\varphi_U(q(t), u^{-1}), \psi(t))$

it is equivalent to say that the measured value of the state is $\vartheta(u(t))\psi_0$ with some $u(t) \in U$. This is the generalized principle of equivalence.

The quantity which is measured is $\psi(q(t)) = \vartheta(u(q(t)))\psi_0$.

Its change $\delta\psi = \vartheta'(1)(X)\psi_0$ is imputed to a change $X \in T_1U$ in the value of the force field, along the trajectory. So, to the curve on M given by the trajectory with tangent V is associated a unique vector on the tangent space TP_U to the principal bundle. At any point $p \in P_U$ a vector of the tangent bundle TP_U reads :

$V_U = \varphi'_{Um}(m, u)V + \zeta(X)(p)$

where $\zeta : T_1U \to VP_U$ is a map, depending on the fiber bundle, from the Lie algebra to the vertical bundle $VP_U = \ker \pi'_U$

The curve on M is lifted as a unique curve on P_U. The vector V_U associated to V must have a precise decomposition in its components $V \in T_mM, X \in T_1U$. There is no such canonical decomposition (that is which does not depend on the choice of a trivialization). The vertical space V_pP_U, is uniquely defined independently on the trivialization, and is isomorphic to the Lie algebra of $\widehat{U}$. A connection Φ is a projection $\Phi : TP_{\widehat{U}} \to VP_U$ from the tangent space T_pP_U to the vertical space V_pP_U . This is a tensor on the tangent bundle TP_U. The lift should not depend on the gauge, in the meaning that, in a global change of gauge by the observer, the result should be adjusted similarly to account for the change of gauge : the connection on $P_{\widehat{U}}$ must be equivariant. This is a principal connection. It is expressed through a map, the potential $\grave{A} \in \Lambda_1(M; T_1U)$ such that at $p = \varphi_U(m, u) : \Phi(\varphi'_{Um}(m, u)v_m + \zeta(u)(p)) = \zeta\left(u + Ad_{g^{-1}}\grave{A}(m)v_m\right)(p)$

In a change of gauge $p = \varphi_U(m, g) \to \widetilde{p} = \widetilde{\varphi}_U(m, g) = \varphi_U\left(m, g \cdot \varkappa(m)^{-1}\right)$ the potential follows an affine law :

$\widetilde{\grave{A}}(m) = Ad_{\varkappa}\left(\grave{A}(m) - L'_{\varkappa^{-1}}\varkappa(\varkappa')\right)$

A principal connection defines a linear connection on any associated bundle $P_U[E, \vartheta]$:

$\widehat{\Phi}(q)(v_m, \delta\psi) = \delta\psi + \vartheta'_g(1)\left(\grave{A}(m)v_m\right)\psi \in V_qE_M$

So the natural representation of the force fields is by a principal connection. This is a tensor, and the potential is a 1-form, valued in the Lie algebra T_1U, acting on vectors tangent to M, which is necessary because the Universe is not isotropic and the change in the value of the field is measured along the trajectory of the particle.

The linear differential operator which represents the action of the fields on the particle, represented by a section ψ on $P_U[F, \vartheta]$, in a continuous process, is then the covariant derivative :

$\nabla : \mathfrak{X}(P_U[F, \vartheta]) \to \Lambda_1(TM; P_U[F, \vartheta]) :: \nabla\psi = \psi^*\widehat{\Phi}$

$\nabla_v\psi = \widehat{\Phi}(q)(\psi'v) = \vartheta'_g(1)\left(\grave{A}(m)v\right)\psi + \psi'(m)v \in F$

Notice that the covariant derivative involves a vector field V : $(\nabla_\alpha \psi)_{\alpha=0..3}$ are just the components of the 1 form. We need V to get a vector $\nabla_V \psi \in F$. The explicit introduction of V is essential in the determination of the trajectory of the particle under the action of the field. There is no need for a “coupling constant” or the imaginary i.

For a given principal bundle P_U, for any curve $q(t)$ there is a unique curve $\psi(t)$ lifted to V, corresponding to the condition $\nabla_V \psi = 0$. However it is not difficult to see that this condition is usually not compatible with the other condition $\psi(t) = \vartheta(g(q(t)))\psi_0$. The physical reason is that the particle interacts with the field, at its location, and this interaction is represented in a more complicated way through the Principle of Least Action.

Particles of the same type k, in a given environment, have similar behaviors. If there is no collision their trajectories can be represented as integral curves of a common vector field V_k, and one can introduce a section $\psi_k \in \mathfrak{X}(P_U[F,\vartheta])$, then the state of a given particle j is represented by $\psi_k(q_j(t))$, where the trajectory $q_j(t)$ is actually defined by the initial conditions. The state of each particle is then represented by the value of ψ_k at the location of the particle. This can be seen as replacing the individual particles by a “field” (in the common meaning) of particles represented by ψ_k, which is nothing more than a general solution of the problem.

1.4.3 Propagation of the fields

Conversely the particle changes the value of the field. The change of the value of the field, occurring at the point where the particle is located, is smeared out by propagation. To model the propagation of the field one needs a derivative of the connection. Because this is a tensor, its computation needs a special approach, using the Lie derivative, and the result is a 2 form on M valued in the Lie algebra, the strength of the field $\mathcal{F}$:

$\mathcal{F} = \sum_{a=1}^{m} \sum_{\{\alpha,\beta\}} \mathcal{F}^a_{\alpha\beta} d\xi^\alpha \wedge d\xi^\beta \otimes \overrightarrow{\kappa}_a$

$\mathcal{F}^a_{\alpha\beta} = \partial_\alpha \grave{A}^a_\beta - \partial_\beta \grave{A}^a_\alpha + \left[\grave{A}_\alpha, \grave{A}_\beta\right]^a$

with ordered indices $\alpha, \beta = 0..3$, vectors $\overrightarrow{\kappa}_a$ of the Lie algebra T_1U and the bracket [] on T_1U.

In a change of gauge $\mathcal{F}$ transforms linearly with the adjoint map on T_1U : $\mathcal{F}_{\alpha\beta} \to \widetilde{\mathcal{F}}_{\alpha\beta} = Ad_u \mathcal{F}_{\alpha\beta}$ so it can be seen as a section of the adjoint bundle $P_U[T_1U, Ad]$

The interaction particle / field is not a symmetric process. Meanwhile the particle carries away the change of its momentum, the impact of the interaction on the field is carried away by propagation, which is not an instantaneous phenomenon. The effect can be measured directly in some experiments, such that the Bremstrahlung, but the main variable is the energy exchanged between the field and the particle. In a collider, the energy necessary locally to increase the kinetic energy of a particle is borrowed from the field and compensated through its propagation, which is at a finite speed. This is why the acceleration becomes more difficult when the speed is close to c. The concept of particles localized at a point introduces a discontinuity in the field, which can be smeared out if the exchange of energy is small, but can manifest itself as a discontinuity which propagates, and this gives a boson. We will see how they can be introduced in the model.

So the architecture of the model is based on a group U, a vector space F and a representation (F, ϑ), a principal bundle $P_U(M, U, \pi_U)$ and the associated bundles $P_U[F, \vartheta], P_U[T_1U, Ad]$. We need a motor to animate this representation and provide equations, and it is given by the Principle of Least Action.

1.4.4 The Principle of Least Action

Lagrangian

The Principle of Least Action is an essential tool in Physics, in that it provides the most general equations linking the different parts of a system. It is in some way a dynamic version of the Principle of Conservation of Energy. It expresses that, for a given system and over the domain which it covers in space and time, the energy exchanged between the objects of the system must be balanced.

For any system represented in a model by variables $Z = \left(z^i\right)_{i=1}^n$ over an area $\Omega \subset M$, it postulates the existence of a real function $L\left(z^i, z^i_\alpha\right)$, the Lagrangian, such that, at equilibrium, the action $\int_\Omega L\left(z^i, z^i_\alpha\right)\omega_0$ is stationary. ω_0 is a volume form, given by the metric, and $\left(z^i, z^i_\alpha\right)$ are the coordinates of a section of a 1st jet bundle (the Lagrangian can be at an order higher than1, but this is not necessary here because only the first derivatives are involved). The Principle gives only the necessary conditions for an equilibrium, which is dynamic because it encompasses an area Ω of the space-time. It provides equations relating the coordinates $\left(z^i, z^i_\alpha\right)$. In continuous processes the solution is the 1st jet extension of a section, and the quantities z^i_α are replaced by the partial derivatives. The Lagrangian is real valued, usually involves complex variables, but is not necessarily itself holomorphic. The Principle of Least Action does not give the specification of the Lagrangian, which sums up most of the Physics of the model. It is the only general law which models the interactions fields / fields and particles / fields.

For a system of particles and fields interacting the Lagrangian splits in a part related to the fields (which is the only one containing $\mathcal{F}^a_{\alpha\beta}$) and a part related to the particles (which is the only one containing ψ):

$\sum_{j=1}^N \int_0^T L_P dt$ where each particle is represented by a map ψ_j defined on some interval $[0, T] \subset \mathbb{R}$ and valued in $P_U[F, \vartheta]$, with velocities V_j.

$\int_\Omega L_F \varpi_0$ where the density ϖ_0 is related to the metric, and Ω is some compact area of M.

The fact that the two integrals have not the same dimension (over the time period $[0, T]$ and Ω) raises mathematical and physical problems, which are not solved by the naive method of introducing Dirac's function. A rigorous solution can be found, in the most general case, by the method of variational derivatives, which can be extended to fiber bundles. But the solutions are "distributions" (or "generalized functions"), which are well defined and have a physical meaning

: whenever an experiment is done, with known fields defined over an arbitrary compact area $\omega \subset \Omega$, the distribution ψ gives the "right" value, and similarly for the potential. One retrieves the common interpretation of Quantum Physics by operators acting on families of functions.

The state of the particle is represented only in the second integral, thus the condition is that $\int_0^T L_P dt$ is stationary with the value of the field accounting for the interaction with the particle.

The equilibrium itself does not depend on the observer, so the Lagrangian must be invariant in a change of standards. This imposes conditions on the specification : the potential cannot appear explicitly in the Lagrangian. The Lagrangian has necessarily the form $L_P = L\left(\psi^{ij}(m), \nabla_V \psi^{ij}(m), g_{\alpha\beta}\right)$ for the particles and $L_F = L\left(\mathcal{F}^a_{\alpha\beta}(m), g_{\alpha\beta}\right)$ for the fields, including the metric g evaluated at each point. V stays an independent variable, related to the motion, which must be itself represented in the state ψ through a section ψ.

Charges

Let us consider a particle with trajectory $[0,T] \to \Omega :: q(t)$. Assuming that $\psi(t) = \vartheta(u(t))\psi_0$ then the variable becomes $u(t) \in U$ which is a Lie group, thus a manifold with a chart : $\varphi : \mathbb{R}^m \to U : \varphi(\zeta_1, .\zeta_a ... \zeta_m) = u$, and its derivative can be expressed in its Lie algebra T_1U.

The covariant derivative $\nabla_V \psi$ reads :

$\nabla_V \psi = \vartheta'_g(1)\left(\grave{A}(q(t))V\right)\psi(t) + \frac{d\psi}{dt}$

$= \vartheta'_g(1)\left(\grave{A}(q(t))V\right)\vartheta(u(t))\psi_0 + \vartheta'(u(t))\left(\frac{du}{dt}\right)\psi_0$

Using the derivatives $L'_{u^{-1}}u, R'_u 1$ of the translations and the adjoint map Ad_u we have the identities (Maths.1882) :

$\vartheta'(u) = \vartheta'(1) R'_{u^{-1}}u = \vartheta(u)\vartheta'(1) L'_{u^{-1}}u$

Thus :

$\vartheta'_g(1)\left(\grave{A}(q(t))V\right)\vartheta(u(t))\psi_0 + \vartheta'(u(t))\left(\frac{du}{dt}\right)\psi_0$

$= \vartheta(u)\vartheta'(1) L'_{u^{-1}}u\left((R'_u 1)\left(\grave{A}(q(t))V\right) + \left(\frac{du}{dt}\right)\right)\psi_0$

$L_P(\psi(q(\tau)), \nabla_V \psi(q(\tau)))$

$= L_p\left(\vartheta(u)\psi_0, \vartheta(u)\vartheta'(1) L'_{u^{-1}}u\left((R'_u 1)\left(\grave{A}V\right) + \left(\frac{du}{dt}\right)\right)\psi_0\right)$

$= L_P\left(\psi_0, \vartheta'(1) L'_{u^{-1}}u\left((R'_u 1)\left(\grave{A}V\right) + \left(\frac{du}{dt}\right)\right)\psi_0\right)$

$= L_P\left(\psi_0, \vartheta'(1)\left(Ad_{u^{-1}}\left(\grave{A}V\right) + L'_{u^{-1}}u\left(\frac{du}{dt}\right)\right)\psi_0\right)$

The quantity $X(t) = Ad_{u^{-1}}\left(\grave{A}V\right) + L'_{u^{-1}}u\left(\frac{du}{dt}\right) = \widehat{\nabla}_{V_U} p(u)$ is the covariant derivative of the section $m \to p(u(m)) \in P_U$ along the curve in P_U. It is valued in T_1U. The field exists everywhere, we can then assume the existence of a section $\mathbf{U} \in \mathfrak{X}(P_U)$ which gives the value of u at each point m, that is of a map $M \to U :: u(m)$ which is differentiable. The quantity $X(t)$ can then be expressed as $\sum_{\alpha=0}^{3} T_\alpha(m) V^\alpha \in T_1U$ with a 1 form $\Lambda_1(M; T_1U)$ which takes the place of the potential. The Lagrangian then reads :

$L_P\left(\psi_0, \vartheta'(1)\left(\sum_{\alpha=0}^{3} T_\alpha(q(\tau))V^\alpha\right)(\psi_0)\right)$

In a change of gauge T changes with an affine map as the potential.

The partial derivative $Q(\psi_0) = \frac{\partial L_p}{\partial X}$ is a linear map from T_1U to $\mathbb{R}$, that is an element of the dual T_1U and $Q: V \to T_1U^*$ gives the charges of the particles, in the meaning that for a variation δX of the force fields the energy exchanged with the particle is $Q(\psi_0)(\delta X)$.

If the Lagrangian is expressed as a scalar product then $L_p(\psi_0, \vartheta'(1)(X)\psi_0) = Q(X) = \sum_{a=1}^{m} Q_a X^a$ with a basis $(\kappa_a)_{a=1}^{m}$ of T_1U. The charges Q_a are scalars, invariant in a change of gauge, and there are as many charges as the dimension of T_1U. A particle such that $Q_a = 0$ is insensitive to the field represented by the component a.

In a given environment the Principle of Least Action tells that the action is stationary. ψ appears only in the second integral and the condition is just that the trajectories are such that $\int_0^T L_p(\psi_0, \vartheta'(1)(X)\psi_0)\,dt$ is stationary.

In a system composed of particles, without collisions, one can associate a section $\psi_k \in \mathfrak{X}(P_U[F,\vartheta])$ to represent each type of particles, and the velocity belongs to a common vector field V_k such that the trajectories are the integral curves of V_k. The action for the particles reads :

$\sum_k \int_0^T L_p(\psi_{0k}, \vartheta'(1)(X_k(p_k(t)))(\psi_{0k}))\,dt$ with the variables $X_k : M \to T_1U : X_k(q_k(t))$.

The variation of ψ_k is given by the vector field $T(q_k(t))$, common to the particles of the same type, and are such that L_p is the same for each type of particles, and constant on the trajectories. At equilibrium in continuous processes the trajectories correspond to constant energy for the different types of particles.

1.5 The challenges of a GUT

1.5.1 The representation of the motion

The classification of particles, that is their identification, is first done through their trajectories : particles of the same type must have similar trajectories. Because the motion, and thus their trajectories, are represented with standards given by the principal bundle P_G it should be involved in the definition of P_U. In a GUT the group U should account for the group $Spin(3,1)$, which provides the standards for the motion.

1.5.2 Family of particles and charges

In a unified theory of fields, all the components of the fields act simultaneously, however we start from experiments in which families of particles are distinguished by their behavior under some components of the fields, and these characteristics are represented by charges. The challenge of a GUT is to provide a unified representation which accounts for the specificities of these force fields on one hand, and distinguish families of particles according to their behavior under the action of these different fields, on the other hand. The force fields are represented through groups, so each type of field should be a subgroup U_j of U. Their action is represented through connections, valued in the Lie algebras. The vector space which is associated to the force fields is the Lie algebra T_1U, and each type of field is associated to a Lie subalgebras $T_1U_j \subset T_1U$. Each elementary particle is identified by a fundamental vector $\psi_0 \in V$, but each family k of particle, associated to the same values of the charges, and then with the same behavior under the action of a type of field, is identified by a vector subspace $F_k \subset F$. Under the action of a field j the states of particles of a family k stay in the orbit $\{\vartheta(u)\psi_{0k}, u \in U_j\} \subset F_k$. If the particles of a family k have the same behavior under a field j, then the fundamental vectors ψ_{0k} must belong to a vector subspace which is globally invariant by the action of U_j, and the vector subspaces must be disconnected (except for 0) for $j \neq j'$.

1.5.3 The gravitational field

The interaction particle / field is usually interpreted as a force (or a torque) F exercised on the particle, depending on the charge q of the particle, which changes the momentum p of the particle $F = \delta p$. The momentum and the kinetic energy are represented through to the inertial mass M_p. In Classical Physics gravitation is a force field as the others. So there there should be a gravitational charge μ, however experiments show that the gravitational charge is equal to the inertial mass : $M_p = \mu$.

To solve the conundrum Einstein proposed in his theory of gravitation to forget both the gravitational field and the gravitational charge : the gravitational forces are nothing more than the inertial forces appearing in a curved trajectory. It requires 2 assumptions : that these trajectories are geodesics (we need to explain why the trajectories are not straight lines), defined from the metric, and a law, involving only the inertial charges, explaining how the metric changes from one point to another, and this is done through the scalar curvature.

The difficulties to explain the motion of stars in the Galaxy with this theory show that it is difficult to get rid of the gravitational field. In a model of great unification, including the gravitational field, this is the inertial mass which must disappear. To keep the concepts of momentum and kinetic energy we must acknowledge that the spinor S represents both the inertial and gravitational characteristics of the particle. In particular the kinetic energy should depend on S only. The spinor is then represented in a vector subspace E of F, globally invariant by the action of $Spin(3,1)$, and the other characteristics of elementary particles are represented in another vector subspace of F on which act the other force fields represented by a group U. This is the genuine interpretation of the Principle of Equivalence : a change of observer (an "accelerated frame") is similar to the action of a gravitational field. This is not a return to Einstein's theory of gravitation : the metric does not play any role in the picture and we will assume that its value is given in each environment (how the metric changes with matter and fields is another problem).

1.6 Quantization

1.6.1 Quantum Mechanics

The universal use of mathematical models in Physics makes necessary to investigate the properties of these models, and this can be done with a common method : models are mathematical constructs, which can be studied using Mathematics. And one can prove that they have properties of their own.

The "Axioms" of Quantum Mechanics (Hilbert space, eigen values, operators, Heisenberg law,...) are actually mathematical theorems, which can be proven as such, and apply to models which have some precise characteristics.

The models considered must involve only vectorial variables $X_1, ..., X_k$ collectively denoted X, belonging to an open subset of a vector space V which must be infinitely dimensional, separable and be a Fréchet space. These conditions are usually met whenever the variables are differentiable and such that the integral of their norm is bounded. Usually X are maps, defined over the time or some spatial area, they represent the state of a system over this area. There is no assumption about the laws followed by the variables : we stay at the level of the description of the system.

Then :

1. There is a separable Hilbert space H and a linear isometry $\Upsilon : V \to H$ which associates to each state represented by X a vector ψ belonging to an open subset of H containing 0. For any basis $(e_i)_{i \in I}$ of V there are unique families $(\varepsilon_i)_{i \in I}, (\phi_i)_{i \in I}$ of linearly independent vectors of H such that : $\varepsilon_i = \Upsilon(e_i), \Upsilon(X) = \sum_{i \in I} \langle \phi_i, \Upsilon(X) \rangle_H \varepsilon_i, \forall i, j \in I : \langle \phi_i, \varepsilon_j \rangle_H = 0.$

2. If two similar systems, represented by the variables X, X', interact, it is possible to represent the total system by the tensorial product $Y = X \otimes X'$, then the corresponding Hilbert space is the tensorial product $H \otimes H'$. The tensors are decomposable if there is no interaction.

3. Because the vector space is infinite dimensional, the value of X is estimated from a batch of data using a simplified definition $\Phi(X)$ of X, an observable, valued in a finite dimensional vector space. The simplest solution is to take a primary observable $Y_J(X) = \sum_{i \in I} X^i e_i$ where J is a finite subset of I. $Y_J(X)$ provides an optimal estimator of X, to any primary observable is associated uniquely a self-adjoint compact, trace-class, operator $\widehat{Y}_J$ on H such that the measure $Y_J(X) = \sum_{i \in I} \left\langle \phi_i, \widehat{Y}_J(\Upsilon(X)) \right\rangle_H e_i$. The value $Y_J(X)$ which is measured is an eigen vector of Y_J and the probability that the measure is equal to $Y_J(X)$ if the system is in the state X is : $\Pr(Y_J(X) | X) = \frac{\left\| \widehat{Y}_J(\Upsilon(X)) \right\|_H^2}{\|\Upsilon(X)\|_H^2}$

4. If a system can be represented by the variables $X \in V, Y \in V$ meeting the conditions above, such that $Y = \vartheta(g)(X)$ where ϑ is the action of a group G (meaning that X and Y represent the same state) then :

i) $\left(H, \widehat{\vartheta}\right)$ is a unitary representation of the group G, with $\widehat{\vartheta} = \Upsilon \circ \vartheta(g) \circ \Upsilon^{-1}$

ii) For any observable $\Phi \in \mathcal{L}(V; W)$ the vector space $W \subset V$ is globally invariant by G and $\left(H_\Phi, \widehat{\vartheta}\right)$ with $H_\Phi = \widehat{\vartheta}(g) \circ \widehat{\Phi} \circ \widehat{\vartheta}\left(g^{-1}\right)(W)$ is a finite dimensional unitary representation of G

iii) If there is an additional variable Z, defined by a fixed map $Z = f(X)$ then it defines a relation of equivalence between the states represented by $X, X\prime$: $X \sim X' \Leftrightarrow f(X) = f(X')$. To each class of equivalence is associated a class of equivalence in H with the relation : $\psi \sim \psi' \Leftrightarrow \exists g \in G : \psi' = \widehat{\vartheta}(g)\psi$. Each class of equivalence in H is globally invariant by $\widehat{\vartheta}$.

5. Evolution of a system : the model meets the additional conditions : X is a map $X : R \rightarrow F$ from an open subset of $\mathbb{R}$ to a normed vector space F; $\forall t \in R$ the evaluation map $\mathcal{E}(t) : V \rightarrow F$ is continuous ; two states X, X' of the system are deemed equivalent if the set $\omega = \{t : X(t) = X'(t)\}$ has a null Lebesgue measure. Then there is a map $\Theta : R \rightarrow \mathcal{L}(F; F)$ such that $\Theta(t)$ is unitary, $X(t) = \Theta(t) X(0)$, for each t there is an isometry $\widehat{\mathcal{E}}(t) \in \mathcal{L}(H; F)$ such that $\widehat{\mathcal{E}}(t)(\Upsilon(X)) = X(t)$.

There are other theorems but we will not use them here.

1.6.2 Observable of the type of a particle

We will proceed by steps.

The assumptions are :

i) The state of a particle is measured in a finite dimensional, normed, vector space F

ii) Over its trajectory in a given environment the state of a particle follows a continuous map : : $\psi : [0, T] \rightarrow F$

iii) With a norm on F the maps $\psi(t)$ are such that :
$\int_0^T \max\left(\|\psi(t)\|, \left\|\frac{d\psi(t)}{dt}\right\|\right) dt < \infty$.

iv) Two particles with maps ψ, ψ' are deemed belonging to the same type if the set $\{t \in [0, T] : \psi(t) = \psi'(t)\}$ has a non null Lebesgue measure.

The set $\mathcal{S}$ of maps $\psi : [0, T] \rightarrow F$ which meet the condition iv) is an infinite dimensional, separable, Fréchet space. Moreover the evaluation map : $\mathcal{E}(t)$: $\mathcal{S} \rightarrow V :: \mathcal{E}(t)\psi = \psi(t)$ is then continuous.

$\mathcal{S}$ is isomorphic to an open of a Hilbert space H with an isometry : $\Upsilon : \mathcal{S} \rightarrow H :: \Upsilon(\psi) = \widehat{\psi}$

There is a map $\Theta : \mathbb{R} \rightarrow \mathcal{L}(F; F)$ such that $\psi(t) = \Theta(t)\psi(0)$ and $\Theta(t)$ is unitary

Practically particles are submitted to different environments represented by $\Theta(t)$, and from the set of data $\{\psi(t) \in F, t \in [0, T]\}$ which is collected, one estimates a vector ψ_0 with a linear model $[\psi(t)] = [\Theta(t)][\psi_0]$. Each type k of particle is characterized by a single vector $\psi_{0k} \in F$, which does not depend on the environment and represents its fundamental state. The map : $\Pi : \mathcal{S} \rightarrow V ::$ $\Pi(\psi) = \psi_{0k}$ is an observable.

We add now the assumption :

v) The state of the particle is measured with respect to standards following the rules of a group U, that is in a vector bundle $P_U[F, \vartheta]$ associated to a principal bundle with Lie group U.

Then the map $\Theta(t)$ can be expressed as $\vartheta(u(t))$ where $u : [0,T] \to U$ is a map which depends on the environment. And $\psi(t) = \vartheta(u(t))\psi_{0k}$.

The space $\mathcal{S}$ is invariant by the global action ϑ of U on F, $(\mathcal{S},\vartheta)$ is a representation of U. We can implement the theorem 4 : $\left(H, \widehat{\vartheta}\right)$ is a unitary representation of U, with $\widehat{\vartheta} = \Upsilon \circ \vartheta \circ \Upsilon^{-1}$. Υ is an isometry, which induces an inner product on the space $\mathcal{S}$ which is itself a unitary representation of U . A unitary representation is the sum of irreducible, orthogonal, unitary representations : $\left(H, \widehat{\vartheta}\right) = \oplus_k \left(H_k, \widehat{\vartheta}_k\right)$ and similarly : $(\mathcal{S},\vartheta) = \oplus_k (\mathcal{S}_k, \vartheta_k)$ isomorphic to $\left(H_k, \widehat{\vartheta}_k\right)$ by Υ. Each type of particle generates, by $\{\vartheta(u)\psi_{0k}, u \in U\}$ a vector space, invariant by U, which belongs necessarily to one of the finite dimensional irreducible representations of U. These representations can be expressed as tensorial products of fundamental representations. Composite particles are associated to tensorial products of representations, and elementary particles are represented by vectors of the standard (or spin) representation of the group U and its contravariant.

vi) A key feature measured in the experiments is the kinetic energy, linked directly to the motion. In the model above, representing the measures done on a particle along its trajectory, if we add the measure of the kinetic energy K of the particle, the model is no longer strictly linear : the kinetic energy is a scalar function depending non linearly on the state ψ. So the general theorem for a change of standard does not apply, but we can implement the theorem 4.iii) : to each irreducible representation is associated a given value of the kinetic energy. We have generations of particles, differentiated by their mass (or gravitational charge) and behaving similarly with respect to the other fields.

So quantization gives a picture which is consistent with what has been assumed previously.

Moreover we can precise the definition of the observable $\widehat{\psi_0}$ of the fundamental state of a particle. The Physicist proceeds to an experiment in which he submits the particle to different, known, values of the force fields, say $u_n \in U, n = 1...N$ and measures the states $X_n \in F, n = 1...N$ of the particle. With the previous hypotheses F is a normed vector space. The map : $\varepsilon(X) = \sum_{n=1}^{N} \|\vartheta(u_n)(X) - X_n\|$ is continuous, bounded from below, it has a minimum in F and $\widehat{\psi_0} = X$ for which $\varepsilon(X)$ is minimum.

However it raises several issues, which are common in quantization.

Let us consider experiments in which one tries to identify a particle. In the theory there are p particles, identified by their known fundamental state $(\psi_k)_{k=1}^{p}$. The experiment can be summed up to submit the particle to known forces fields, that is a set $u_n \in U$ of N values of u, and to collect the values $X_n \in F$ of the state of the unknown particle under u_n. One can compute the quantities : $\varepsilon_k = \sum_{n=1}^{N} \|X_n - \vartheta(u_n)\psi_k\|$ and the particle will be identified to the family k such that ε_k is minimum. The procedure provides always an answer. The quality of the experiment depends on N, but also crucially on p

: by increasing the number of families of particles, one increases the assumed efficiency of the experiment. To see this, we could improve the experiment by checking the two hypotheses :

H_0 : the particle is new

against H_1 : the particle belongs actually to one of the known families.

Under H_0 we can estimate the value $\widehat{\psi}_0$ of the fundamental state of the new particle and we accept H_0 if $\varepsilon_0 = \sum_{n=1}^{N} \left\| X_n - \vartheta(u_n) \widehat{\psi}_0 \right\| \leq \varepsilon_k$ for $k = 1...p$. But by construct $\forall \psi \in F : \varepsilon_0 \leq \sum_{n=1}^{N} \| X_n - \vartheta(u_n) \psi \| \Rightarrow \varepsilon_0 \leq \varepsilon_k$.

This is the classic delusion of Quantum Physics, which claims that its computations are always checked with great accuracy. It always provides a result, but we do not know if this is all the story.

The classification depends on the choice of the vector space F and the group U. One guesses that more sophisticated experiments, in exotic conditions, lead to account for new phenomena, which is done by enlarging F and U. It has been the path followed by Particles Physics. The classification becomes more complicated, new particles as well as new force fields are introduced. But this raises some issues. The only force fields of which we have a sensible knowledge are the EM and the gravitational fields, which are at the foundation of the concept and of the theory of fields. Enlarge the group U to account for experimental facts and pretend that the extension is the manifestation of "new" force fields is an assumption consistent with the gauge theory, but is not the evidence of the existence of separate, distinct, force fields. Actually the idea of a unified theory of fields leads to the contrary. So we need to clarify the physical meaning of the enlargement of the mathematical representation, which leads to look at the idea of symmetry.

1.7 Symmetries

Symmetries are essential in Particles Physics, and their study requires a consistent mathematical background.

1.7.1 Orbits and classes of conjugacy

Let $x \in E$ be a variable representing some physical property of an object. The value of x is measured with respect to a gauge given by a group G : there is an action ϑ of G on E and $\vartheta(g) : E \to E$ is a map which follows the usual rules. The measure X of a variable $x \in E$ changes with the gauge g as : $X = \vartheta(g)(x)$.

The orbit of $x \in E$ is the set $O(x) = \{\vartheta(g)(x), g \in G\}$. The relation of equivalence on E : $X \sim Y \Leftrightarrow \exists g \in G : Y = \vartheta(g)(X) \Leftrightarrow Y \in O(X)$ defines a partition of E denoted E/G. In a gauge theory the orbits define the physical objects which can be distinguished : the orbits of two distinct physical objects must be disjointed.

We say that there is a symmetry at $x = a$ if, for some values $g \in S \subset G : \vartheta(g)(a) = a$. Obviously the set S is a subgroup of G, called the isotropy subgroup of G with respect to a : $G(a) = \{g \in G : \vartheta(g)(a) = a\}$. It contains at least $g = 1$ so it is never an empty set.

If $x, y \in E$ belong to the same orbit, that is if $\exists g \in G : y = \vartheta(g)(x)$, it is easy to check that the isotropy subgroups of x, y are related : $G(y) = g \cdot G(x) \cdot g^{-1}$. Conjugation is the operation on G : $J : G \to G :: J(g)(h) = g \cdot h \cdot g^{-1}$, it defines a partition of G in classes of conjugacy, denoted G/J. The classes of conjugacy are an essential characteristic of the structure of a group.

From the result above : $O(x) = O(y) \Rightarrow G(y) = J(G(x))$: if $x, y \in E$ belong to the same orbit in E, then $G(x), G(y)$ belong to the same class of conjugacy in G. There is a bijection between the orbits, that is the sets E/G - which define the physical objects which can be distinguished - and the classes of conjugacy G/J of the gauge group. *The power of discrimination of the representation depends on the gauge group.*

One useful tool to study the classes of conjugacy is the character. The character of a geometric representation (E, f) of the group G is the map : $G \to \mathbb{C} :: \chi(g) = Tr(f(g))$. The character depends only on the class of conjugacy of g : $\chi(g) = \chi\left(h \cdot g \cdot h^{-1}\right)$. If $(E_1, f_1), (E_2, f_2)$ are unitary representations of G then the character of the sum of the representations is the sum of the characters, and the character of the tensorial product of the representations is the product of the characters.

1.7.2 Symmetries for elementary particles

In our model elementary particles and force fields are represented through a gauge group U acting on a vector space F by ϑ.

Mathematics tell us that the Lie algebra T_1U is the direct sum of finite dimensional Lie subalgebras $(T_1U)_{j=0}^{p}$ which are also ideals. The first Lie algebra U_0 corresponds to the radical, which can possibly be null, and the others are ideals which, by the exponential, give normal subgroups U_j of U and U is the product $U = U_0 \times U_1 \times U_2 ... \times U_p$. The decomposition is unique, but there is no simple rule to find it, and moreover, *this mathematical feature by itself does not imply that these subgroups U_j correspond to physical symmetries.* Indeed the group U is defined by the observer, adjusted to fit the experimental results, but is somewhat arbitrary and the mathematical decomposition does not necessarily corresponds to a significant physical feature.

In Physics a symmetry is always related to a change of standard : if one goes from 1 to $\vartheta(g)$ one gets the same result. In the model above, the observable is ψ and there is a symmetry if $\vartheta(u)\psi_0 = \psi_0$, which occurs for some types of particles and for some values of u. This feature is, strangely, called a "symmetry breakdown" in Quantum Physics. The set $U(\psi_0) = \{u \in U : \vartheta(u)\psi_0 = \psi_0\}$ is the isotropy subgroup with respect to ψ_0, and it does not necessarily coincides with one of the subgroups U_j. The Lie algebra of $U(\psi_0)$ is a Lie subalgebra of T_1U, and the action of $T \in T_1U(\psi_0)$ is null : $\vartheta'(1)(T)\psi_0 = 0$ which is equivalent to say that the charge of the particles with fundamental state ψ_0 is null with respect to the force fields represented by $U(\psi_0)$. The orbits $O(\psi_0)$ are, from an experimental point of view, at the foundation of the classification of particles. The orbits are not vector spaces and, by definition, the orbits of two distinct types of particles are disjointed sets : if $\exists u, u' \in U : \vartheta(u)\psi_k = \vartheta(u')\psi_l \Rightarrow \psi_l = \vartheta\left(u'^{-1}u\right)\psi_k \Rightarrow O(\psi_l) \equiv O(\psi_k)$. The orbits are in bijective correspondence with the classes of conjugacy of U. So, for a given representation (F, ϑ) of U the possible families of elementary particles are in correspondence with the classes of conjugacy of U, and with the values of the character $\chi(g)$. However the mathematical possibilities do not correspond necessarily to physical realizations (usually there is an infinite number of classes of conjugacy), even if this leads some enthusiast Physicists to predict the existence of new particles.

The values of the charges depend on the units. In a GUT there should be a unified set of units, and the charges are scalars defined up to a fixed quantity. Each particle is characterized by a set of charges $q_1, ...q_p$, different for two distinct types of particles. Some particles share the same value q_j for some type j of charge, this feature leads to define families of particles, and further to distinguish specific force fields. Two particles with fundamental states ψ_k, ψ_l are deemed identical with respect to a field represented by a group S if ψ_k, ψ_l belong to the same orbit : $\exists g \in S : \psi_l = \vartheta(g)\psi_k \Leftrightarrow \psi_l \in O_S(\psi_k)$: their behavior is the same under the action of $u \in S$, they have the same charge and belong to the same family. Conversely the values $u \in S$ can be seen as specific to a force field, and the existence of different families as the evidence of the existence of different force fields, which act independently on particles. Force fields are then organized along the families of particles, that is the partition F/S. We could expect that the partition F/S corresponding to the families of particles with the same charges with respect to S, is coarser than F/U, but actually we have the opposite :

$O_S(\psi_0) = \{\vartheta(u)\psi_0, u \in S\} \subset O_U(\psi_0) = \{\vartheta(u)\psi_0, u \in U\}$

the partition given by S is finer than the partition given by U.

The issue comes from the difference between a GUT, where it is assumed that the group U and the space F are known from the beginning, and the experimental path which proceeds by enlarging step by step the group and the vector space : at each step n one specifies a group U_n and a vector space F_n with respect to which the fields u and the states ψ are estimated. Then a sensible representation accounting for all the fields would be $U = U_1 \times ... \times U_n$ and $F = F_1 \oplus ... \oplus F_n$. But it would be unified in name only, and the difference has a physical meaning : the force fields in the product $U_1 \times ... \times U_n$ appear to be independent, as if it was possible to produce the field U_n without U_1, which is contrary to the experiments done.

To conciliate the assumption of the existence of a unified field, represented by a group U acting on a unique vector space F, and the experimental evidences of the existence of families of particles which have a similar behavior under some value of the field, it is necessary to adjust the model. This can be done, in the general framework, as follows.

We keep the representation of the state of each elementary particle by a single vector ψ_k belonging to a unique vector space F, with the action ϑ of a group U. The representation is unitary, there is a Hermitian scalar product on F, and for each $u \in U$, the matrix $[\vartheta(u)]$ is unitary, then it is diagonalizable :

$[\theta(u)] = [P(u)][D(u)][P(u)]^{-1}$ with $[P(u)]^{-1} = [P(u)]^*$

Let us denote :

$(\lambda_k(u))_{k=1}^p$ the eigen values with their order of multiplicity m_k.

$B_k(u)$ the matrix deduced from $[D(u)]$ by subtituting the eigen values λ_k by 1, and by 0 the other terms

$E_k(u) = [P(u)][B_k(u)][P(u)]^{-1}$

then :

$[\theta(u)] = \sum_k \lambda_k(u)[E_k(u)]$

$[E_k][E_l] = \delta_{kl}I, \sum_k [E_k] = I$

$\vartheta(u)$ is the sum of the orthogonal projections on the eigen subspaces $S_k(u)$.

If for some family the particles have the same behavior with respect to a field represented by a subgroup U_k one can say that the state ψ_0 of these particles belong to a vector space $S_k(u)$ common to the subgroup U_k. Or, equivalently, ϑ is the sum of orthogonal projections on vector subspaces which are associated to a type of charge and a type of field.

Which leads to the following assumptions :

i) There are N types of charges, which can be seen as corresponding to N types of force fields.

ii) Each charge $i = 1..i_n$ of the type n is represented by a vector $\phi_n^i \in F$, the null charge is represented by $\phi_n^{i_n} = 0$

iii) For each particle $\psi_k = \sum_{n=1}^N \phi_n^{i_k}$ with a combination $\left(\phi_1^{i_1}, \phi_2^{i_2}, ...\phi_N^{i_N}\right)$ specific for each particle

iv) $F_n = Span\left(\phi_n^i\right)_{i=1...i_n}$, and the vector spaces F_n are orthogonal to each other : $\forall p, q = 1...N : F_p \perp F_q$ so that, with the orthogonal projection $\pi_n : F \to F_n$ the state of each particle is projected for the field n as : $\pi_n(\psi) = \phi_n^{i_k}$

v) The representation of the field of type n is (F_n, ϑ_n) with the action $\vartheta_n(u)(\psi) = \vartheta(u)\pi_n(\psi)$, which gives an apparent action of the field

$\vartheta'(1)(T)\pi_n(\psi) = \vartheta'(1)(T)\left(\phi_n^{i_k}\right)$.

$\vartheta(u)(\psi_k) = \sum_{n=1}^N \vartheta_n(u)(\psi) = \sum_{n=1}^N \vartheta(u)\pi_n(\psi_k) = \sum_{n=1}^N \vartheta(u)\left(\phi_n^{i_k}\right)$

The elements of U for which F_n is globally invariant by ϑ constitute a subgroup U_n of U, so we can say equivalently that the field of type n is represented by the group U_n. To be more precise :

The group U acts unitarily on F, it transforms, by $\vartheta(u)$ orthonormal basis of F into orthonormal basis. So we can assume that $SU(m) \subset U$. If $\dim F = p, \dim F_n = q$, we take an orthonormal basis $(e_i)_{i=1}^q$ of F_n, completed by $p - q$ vectors to give an orthonormal basis $(e_i)_{i=1}^p$ of F. There is a unique matrix $J \subset SU(m)$ which transforms by ϑ the initial basis $(\varepsilon_i)_{i=1}^p$ into $(e_i)_{i=1}^p$: its columns are just the components of the vectors e_i.

The matrices which keep invariant $(\varepsilon_i)_{i=q+1}^p$ and transform the basis $(\varepsilon_i)_{i=1}^p$ in another orthonormal basis are of the form :

$$[M] = \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix}$$

with any matrix $N \in SU(q)$. So the matrices $[\vartheta(u)] = [J]^{-1}[M][J]$ which leave globally invariant F_n constitute a subgroup of U, isomorphic to $SU(q)$.

The vector spaces F_n are not invariant under the action of the whole of U, the representation (F, ϑ) is irreducible, as it was assumed.

Each apparent force fields corresponds to a "layer" of features, measured in a set of experiments, and associated on one hand to a projection from F to a vector subspace, according to the symmetries and the charges, and on the other hand to the subgroup of U which preserves these symmetries.

The key to a GUT is then, not to try to find the "right" subgroups by a pure mathematical analysis of a group U, but, starting from a group which seems adequate and large enough, to define subgroups, isomorphic to the groups usually associated to the different force fields, from the symmetries found with the charges of the particles. This process is simpler, and actually closer to the physical facts.

1.7.3 The radical and the EM field

In a group G the center is the set, which can be empty, of elements of G which commute with all the elements of G. They belong to all the classes of conjugacy so, in a physical gauge theory, they do not enable us to distinguish different objects if this is done in the same representation, that is with the same vector space F and action ϑ.

In a model based on a representation (F, ϑ) of U this is the radical T_1U_0 of T_1U , which can possibly be null, which has a special role. $T_1U = T_1U_0 \oplus T_1U$ and only $T_1U = T_1U/T_1U_0$ is a semi simple Lie algebra, to which the general results about root spaces decomposition hold. The radical has specific properties : this is a subalgebra and an ideal such that some of its power (by the map ad) is abelian - and the corresponding group is commutative. We will limit ourselves to the case where T_1U_0 is itself abelian, because it is the most common[2]. Then the group U_0 is commutative. There is a comprehensive theory of commutative groups and their representations (Maths.1821,1911).

The exponential map : $\exp : T_1U_0 \to U_0$ is onto and its kernel is a discrete subgroup of U_0. There are $p \leq \dim T_1U_0$ linearly independent vectors T_j of T_1U_0 such that $\ker \exp = \left\{ T = \sum_{j=1}^{p} z_j T_j, z_j \in \mathbb{Z} \right\}$ so that $\vartheta\left(\exp T\right)\psi = \psi$. Because T_1U_0 is an ideal $\forall g \in U$ can be written $g = u_0 \cdot u = \exp T \cdot u$ with $T \in T_1U_0, u \in U$. An experiment which differentiates the particles according to U_0 keeps $u = 1$ and varies $T = T_1, ..., T_N$, the measures are then $\psi_n = \vartheta\left(\exp T_n\right)\psi_0$. The states of the particles show a periodic distribution.

The irreducible finite dimensional representations of an abelian group are necessarily unidimensional : $\vartheta\left(u\right)\psi = \lambda\left(u\right)\psi$ with a fixed vector ψ, and a scalar function $\lambda\left(u\right) = \exp\left(i\varkappa\left(u\right)\right)$ with $\varkappa : U_0 \to \left(\mathbb{R}^p, +\right)$ a morphism. If $p = 1$ then $U_0 = U\left(1\right)$.

We cannot, in a unique representation, account for the fields which are represented by a commutative group such as $U(1)$. However, for $U(1)$, there are 3 possible, distinct, unitary irreducible representations :

[2]Notice that a Cartan algebra is abelian, but not an ideal, which implies that $\forall X \in L_0, Y \in L : [X, Y] \in L_0$

the standard representation : $\psi_k = e^{iT}\psi_{0k}$ which gives the action $\vartheta'(1)(T)\psi_{0k} = iT\psi_k$

the contragredient representation : $\psi_k = e^{-iT}\psi_{0k}$ which gives the action $\vartheta'(1)(T)\psi_{0k} = -iT\psi_k$

the trivial representation : $\psi_k = \psi_{0k}$ which gives the action $\vartheta'(1)(T)\psi_{0k} = 0$

So the solution is practically to keep a single vector space F, associate $U(1)$ to the action $\vartheta_1\left(e^{iT}\right)(\psi_{0k}) = e^{iq_k T}\psi_{0k}$ with $q_k = +1, -1$, or 0, which represents the charge of the particle. There are only 3 possible values for the charge, and the neutral particles are not sensitive to the field.

We get back the usual representation of the EM field by the group $U(1)$ and the scalars are the electric charges. In this model the electric charge is a universal constant. For composite particles the total charge is an integer multiple of the elementary charge.

1.8 The Standard model

The Standard Model is only a part, but an important one, of Quantum Theory of Fields. It is built in a common interpretation of Quantum Physics, but its main features are in line with the picture that we have given previously, and must be integrated in a GUT.

In the Standard Model there are 4 force fields which interact with particles (the gravitational field is not included) :

- the electromagnetic field (EM)
- the weak interactions
- the strong interactions
- the Higgs field

and two classes of elementary particles, fermions and bosons, in distinct families.

1.8.1 Geometry

The Standard Model is based on the Geometry of Special Relativity, and the usual presentations are with the signature (1,3). For some strange reason Particle Physicists measure lengths in space with a definite negative metric. Thus the metric is assumed to be constant. The standard for the observation of the motion is an "inertial frame", and a change of standard is given by the Poincaré group. The Principle of Relativity imposes that the variables are equivariant under the action of this group. The Poincaré group is a semi-group, which entails complications in the computation of derivatives, and the Lorentz group $SO(1,3)$ cannot fully account for the rotational motion, which requires the introduction of the spin as an extra variable.

These limitations in the representation of the changes in the motion, which is essential in the concept of force and momentum, seem alleviated in the common picture of QM : particles have no precise location, any measure of a physical quantity is given by an operator acting on a wave function. But the result of the measure is an eigen value, meaning that it corresponds to a "constant" instantaneous motion. From a physical point of view the model cannot account explicitly for the acceleration. These limitations are not crucial as the model is used essentially to represent discontinuous processes, but they are in view of a GUT.

1.8.2 Fermions

Generations

The matter particles, called fermions, are organized in 3 generations with, for each one, 2 leptons and 2 quarks :

- First generation : quarks up and down; leptons : electron, neutrino.
- Second generation : quarks charm and strange; leptons : muon, muon neutrino
- Third generation : quarks top and bottom; leptons : tau and tau neutrino

Their stability decreases with each generation, the first generation constitutes the usual matter. Each type of particle is called a flavor.

Generation	I	II	III
Quarks			
	u up	c charm	t top
mass (Mev/c^2)	2.44	1.275	172.44
charge	2/3	2/3	2/3
	d down	s strange	b bottom
mass (Mev/c^2)	4.8	95	4.18
charge	-1/3	-1/3	-1/3
Leptons			
	e electron	μ muon	τ tau
mass (Mev/c^2)	0.511	105.67	1.7768
charge	-1	-1	-1
	ν_e electron nutrino	ν_μ muon neutrino	ν_τ tau neutrino
mass (Mev/c^2)	$<2.2.10^{-6}$	<1.7	<15.5
charge	0	0	0

Fermions interact with the force fields, according to their charge, which are

- color (strong interactions) : each type of quark can have one of 3 different colors (blue, green, red) and they are the only fermions which interact with the strong field
- hypercharge (electroweak interaction) : all fermions have an hypercharge (-2,-1,0,1,2) and interact with the weak field
- electric charge (electromagnetic interactions) : except the neutrinos all fermions have an electric charge and interact with the electromagnetic field. However the electric charge for the quarks is somewhat conventional because it is built up from the electroweak charge and they cannot be observed directly.

Each fermion (as it seems also true for the neutrinos) has a mass and interacts also with the gravitational field.

Each fermion has an associated antiparticle, which is represented by conjugation of the particle. In the process the charge changes (color becomes anticolor which are different, hypercharge takes the opposite sign), left handed spinors are exchanged with right handed spinors, but the mass is the same.

Fermions of the same type in different generations differ only by their mass, all the other features, such as the charges, involving the interactions with the fields, are identical.

Over all we have 24 elementary particles, and 24 antiparticles :

3 generations of pairs of leptons : e, ν; leptons of the same flavor have the same electroweak charge ;

3 generations of 6 quarks, differentiated by the flavor (u, d) and the color (r, g, b). Quarks of the same color, flavor and generation have the same mass; quarks of the same flavor have the same electroweak charge.

Spinor part

The spinor comes from the Dirac's equation, which is still a key component of the Standard Model. At the early stages of Relativity Physicists extended simply the definition of the momentum to a 4 dimensional vector : $P = mU$ with the 4 dimensional velocity : $U = \frac{1}{\sqrt{1-\frac{\|v\|^2}{c^2}}}\left(\overrightarrow{v} + c\varepsilon_0\right)$ where ε_0 is the time vector associated to the observer. When

$\|v\| \ll c : E = mc^2 \frac{1}{\sqrt{1-\frac{\|v\|^2}{c^2}}} \simeq \frac{1}{2}m\|v\|^2 + mc^2$

it was then postulated that E represents the "energy" of the particle, mc^2 its "energy at rest". With these conventions the 4 dimensional relativist momentum $P = \frac{1}{\sqrt{1-\frac{\|v\|^2}{c^2}}} m\overrightarrow{v} + \frac{E}{c}\varepsilon_0$ is composed of a "relativist momentum" $p_r = \frac{1}{\sqrt{1-\frac{\|v\|^2}{c^2}}} m\overrightarrow{v}$ and its 4th component is the energy.

In common QM a particle has no precise location, and its state is measured through a wave function ψ. In a relativist picture ψ would follow some equation derived from $E^2 = c^2\|\overrightarrow{p}_r\|^2 + m^2c^4$. Its "Quantization", that is applying the "minimal substitution rule", the operation where mathematical symbols are substituted to other symbols, $E \rightarrow i\hbar\frac{\partial}{\partial t}; p_{r\alpha} \rightarrow -i\hbar\partial_\alpha$ gives the Klein-Gordon equation : $\left(\Box + m^2\right)\psi = 0$ which provides wrong results. Dirac proposed another equation, assuming that : $E = A.p_r + Bm$ the substitution gives : $i\hbar\frac{\partial\psi}{\partial t} = (Ai\hbar\nabla + Bm)\psi$. But one can check that this is possible only if ψ is a vectorial quantity (and no longer a scalar function). Moreover to be Lorentz equivariant A, B must be 4×4 complex matrices, built from a set of matrices $\gamma = (\gamma_j)_{j=0..3}$ meeting the relation : $\gamma_i\gamma_j + \gamma_j\gamma_i = 2\eta_{ij}I_4$ where $[\eta]$ is the matrix of the metric. The wave function $\psi(t, \xi_1, \xi_2, \xi_3)$ is then a vector, called spinor, belonging to a 4 dimensional complex vector space E. The Dirac's equation reads : $i\frac{\partial\psi}{\partial t} = -i\sum_{\alpha=1}^{3}\gamma_\alpha\frac{\partial\psi}{\partial\xi_\alpha} + m\gamma_0\psi$. It is usually written : $\left(i\sum_{\mu=0}^{3}\gamma^\mu\partial_\mu - m\right)\psi = 0$

with $\gamma^\mu = \eta^{\mu\mu} [\gamma_0] [\gamma_\mu], [\gamma_0]^2 = 1$ and can be seen as a propagation equation for ψ or as a substitute for the Schrödinger equation. Its eigen values correspond to the energy. Its eigen vectors, which provide a basis for observable quantities, correspond to "plane waves". The existence of solutions with negative energy leads to antiparticles. The proof of their existence has not closed the issue of the interpretation of these solutions, the most common being that antiparticles are "holes" in a sea of virtual particles, and that they moved backwards in time.

The spin is represented by a quantum number (spin up or down) which is added to the representation.

$\rho = \psi(t,x)^* \psi(t,x)$ gives the probability to find the particle at (t,x). Then the Dirac's currents $j^a = \overline{\psi}^t \gamma_a \psi$ gives the probability to find the particle in $\xi^a, a = 1,2,3$ and the solutions of the Dirac's equation meet the continuity equation : $\frac{\partial \rho}{\partial t} + \sum_{\alpha=1}^3 \frac{\partial j_\alpha}{\partial \xi_\alpha} = 0$

By itself the spinor does not involve any motion of a particle, or of its momentum. As all other observables these quantities are operators, which are obtained from the fields equations by Fourier decomposition or path integrals of the Spinor functions.

It happens that an algebra of 4×4 complex matrices generated by 4 matrices $(\gamma_j)_{j=0}^3$ meeting $\gamma_i \gamma_j + \gamma_j \gamma_i = 2\eta_{ij} I_4$ is a representation of the Clifford algebra $Cl(1,3)$, through its complexified $\mathbb{C} \otimes Cl(1,3)$. The matrices γ usually chosen, with the signature $(1,3)$, are :

$\gamma_0 = \begin{bmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{bmatrix}, \gamma_j = \begin{bmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{bmatrix}$ for $j = 1,2,3$

with the Dirac's matrices. Spinors are then vectors of a 4 dimensional complex vector space E, such that (E, γ) is a complex representation of the Clifford algebra $Cl(3,1)$. Spinors are equivariant under the action of $Spin(3,1)$, and then of the Lorentz group, but the group itself does not play any further role in the model. From a mathematical point of view spinors belong to a vectorial bundle on the Minkovski space.

Each elementary particle is associated to a different spinor ψ_k, which are 4×1 matrices of complex valued functions defined on the Minkovski space.

The spinors do not represent the inertial mass, which are independent variables, with which the spinor should be multiplied.

There is a scalar product on the vector space E, given by the matrix γ_0, which is invariant by $Spin(1,3)$.

Antiparticles

According to the CPT conservation principle in a "time inversion" particles transform in anti-particles, and as a consequence the spinors ψ_k^c of antiparticles are related to the spinor of the particle ψ_k by : $\psi_k^c = -i\gamma_2 \left(\overline{(\psi_k)} \right)$

Chirality

Elementary particles have a feature called "chirality" : they can be left handed or right handed. In the Clifford algebra $Cl(\mathbb{C}, 4)$ there is an element ω such that $\left[\gamma\left(\omega^2\right)\right] = \left[\gamma\left(\omega\right)\right]^2 = I_4$ and it induces a decomposition of any spinor in a left and right part :

$\psi^R = \frac{1}{2}\left(\psi + \gamma\left(\omega\right)\psi\right), \psi^L = \frac{1}{2}\left(\psi - \gamma\left(\omega\right)\psi\right)$ called Dirac's chiral spinors (which are 4×1 matrices)

$E = E_R \oplus E_L$

It appears that the weak field acts only on the left part of the spinors, so that actually in the Standard Model the state of particles are represented through ψ^R, ψ^L, which are 4×1 matrices.

Multiplets

Fermions are regrouped in "multiplets", which are sub-families of particles whose behavior is similar in discontinuous processes. Inside the same generation, electrons and neutrinos can change in each other in weak interactions, meanwhile the quarks (u, d) of different colors stay together in strong interactions. So leptons (e, ν) constitute a doublet, and the quarks u (red,blue,green), d (red,blue,green) constitute 2 triplets. Accounting for chirality, the corresponding mathematical objects (one object for each generation) are :

Singlets :

$\left[\psi_e^R\right]_{4\times 1}, \left[\psi_\nu^R\right]_{4\times 1}$, representing the right part of the electron and neutrino

$\left[\psi_u^R\right]_{4\times 3}, \left[\psi_d^R\right]_{4\times 3}$ representing the right part of the quarks u and d

doublet :

$\begin{bmatrix} \psi_e^L \\ \psi_\nu^L \end{bmatrix}_{8\times 1}$, with $\left[\psi_e^L\right]_{4\times 1}, \left[\psi_\nu^L\right]_{4\times 1}$ representing the left part of the electron and the neutrino

triplets :

$\begin{bmatrix} \psi_u^L \\ \psi_d^L \end{bmatrix}_{12\times 3}$, with $\left[\psi_u^L\right]_{4\times 3}, \left[\psi_d^L\right]_{4\times 3}$ representing the left part of the quarks u and d

1.8.3 Action of the force fields

The force fields act on the state of particles, that is the spinor, through a connection, and the field is represented by the potential. In the Standard Model potentials are bosons, but from a mathematical point of view this does not matter. The group is $U = U(1) \times SU(2) \times SU(3)$, with $U(1)$ associated to the electromagnetic field, $U(1) \times SU(2)$ associated to the electroweak field, and $SU(3)$ to the strong field. The EM field acts on all particles, the electro-weak field acts on doublets of left handed leptons and quarks, the strong field acts on triplets of quarks. However there are several issues, theoretical and practical. If the model is based on a gauge field, there is no principal bundle associated to the groups, (E, γ) is a representation of $Cl(\mathbb{C}, 4)$ and has no explicit relation with U. The fundamental representations of $U(1), SU(2), SU(3)$ which are involved are the standard representations : to each vector of the basis of the Lie algebras is associated a matrix, proportional to the 3 2×2 Dirac's matrices for $SU(2)$ and 8 3×3 Gell Mann matrices for $SU(3)$, which act as such on column matrices. So an adjustment in the dimensions is necessary, done through Kronecker products and cross diagonal matrices.

Moreover there are coupling constant, which are necessary for mathematical reasons (the matrices used are Hermitian and not skew Hermitian as they should be), historical reasons, and also because of the mix EM / weak fields.

The Dirac's operator

The Standard Model does not acknowledge trajectories so the covariant derivative $\nabla_V \psi$ is replaced by the Dirac's operator, which is a differential operator which can be defined in a general mathematical context (Th.Ph.p.260). It sums to take the average value of the covariant derivative along the 4 directions $\partial\xi^\mu$:

$D\psi = \sum_{\mu=0}^{3} \gamma\left(d\xi^\mu\right)\left[\nabla_\mu \psi\right]$

The interactions term are then $\langle \psi, D\psi \rangle = [\psi]^* [\gamma_0] [D\psi]$ with the adjustments necessary to account for the splitting of the vectors in multiplets.

1.8.4 Bosons

The potentials are assumed to behave like particles, and possibly have a mass and a charge, but there is no term to account for them, other than the interaction with the fermions, their propagation, and the Higgs mechanism.

There are :

- 8 gluons linked to the strong interactions : they have no electric charge but each of them carries a color and an anticolor, and are massless. They are their own antiparticles.

- 3 bosons W^j linked with the electroweak field, which carry weak hypercharge and have a mass.

- 1 boson B, specific to the electromagnetic field, which carries a hypercharge and a mass.

- 1 Higgs boson, which has two bonded components, is its own antiparticle and has a mass but no charge or color

The bosons W, B combine to give the photon, the neutral boson Z and the charged bosons $W^\pm$. The photon and Z are their antiparticle, $W^\pm$ are the antiparticle of each other. So in the Standard Model photons are not elementary particles (at least when electroweak interactions are considered).

The propagation of the fields is modeled through the strength of the fields, defined as indicated previously : $\mathcal{F}^a_{\alpha\beta} = \partial_\alpha \grave{A}^a_\beta - \partial_\beta \grave{A}^a_\alpha + \left[\grave{A}_\alpha, \grave{A}_\beta\right]^a$, all variables being valued in the Lie algebras, with coupling constants. There is a scalar product, for 2 forms valued in the Lie algebras, and in the Lagrangian the terms for the propagation of the fields are just $\langle \mathcal{F}, \mathcal{F} \rangle$.

So, meanwhile the boson can be seen as the value of the potential at the location of the particle, the propagation is actually treated in a classic manner, with continuous variables $\grave{A}^a_\beta$ defined everywhere.

1.8.5 Masses and the Higgs mechanism

Because of the introduction of multiplets and the combination of the EM and weak field, the masses of fermions do not appear as multipliers of the spinors as could be expected, but in separate terms of the Lagrangian, modeling the action of the Higgs field, in a complicated way, with additional parameters.

The introduction of the Higgs mechanism, interpreted as a 5th field and the "polarization of the vacuum", is primarily motivated by the mathematical inconsistencies coming from the formulation of the Lagrangian. To respect equivariance the potentials (whatever their denomination) should "factor" in the covariant derivatives. Because it is replaced by the Dirac's operator they are left alone (in the bosons), and the equivariance is broken if the bosons have a mass. Because the spinors are expressed in multiplets, to be equivariant, the masses of the fermions of each multiplet should be equal.

1.8.6 Composite particles

Elementary particles can be combined together to give other particles, which have mass, spin, charge,... and behave as a single particle. Quarks cannot be observed individually and group together to form hadrons : a meson (a quark and an anti-quark with opposite color) or a baryon (3 quarks) : a proton is composed of 3 quarks *udd* and a neutron of 3 quarks *uud*. A particle can transform itself into another one, it can also disintegrate in other particles, and conversely particles can be created in discontinuous processes, notably through collisions. The weak interaction is the only field which can change the flavor in a spontaneous, discontinuous, process, and is responsible for natural radioactivity. The decays of particles can then be studied through the equivalence of tensorial representations into some of simpler representations.

1.8.7 Requirements and assumptions of the GUT model

A GUT model must account for the experimental facts as they appear in the Standard Model. They lead to the following assumptions.

1) We have a n dimensional complex vector space F endowed with a Hermitian scalar product and a unitary representation (F, ϑ) of the m dimensional Lie group U

$(\kappa_a)_{a=1}^m$ a basis of T_1U

a Hermitian scalar product on F, which is preserved by ϑ : $\langle \vartheta(u)\psi_1, \vartheta(u)\psi_2 \rangle = \langle \psi_1, \psi_2 \rangle$

$\Rightarrow \langle \vartheta'(1)(T)\psi, \psi \rangle = -\langle \psi, \vartheta'(1)(T)\psi, \psi \rangle$: we take as Lagrangian

$L_p = \frac{1}{i}\langle \psi, \vartheta'(1)(T)\psi \rangle$

The charges are then : $Q_a(\psi) = \frac{1}{i}\langle \psi, \vartheta'(1)(\kappa_a)(\psi) \rangle = \frac{1}{i}[\psi]^*[\vartheta_a][\psi]$ which implies that the charges are real

2) Antiparticles are represented in the contragredient representation $\left(F, \overline{(\vartheta)}\right)$.

The conjugate $\overline{(\vartheta)}$ of the map ϑ is defined by $\overline{(\vartheta)}(u)(\psi) = \overline{(\vartheta(u))\overline{(\psi)}}$ so this is equivalent to represent the state ψ^c of the antiparticle associated to ψ by its conjugate. Then :

$Q_a(\psi^c) = \frac{1}{i}[\psi^c]^*[\vartheta_a][\psi^c] = \frac{1}{i}[\psi]^t[\vartheta_a]\overline{[\psi]} = \left(\frac{1}{i}[\psi]^t[\vartheta_a]\overline{[\psi]}\right)^t = \frac{1}{i}[\psi]^*[\vartheta_a]^t[\psi] = -\frac{1}{i}[\psi]^*[\vartheta_a][\psi] = -Q_a(\psi)$

because $[\vartheta_a]$ is skew-symmetric.

3) The force fields associated to commutative groups must be treated in a distinct representation. The only identified case is the EM field associated to $U(1)$, the electric charge is a distinct feature, and we have seen above how to deal with it.

4) So, in the core of the model we are concerned only with particles, distinguished by the apparent action of the gravitational, weak and strong fields in 3 layers, along which are organized the known fermions.

In the standard model left and right parts are represented by distinct spinors. We have in one generation $a = 1, 2, 3$:

2 electrons right and left :e^a_R, e^a_L

2 neutrinos right and left : ν^a_R, ν^a_L

6 quarks u right and left : $(u^a_{Rb}, u^a_{Rg}, u^a_{Rr}), (u^a_{Lb}, u^a_{Lg}, u^a_{Lr})$

6 quarks d right and left : $(d^a_{Rb}, d^a_{Rg}, d^a_{Rr}), (d^a_{Lb}, d^a_{Lg}, d^a_{Lr})$

4a) The gravitational field leads to a first projection for $a = 1, 2, 3$

$\pi_G(e^a_L) = \pi_G(e^a_R) = e^a$

$\pi_G(\nu^a_L) = \pi_G(\nu^a_R) = \nu^a$

$\pi_G(u^a_{Rb}) = \pi_G(u^a_{Rg}) = \pi_G(u^a_{Rr}) = \pi_G(u^a_{Lb}) = \pi_G(u^a_{Lg}) = \pi_G(u^a_{Lr}) = u^a$

$\pi_G(d^a_{Rb}) = \pi_G(d^a_{Rg}) = \pi_G(d^a_{Rr}) = \pi_G(d^a_{Lb}) = \pi_G(d^a_{Lg}) = \pi_G(d^a_{Lr}) = d^a$

that is 12 distinct vectors, upon which acts u by $\vartheta(u)\pi_G(\psi)$ and this action is equivalent to $Spin(3,1)$ acting on the vector subspace $F_G = Span\{e^a, \nu^a, u^a, d^a, a = 1, 2, 3\}$

4b) The weak field leads to a second projection

The vectors $e^a_R, \nu^a_R, u^a_{Rb}, u^a_{Rg}, u^a_{Rr}, d^a_{Rb}, d^a_{Rg}, d^a_{Rr}$ are invariant by a subgroup U_W of U

Moreover $(u^a_{Lb}, u^a_{Lg}, u^a_{Lr})^3_{a=1}$ are projected as a vector u_L, $(d^a_{Lb}, d^a_{Lg}, d^a_{Lr})^3_{a=1}$ are projected as a vector d_L

The subgroup U_W of U isomorphic to $SU(2)$ acts on the vectors e^a_L, ν^a_L, u_L, d_L

We can assume that there is a projection :

$\pi_W(e^a_R), \pi_W(\nu^a_R), \pi_W(u^a_{Rb}), \pi_W(u^a_{Rg}), \pi_W(u^a_{Rr}), \pi_W(d^a_{Rb}), \pi_W(d^a_{Rg}), \pi_W(d^a_{Rr}) = 0$

$\pi_W(u^a_{Lb}) = \pi_W(u^a_{Lg}) = \pi_W(u^a_{Lr}) = u_L$

$\pi_W(d^a_{Lb}) = \pi_W(d^a_{Lg}) = \pi_W(d^a_{Lr}) = d_L$

The action of $\vartheta(u)\pi_W$ is equivalent at $SU(2)$

4c) The strong field leads to a third projection.

The vectors $e^a_R, e^a_L, \nu^a_R, \nu^a_L$ are invariant by a subgroup U_S of U

The vectors $u^1_{Rb}, u^2_{Rb}, u^3_{Rb}$ are projected as a vector u_{Rb},the vectors $u^1_{Rg}, u^2_{Rg}, u^3_{Rg}$ are projected as a vector u_{Rg}, the vectors $u^1_{Rr}, u^2_{Rr}, u^3_{Rr}$ are projected as a vector u_{Rr}, the vectors $u^1_{Lb}, u^2_{Lb}, u^3_{Lb}$ are projected as a vector u_{Lb},the vectors $u^1_{Lg}, u^2_{Lg}, u^3_{Lg}$ are projected as a vector u_{Lg}, the vectors $u^1_{Lr}, u^2_{Lr}, u^3_{Lr}$ are projected as a vector u_{Lr},

The vectors $d^1_{Rb}, d^2_{Rb}, d^3_{Rb}$ are projected as a vector d_{Rb},the vectors $d^1_{Rg}, d^2_{Rg}, d^3_{Rg}$ are projected as a vector d_{Rg}, the vectors $d^1_{Rr}, d^2_{Rr}, d^3_{Rr}$ are projected as a vector d_{Rr}, the vectors $d^1_{Lb}, d^2_{Lb}, d^3_{Lb}$ are projected as a vector d_{Lb},the vectors $d^1_{Lg}, d^2_{Lg}, d^3_{Lg}$ are projected as a vector d_{Lg}, the vectors $d^1_{Lr}, d^2_{Lr}, d^3_{Lr}$ are projected as a vector d_{Lr}.

We can assume that there is a projection π_S which accounts for these results. The action of $\vartheta(u)\pi_S$ is equivalent at $SU(3)$

5) All together the states of the 48 particles are defined by :

12 vectors (e^a, ν^a, u^a, d^a) such that

$\pi_G(e^a) = e^a, \pi_G(\nu^a) = \nu^a, \pi_G(u^a) = u^a, \pi_W(d^a) = d^a$

$\pi_W(e^a) = \pi_W(\nu^a) = \pi_W(u^a) = \pi_W(d^a) = 0$

$\pi_S(e^a) = \pi_S(\nu^a) = \pi_S(u^a) = \pi_S(d^a) = 0$

4 vectors (e_L, ν_L, u_L, d_L) such that

$\pi_G(e_L) = e_L, \pi_G(\nu_L) = \nu_L, \pi_G(u_L) = u_L, \pi_W(d_L) = d_L$

$\pi_W(e_L) = \pi_W(\nu_L) = \pi_W(u_L) = \pi_W(d_L) = 0$

$\pi_S(e_L) = \pi_S(\nu_L) = \pi_S(u_L) = \pi_S(d_L) = 0$

4 vectors (e_R, ν_R, u_R, d_R) such that

$\pi_G(e_R) = \pi_G(\nu_R) = \pi_G(u_R) = \pi_W(d_R) = 0$

$\pi_W(e_R) = \pi_W(\nu_R) = \pi_W(u_R) = \pi_W(d_R) = 0$

$\pi_S(e_R) = \pi_S(\nu_R) = \pi_S(u_R) = \pi_S(d_R) = 0$

3 vectors (q_r, q_b, q_r) such that

$\pi_G(q_b) = \pi_G(q_g) = \pi_G(q_r) = 0$

$\pi_W(q_b) = \pi_W(q_g) = \pi_W(q_r) = 0$

$\pi_S(q_b) = q_b, \pi_S(q_g) = q_g, \pi_q(q_r) = q_r$

With orthogonal projections :

$\pi_G : V \to Span(F^a, \nu^a, u^a, d^a) = F_G,$

$\pi_W : V \to Span(F_L, \nu_L, u_L, d_L) = F_L$

$\pi_S : V \to Span(q_r, q_b, q_r) = F_q$

$F_R = Span(F_R, \nu_R, u_R, d_R)$

$F_G \perp F_L, F_G \perp F_q, F_L \perp F_q, F_R \perp F_G, F_R \perp F_L, F_R \perp F_q$

The Left and right hand part, defined in an even dimensional Clifford algebra are orthogonal, so $F_R \perp F_L$.

$(F_G, \vartheta \circ \pi_G)$ is equivalent to an irreducible representation of $Spin(3,1)$

$(F_W, \vartheta \circ \pi_W)$ is equivalent to an irreducible representation of $SU(2)$

$(F_S, \vartheta \circ \pi_S)$ is equivalent to an irreducible representation of $SU(3)$

2 THE GROUP U

The specification of the group U is of course essential. From the previous results we have some hints.

1) The first, and main features are linked to the motion and kinetic energy. We have already a strong model, based on spinors and the Spin group, which come both from the Clifford algebra $Cl(3,1)$. In the Standard Model Spinors - even if this is not explicit - come from a representation of $Cl(1,3)$. The tetrad at each point defines a copy of the Clifford algebra so, as a standard, it has a physical ground.

2) The groups $U(1), SU(2), SU(3)$ involve a complex structure. Any practical Lagrangian is based on a Hermitian product. The representation is unitary. All these elements lead to consider a complex structure, and any real Clifford algebra can be complexified.

3) The action of the weak field depends on the chirality, which is a specific feature defined through the Clifford algebra $Cl(\mathbb{C},4)$.

4) Clifford algebras are rich mathematical structures, which contain themselves many subgroups : indeed $Cl(\mathbb{C},4)$ incorporates obviously $U(1)$ and $SU(2) \sim Spin(3)$, as well as $Spin(3,1)$. A, non exhaustive, list of groups in a Clifford algebra can be found in Shirokov.

5) The groups are a necessary component of the representation, however the fields are themselves valued in the Lie algebra. It is natural to want that, at each point m, the groups and their Lie algebras are defined in the same mathematical object. The Lie algebras of groups defined from a Clifford Algebra belong to the Clifford Algebra : we have a single structure, with many properties, which contains everything.

So our leading assumption is that the force fields can be represented as a group defined on $Cl(\mathbb{C},4)$ acting itself on $Cl(\mathbb{C},4)$.

Clifford algebras have become common with their use in engineering and the conception of software but it is useful to remind their basic properties, and I will introduce some results which are new in Mathematics.

2.1 General properties of Clifford algebras

A Clifford algebra is defined by introducing, in a n dimensional vector space F on a field $K = \mathbb{R}, \mathbb{C}$, endowed with a bilinear *symmetric* form $\langle\rangle$, a new operation : the product between vectors, denoted $\cdot$, with a specific property :

$$\forall u, v \in F : u \cdot v + v \cdot u = 2 \langle u, v \rangle$$

A Clifford algebra is 2^n dimensional, it contains the scalars K, the vectors of F, the product of vectors of F (called homogeneous elements) and the sum of homogeneous elements.

All real Clifford algebras with a form with the same signature (p, q) are isomorphic. All complex Clifford algebras of the same dimension are isomorphic.

$Cl(\mathbb{C},4)$ corresponds to $\mathbb{C}^4$ with the form

$\langle X,Y\rangle = \sum_{k=1}^{4} X_k Y_k \Leftrightarrow \langle\varepsilon_j,\varepsilon_k\rangle = \delta_{jk}$
$Cl\,(3,1)$ corresponds to $\mathbb{R}^4$ with the form
$\langle X,Y\rangle = \sum_{k=0}^{3} X_k Y_k - X_0Y_0 \Leftrightarrow \langle\varepsilon_j,\varepsilon_k\rangle = \eta_{jk}, \eta_{00} = -1$
$Cl\,(1,3)$ corresponds to $\mathbb{R}^4$ with the form
$\langle X,Y\rangle = X_0Y_0 - \sum_{k=0}^{3} X_k Y_k \Leftrightarrow \langle\varepsilon_j,\varepsilon_k\rangle = \eta_{jk}, \eta_{00} = +1, \eta_{jj} = -1$
$Cl\,(3,1), Cl\,(1,3)$ are not isomorphic.

2.1.1 Basis

A Clifford algebra is a 2^n dimensional vector space, and it is convenient to use bases defined by the ordered product of vectors ε_j of an orthonormal basis of F :

$$E_\alpha = E_{i_1\ldots i_p} = \varepsilon_{i_1}\cdot\varepsilon_{i_2}\ldots\varepsilon_{i_p}, i_1 < i_2\ldots < i_p$$

with the vector $E_0 = 1$ to account for the scalars, so that :
$Z = A + \sum Z_{i_1\ldots i_p} E_{i_1\ldots i_p}$
The scalar component A is denoted $\langle Z\rangle = A$

A Clifford algebra is a finite dimensional Banach vector space, and the product $\cdot$ can be expressed as a bilinear map.

It is convenient to use in the algebras $Cl\,(\mathbb{C},4), Cl\,(3,1), Cl\,(1,3)$ the basis :

$Z = a + v_0\varepsilon_0 + v_1\varepsilon_1 + v_2\varepsilon_2 + v_3\varepsilon_3 + w_1\varepsilon_0\cdot\varepsilon_1 + w_2\varepsilon_0\cdot\varepsilon_2 + w_3\varepsilon_0\cdot\varepsilon_3$
$+r_1\varepsilon_3\cdot\varepsilon_2 + r_2\varepsilon_1\cdot\varepsilon_3 + r_3\varepsilon_2\cdot\varepsilon_1$
$+x_0\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3 + x_1\varepsilon_0\cdot\varepsilon_3\cdot\varepsilon_2 + x_2\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_3 + x_3\varepsilon_0\cdot\varepsilon_2\cdot\varepsilon_1 + b\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3$
and to represent a vector by the notations :
$Z = (a, v_0, v, w, r, x_0, x, b)$ in $Cl\,(\mathbb{C},4)$
$Z = [a, v_0, v, w, r, x_0, x, b]$ in $Cl\,(3,1), Cl\,(1,3)$
with the 4 scalars a, v_0, x_0, b and the 4 vectors $v, w, r, x \in \mathbb{C}^3$ or $\mathbb{R}^3$.

2.1.2 Product

From the basic property of the product, for any orthogonal vectors of F : $u\cdot v + v\cdot u = 0$ but the antisymmetry does not hold for other vectors of the Clifford algebra. For the vectors of the orthonormal basis of F :

$$\varepsilon_j\cdot\varepsilon_k + \varepsilon_k\cdot\varepsilon_j = 2\langle\varepsilon_j,\varepsilon_k\rangle = 2\eta_{jk}$$

with the matrix $[\eta]_F$ of the symmetric form on F.

As a consequence the product of 2 vectors of an orthonormal basis of the Clifford algebra is : $E_\alpha\cdot E_\beta = \epsilon(\alpha,\beta)\, E_\gamma$ where E_γ is another vector of the basis, and $\epsilon(\alpha,\beta) = \pm 1$ depends on both α,β and their order (it is usually not antisymmetric).

$Z = X\cdot Y = \sum_{\alpha,\beta} X_\alpha Y_\beta E_\alpha\cdot E_\beta = \sum_\gamma \left(\sum_{\alpha,\beta}\epsilon(\alpha,\beta)\, X_\alpha Y_\beta\right) E_\gamma$

In a Clifford algebras some elements are invertible for the internal product. The set GCl of invertible elements is a Lie group.

To each element of the Clifford algebra is associated a linear map :

$\pi_L : Cl \to \mathcal{L}(Cl; Cl) :: \pi_L(X)(Y) = X \cdot Y$

and in an orthonormal basis $\pi_L(X)$ is represented by a $2^n \times 2^n$ matrix :

$\pi_L(X)[Y] = [X \cdot Y] = \sum_{\alpha\beta} [\pi_L(X)]^\alpha_\beta [Y]^\beta E_\alpha$

The matrices $\pi_L(X)$ are such that each line is a different permutation of the components of X, with a sign ± 1, and similarly for the columns.

The map π_L is an algebra isomorphism : $\pi_L : Cl \to \mathcal{L}(Cl; Cl)$

$\pi_L(X \cdot Y) = \pi_L(X) \circ \pi_L(Y) ; \pi_L(X^{-1}) = [\pi_L(X)]^{-1} ; \pi_L(1) = I_{2^n}$

Similarly we have :

$\pi_R : Cl \to \mathcal{L}(Cl; Cl) :: \pi_R(Y)(X) = X \cdot Y$

$\pi_R(Y)[X] = [X \cdot Y]$

$\pi_R(Y \cdot Z) = \pi_R(Z)\pi_R(Y) ; \pi_R(Y^{-1}) = \pi_R(Y)^{-1}$

$\pi_L(X) \circ \pi_R(X)(Z) = \pi_R(X) \circ \pi_L(X)(Z) = X \cdot Z \cdot X$

The product is then in $Cl(\mathbb{C}, 4)$:

$(a, v_0, v, w, r, x_0, x, b) \cdot (a', v_0', v', w', r', x_0', x', b') = (A, V_0, V, W, R, X_0, X, B)$

$A = aa' + v_0 v_0' + v^t v' - w^t w' - r^t r' - x_0' x_0 - x^t x' + bb'$

$V_0 = av_0' + v_0 a' - v^t w' + w^t v' - r^t x' - x^t r' + x_0 b' - bx_0'$

$V = av' + a'v + v_0 w' - v_0' w + x_0' r + x_0 r' + b'x - bx' + j(v) r' + j(r) v' - j(w) x' + j(x) w'$

$W = aw' + a'w + v_0 v' - v_0' v + b'r + br' + x_0' x - x_0 x' - j(v) x' + j(w) r' + j(r) w' + j(x) v'$

$R = ar' + a'r - x_0' v - x_0 v' + b'w + bw' + v_0' x + v_0 x' - j(v) v' + j(w) w' + j(r) r' + j(x) x'$

$X_0 = ax_0' + a'x_0 + v_0 b' - bv_0' - v^t r' - r^t v' + w^t x' - x^t w'$

$X = ax' + a'x + b'v - bv' - x_0' w + x_0 w' + v_0 r' + v_0' r + j(v) w' - j(w) v' + j(r) x' + j(x) r'$

$B = ab' + a'b + v_0 x_0' - v_0' x_0 + v^t x' - x^t v' - w^t r' - r^t w'$

with the operator $j : \mathbb{C}^3 \to L(\mathbb{C}, 3)$:

$$j(z) = \begin{bmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{bmatrix}$$

which has many algebraic properties and is very convenient in computations. In particular :

$j(x) y = -j(y) x$

$[j(x)]^t = [j(-x)]$

$j(x) j(y) = yx^t - y^t x$

2.1.3 Involutions

Homogeneous elements are elements which can be written as the product of vectors of the underlying vector space $Z = X_1 \cdot X_2 ... \cdot X_p$. On these elements we have 2 operations, which are extended by linearity to the sum of homogeneous elements, that is to the whole of the Clifford algebra..

Graded involution

The principal involution $\imath : Cl \to Cl$ is the extension to the Clifford algebra of the operation on $F : \varepsilon_j \to -\varepsilon_j$, so that the homogeneous elements of rank even do not change sign, and the homogeneous elements of rank odd change sign. The principal involution is an algebra automorphism $(\imath(X \cdot Y) = \imath(X) \cdot \imath(Y))$, it leads to distinguish 2 vector subspaces $\imath(Z) = \pm Z$ and any Clifford algebra is the sum $Cl = Cl_0 \oplus Cl_1$ of a Clifford subalgebra Cl_0 such that $\imath(Z) = Z$ and a vector subspace $Cl_1 : \imath(Z) = -Z$.

In $Cl(\mathbb{C}, 4)$:

$$\imath(a, v_0, v, w, r, x_0, x, b) = (a, -v_0, -v, w, r, -x_0, -x, b)$$

$Cl_0 = \{(a, 0, 0, w, r, 0, 0, b)\}$
$Cl_1 = \{(0, v_0, v, 0, 0, x_0, x, 0)\}$

Transposition

Transposition is the operation which reverses the order of the product : $Z^t = X_p \cdot X_{p-1} ... \cdot X_1 = (-1)^{\frac{1}{2}p(p-1)} X_1 \cdot X_2 ... \cdot X_p$.

It is not an automorphism.

In $Cl(\mathbb{C}, 4)$:

$$(a, v_0, v, w, r, x_0, x, b)^t = (a, v_0, v, -w, -r, -x_0, -x, b)$$

The symmetric elements are $Cl_S = (a, v_0, v, 0, 0, 0, 0, b)$, and the antisymmetric $Cl_A = (0, 0, 0, w, r, x_0, x, 0)$

Transposition acts by a diagonal matrix D_T on the components : $[Z^t] = [D_T][Z]$, from which one deduces a relation between the matrices π_L, π_R :

$$[\pi_R(Y)] = [D_T]\left[\pi_L\left(Y^t\right)\right][D_T]$$

A homogeneous element Z is invertible iff its scalar product $\langle Z, Z\rangle \neq 0$. Its inverse is then :$Z^{-1} = \frac{1}{\langle Z, Z\rangle} Z^t$

Using these 2 involutions one can decompose any Clifford algebra in subspaces of quaternionic type :

$[Cl^s] = \oplus_{k=s(\mathrm{mod}\, 4)} \left\{\imath(Z) = (-1)^s Z; (Z)^t = (-1)^{\frac{1}{2}s(s-1)} Z\right\}, s = 0..4$

Chirality

The ordered product of all the vectors of a basis of $F : Z = \varepsilon_1 \cdot \varepsilon_2 ... \varepsilon_n$, does not depend on the choice of a basis and has specific properties. On $Cl(\mathbb{C}, 4)$ $\omega = \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$ is such that $\omega^2 = 1, \omega = \omega^t$. It decomposes the Clifford algebra in a right and left part $Cl = Cl_R \oplus Cl_L$:

$Cl_R = \left\{Z = \frac{1}{2}(Z + \omega \cdot Z)\right\} = \{Z : \omega \cdot Z = Z\}$
$Cl_L = \left\{Z = \frac{1}{2}(Z - \omega \cdot Z)\right\} = \{Z : \omega \cdot Z = -Z\}$
Cl_R is a sub Clifford algebra and an ideal : $\forall Z \in Cl_R; Z' \in Cl : Z \cdot Z' \in Cl_R$
$[Cl_R, Cl_R] \subset Cl_R, [Cl_L, Cl_L] \subset Cl_R$

But $Z \in Cl_R, Cl_L$ are never invertible : $\omega \cdot g = \epsilon g \Leftrightarrow \omega \cdot g \cdot g^{-1} = \epsilon = \omega$

In $Cl(\mathbb{C}, 4)$:

$$Cl_R = \{Z : (a, v_0, v, w, w, -v_0, -v, a)\} ; Cl_L = \{Z : (a, v_0, v, w, -w, v_0, v, -a)\}$$

2.1.4 Scalar product

There is a scalar product on the Clifford algebra defined by extension from homogeneous elements :

$$\langle X_1 \cdot X_2 ... X_p, Y_1 \cdot Y_2 ... Y_q \rangle = \delta_{pq} \langle X_1, Y_1 \rangle ... \langle X_p, Y_p \rangle$$

such that a basis is orthonormal.

In an orthonormal basis :

$Z = A + \sum Z_{i_1 ... i_p} E_{i_1 ... i_p}$

$\langle Z, Z' \rangle = AA' + \sum Z_{i_1 ... i_p} Z'_{i_1 ... i_p} \left\langle E_{i_1 ... i_p}, E_{i_1 ... i_p} \right\rangle$

In $Cl(\mathbb{C}, 4)$:

$$\langle Z, Z' \rangle = aa' + v_0 v'_0 + v^t v' + w^t w' + r^t r' + x_0 x'_0 + x^t x + bb'$$

In $Cl(1, 3)$:

$$\langle Z, Z' \rangle = aa' + v_0 v'_0 - v^t v' + w^t w' - r^t r' - x_0 x'_0 + x^t x + bb'$$

For homogeneous elements : $\langle Z \cdot Z', Z \cdot Z' \rangle = \langle Z, Z \rangle \langle Z', Z' \rangle$

The scalar component of the product $Z \cdot Z'$ is related to the the scalar product $\langle Z, Z' \rangle$. In any Clifford algebra :

$$\langle X, Y \rangle = \left\langle X^t \cdot Y \right\rangle$$

As a consequence :

$\langle E_\alpha, E_\beta \rangle = \langle E^t_\alpha \cdot E_\beta \rangle = [\eta]^\alpha_\beta$

$\forall X, Y, Z : \langle X \cdot Y, Z \rangle = \langle Y, X^t \cdot Z \rangle , \langle Y \cdot X, Z \rangle = \langle Y, Z \cdot X^t \rangle$

Transpose and the graded involution preserve the scalar product.

$\langle X^t, Y^t \rangle = \langle X, Y \rangle ; \langle \imath(X), \imath(Y) \rangle = \langle X, Y \rangle$

2.1.5 Transpose of matrices

From these results we have a useful relation between the matrix $[\pi_L(X)]$ and its transpose :

$$\begin{gathered} [\pi_L(X^t)] = [\eta] [\pi_L(X)]^t [\eta] \\ [\pi_R(X)]^t = [\eta] [\pi_R(X^t)] [\eta] \\ [\pi_R(X)] = [D_T] [\eta] [\pi_L(X)]^t [\eta] [D_T] \end{gathered}$$

where $[\eta]$ is the diagonal matrix of the scalar product.

Proof. $[X \cdot Y] = \sum_{\alpha\beta} [\pi_L(X)]^\alpha_\beta [Y]^\beta E_\alpha \Rightarrow [X \cdot E_\beta] = \sum_\alpha [\pi_L(X)]^\alpha_\beta E_\alpha \Rightarrow$ $\langle X \cdot E_\beta, E_\alpha \rangle = [\eta]^\alpha_\alpha [\pi_L(X)]^\alpha_\beta = \left\langle X, E_\alpha \cdot E^t_\beta \right\rangle = [D_T]^\beta_\beta \langle X, E_\alpha \cdot E_\beta \rangle$

using $\langle Y \cdot X, Z \rangle = \langle Y, Z \cdot X^t \rangle, E^t_\beta = [D_T]^\beta_\beta E_\beta$

$[\pi_L(X)]^\alpha_\beta = [\eta]^\alpha_\alpha [D_T]^\beta_\beta \langle X, E_\alpha \cdot E_\beta \rangle$

$[\pi_L(X)]^\beta_\alpha = [\eta]^\beta_\beta [D_T]^\alpha_\alpha \langle X, E_\beta \cdot E_\alpha \rangle$

$E_\alpha \cdot E_\beta = \epsilon(\alpha,\beta) E_\gamma$ with a unique γ and $\epsilon(\alpha,\beta) = \pm 1$

$(E_\alpha \cdot E_\beta)^t = E^t_\beta \cdot E^t_\alpha = \epsilon(\alpha,\beta) E^t_\gamma = [D_T]^\beta_\beta [D_T]^\alpha_\alpha E_\beta \cdot E_\alpha = [D_T]^\gamma_\gamma \epsilon(\alpha,\beta) E_\gamma =$ $[D_T]^\gamma_\gamma E_\alpha \cdot E_\beta$

$E_\beta \cdot E_\alpha = [D_T]^\beta_\beta [D_T]^\alpha_\alpha [D_T]^\gamma_\gamma E_\alpha \cdot E_\beta = \epsilon(\beta,\alpha) E_\gamma = [D_T]^\beta_\beta [D_T]^\alpha_\alpha [D_T]^\gamma_\gamma \epsilon(\alpha,\beta) E_\gamma$

$\epsilon(\beta,\alpha) = [D_T]^\beta_\beta [D_T]^\alpha_\alpha [D_T]^\gamma_\gamma \epsilon(\alpha,\beta)$

$[\pi_L(X)]^\beta_\alpha = [\eta]^\beta_\beta [D_T]^\alpha_\alpha [D_T]^\beta_\beta [D_T]^\alpha_\alpha [D_T]^\gamma_\gamma \langle X, E_\alpha \cdot E_\beta \rangle$

$= [\eta]^\beta_\beta [D_T]^\alpha_\alpha [D_T]^\beta_\beta [D_T]^\alpha_\alpha [D_T]^\gamma_\gamma [\eta]^\alpha_\alpha [D_T]^\beta_\beta [\pi_L(X)]^\alpha_\beta$

$= [\eta]^\alpha_\alpha [\eta]^\beta_\beta [D_T]^\gamma_\gamma [\pi_L(X)]^\alpha_\beta$

$[\pi_L(X^t)]^\alpha_\beta = [\eta]^\alpha_\alpha [D_T]^\beta_\beta \langle X^t, E_\alpha \cdot E_\beta \rangle = [\eta]^\alpha_\alpha [D_T]^\beta_\beta \left\langle X, (E_\alpha \cdot E_\beta)^t \right\rangle$

$= [\eta]^\alpha_\alpha [D_T]^\beta_\beta \left\langle X, [D_T]^\gamma_\gamma E_\alpha \cdot E_\beta \right\rangle$ using $\langle X^t, Y^t \rangle = \langle X, Y \rangle$

$[\pi_L(X^t)]^\alpha_\beta = [\eta]^\alpha_\alpha [D_T]^\beta_\beta [D_T]^\gamma_\gamma \langle X, E_\alpha \cdot E_\beta \rangle = [D_T]^\gamma_\gamma [\pi_L(X)]^\alpha_\beta$

$= [\eta]^\alpha_\alpha [\eta]^\beta_\beta [\pi_L(X)]^\beta_\alpha$

that we can write : $[\pi_L(X^t)] = [\eta] [\pi_L(X)]^t [\eta]$

$[\pi_R(X)] = [D_T] [\pi_L(X^t)] [D_T] = [D_T] [\eta] [\pi_L(X)]^t [\eta] [D_T]$

$[\pi_R(X)]^t = [D_T] [\eta] [\pi_L(X)] [\eta] [D_T] = [\eta] [D_T] [\pi_L(X)] [D_T] [\eta] = [\eta] [\pi_R(X^t)] [\eta]$

■

In $Cl(\mathbb{C},4)$:

$$\left[\pi_L\left(Z^t\right)\right] = [\pi_L(Z)]^t ; \left[\pi_R\left(Z^t\right)\right] = [\pi_R(Z)]^t$$

2.1.6 Morphisms

Definition

A morphism between 2 sets endowed with the same algebraic structure is a map Φ which preserves all the operations. For two algebras this is a linear map such that $\Phi(X \cdot Y) = \Phi(X) \cdot \Phi(Y)$. For Clifford algebras, because of the additional relation $2\langle u, v \rangle = u \cdot v + v \cdot u$ morphisms (that we will called Clifford morphisms) require additional properties. We have 3 main theorems :

i) An algebra morphism between the Clifford algebras $Cl(F_1, \langle\rangle_1), Cl(F_2, \langle\rangle_2)$ on the same field is a Clifford morphism iff it preserves the scalar product : $\langle \Phi(Z), \Phi(Z') \rangle_2 = \langle Z, Z' \rangle_1$.

ii) Any linear map $\varphi : F_1 \to F_2$ between vector spaces on the same field which preserves the scalar product can be extended to a Clifford morphism $\Phi : Cl(F_1, \langle\rangle_1) \to Cl(F_2, \langle\rangle_2)$

iii) If a linear map : $\varphi : F_1 \to Cl\left(F_2, \langle\rangle_2\right)$ is such that $\forall u, v \in F_1 : \varphi(u) \cdot \varphi(v) + \varphi(v) \cdot \varphi(u) = 2\langle \varphi(u), \varphi(v)\rangle_2$ then there is a unique morphism $\Phi : Cl\left(F_1, \langle\rangle_1\right) \to Cl\left(F_2, \langle\rangle_2\right)$ such that $\varphi = \Phi \circ \imath$ where $\imath : F_1 \to Cl\left(F_1, \langle\rangle_1\right)$ is the canonical injection.

These results show that all the Clifford algebras, on a vector space of same dimension and same field, endowed with a scalar product of same signature, are isomorphic.

Consider a change of orthonormal basis defined by a linear map φ on a Clifford algebra $Cl(F, \langle\rangle)$, considered as a vector space : $\varphi(E_\alpha) = \widetilde{E}_\alpha$. It will preserve the scalar product : $\forall Z, Z' \in Cl : \langle \varphi(Z), \varphi(Z')\rangle = \langle Z, Z'\rangle$. However we will not have necessarily :

$\forall u, v \in F : \varphi(u) \cdot \varphi(v) + \varphi(v) \cdot \varphi(u) = 2\langle \varphi(u), \varphi(v)\rangle = 2\langle u, v\rangle = u \cdot v + v \cdot u$

φ is not an automorphism. The new basis $\widetilde{E}_\alpha$ does not have all the usual properties. It will happen only if $\varphi(Z) \cdot \varphi(Z') = \varphi(Z \cdot Z')$ and for this it suffices that φ maps vectors of F on vectors of F. Moreover it must maps 1 to 1 (because $\varphi(1)^n = n\varphi(1) = \varphi(1)$). Which gives a special importance to linear maps which preserve both the scalar product and F. They come in particular from the adjoint map.

Adjoint map

The adjoint map :

$$Ad : GCl \to G\mathcal{L}(Cl; Cl) :: Ad_g Z = g \cdot Z \cdot g^{-1}$$

defines a linear action of the group GCl of invertible elements :

$$Ad_{g \cdot g'} = Ad_g \circ Ad_{g'}; Ad_1 = Id$$

and is such that :

$$Ad_g(X \cdot Y) = Ad_g X \cdot Ad_g Y$$

In any basis E_α of the Clifford algebra :

$[Ad_g](E_\alpha) = [Ad_g]\left(\varepsilon_{j_1} \cdot .... \cdot \varepsilon_{j_q}\right) = [Ad_g](\varepsilon_{j_1}) \cdot ... \cdot [Ad_g]\left(\varepsilon_{j_q}\right)$

the map Ad_g is fully defined by its value for the vectors ε_j of F, that is by its value on F. Moreover $Ad_g 1 = 1$.

This is a projective map, in the meaning : $\forall k \neq 0 \in K : Ad_{kg} = Ad_g$

(Cl, Ad) is a representation of the group GCl. So for any group G of a Clifford algebra, by restriction (Cl, Ad) is a representation of G on the Clifford algebra.

Its matrix in an orthonormal basis is :

$$[Ad_g] = [\pi_L(g)]\left[\pi_R\left(g^{-1}\right)\right] = \left[\pi_R\left(g^{-1}\right)\right][\pi_L(g)]$$

and using the previous result :

$$[Ad_g]^t = [\eta][Ad_{g^t}][\eta]$$

Orthogonal group

In a Clifford algebra the adjoint map preserves the scalar product if :

$\langle Ad_g X, Ad_g Y\rangle = \langle X, Y\rangle$

$\langle Ad_g X, Ad_g Y\rangle = [Ad_g X]^t [\eta] [Ad_g Y] = [X]^t [\eta] [Y] \Leftrightarrow [Ad_g]^t [\eta] [Ad_g] = [\eta]$

thus if $[Ad_{g^t}] = \left[Ad_{g^{-1}}\right] \Leftrightarrow g^t \cdot g \in \mathbb{R}$

The orthogonal group of a Clifford algebra is the group :

$$O(Cl) = \left\{g \in Cl : g^t \cdot g = 1\right\}$$

Then the matrix $[Ad_g]$ belongs to the orthogonal group $O(2^n)$.

The Lie algebra of the orthogonal group is given by :

$T_1 O(Cl) = \{T : T^t + T = 0\}$

$g^t \cdot g = 1$ provides relations between the components.

If the adjoint map preserves also the vector space F then, as the adjoint map is defined by its value on F, the group is, up to a complex scalar, isomorphic to $SO(n)$.

Reflection

The adjoint map for the orthogonal group preserves the scalar product, and its matrix is orthogonal and belongs to $SO(2^n)$, however it does not necessarily preserves the vector space F,or is isomorphic to $SO(n)$. This feature requires additional properties.

In any n dimensional real vector space endowed with a non degenerate scalar product (not necessarily definite positive) a reflection of vector $u, \langle u, u\rangle \neq 0$ is the map : $R(u)v = v - 2\frac{\langle u,v\rangle}{\langle u,u\rangle}u$. Its unique eigen vector is u with eigen value -1 and $\det R(u) = (-1)^n$. It preserves the scalar product and, conversely, any orthogonal map can be written as the product of at most n reflections.

In a real Clifford algebra based on a vector space F of dimension n the reflection of vector $u \in F, \langle u, u\rangle \neq 0$ can be written, using $u \cdot v + v \cdot u = 2\langle u, v\rangle, u^{-1} = \frac{1}{\langle u,u\rangle}u$:

$R(u)v = v - 2\frac{\langle u,v\rangle}{\langle u,u\rangle}u = v - (u \cdot v + v \cdot u) \cdot u^{-1} = -Ad_u v \Leftrightarrow Ad_u v = -R(u)v$

The matrix of the restriction of Ad_u to F has for determinant : $\det [Ad_u]_F = (-1)^n \det [R(u)] = 1$. The map Ad_u can be extended to the Clifford algebra, it preserves the scalar product on Cl, thus it is orthogonal and defines an automorphism. More generally $Ad_{u_1 \ldots u_p}$ defines an automorphism.

Conversely a Clifford automorphism $\vartheta \in \mathcal{L}(Cl; Cl)$ must preserve both the scalar product and be globally invariant on F. Its restriction to F is expressed as the product of $p \leq n$ reflections, that is $[Ad_g]_F = (-1)^p [R(u_1)] \ldots [R(u_p)] = \left[Ad_{u_1 \ldots u_p}\right]$. As the map Ad_g is fully defined by its value on F, any automorphism on a Clifford algebra can be expressed as Ad_g where g is the product of at most n vectors of F. If g is the product of at most p vectors, then its components in the basis are at most of order p, and the corresponding Lie algebra is deduced from $T_1 O(Cl) = \{T : T^t + T = 0\}$ by discarding the components T of order higher than p.

Spin group

A set whose elements are the product of an odd number of vectors of F cannot be a group, as we can see with the graded involution :

$\imath(u_1 \cdot ... \cdot u_{2p+1}) = -u_1 \cdot ... \cdot u_{2p+1} \Rightarrow \imath(g \cdot g') = \imath(g) \cdot \imath(g') = g \cdot g'$

A set whose elements are the product of an even number of vectors of F is a group. When each vector is such that $\langle u_k, u_k \rangle = 1$ we have the Spin group. The Spin group is a subgroup of the orthogonal group, Ad_g preserves the scalar product and maps vectors of F on vectors of F. $\det Ad_g = 1$.

$\forall g \in Spin(Cl) : g^t \cdot g = 1$

This relation gives necessary identities between the components.

The groups $Spin(p,q), Spin(q,p)$ are isomorphic.

2.1.7 Lie algebras

Lie algebra

As any algebra a Clifford algebra is a Lie algebra with the bracket

$$[Z, Z'] = Z \cdot Z' - Z' \cdot Z$$

$[Z^t, Z'^t] = -[Z, Z']^t$

$\imath([Z, Z']) = [\imath(Z), \imath(Z')]$

A Clifford algebra is the Lie algebra of its invertible elements : $Cl = T_1 GCl$.

The map $ad(Z) : Cl \to Cl :: ad(Z)(Z') = [Z, Z']$ is linear and represented in matrix by

$$[ad(Z)] = \pi_L(Z) - \pi_R(Z)$$

The radical is the center Z_{Cl}, composed of the scalars if n is even, of the scalars and the multiple of the volume element ω if n is odd. The quotient Cl/Z_{Cl} is then a semi-simple Lie algebra.

In $Cl(\mathbb{C}, 4)$:

$[Z, Z'] = (A, V_0, V, W, R, X_0, X, B)$

$A = 0$

$\frac{1}{2}V_0 = -v^t w' + w^t v' + x_0 b' - b x_0'$

$\frac{1}{2}V = v_0 w' - v_0' w + b' x - b x' + j(v) r' + j(r) v'$

$\frac{1}{2}W = v_0 v' - v_0' v + x_0' x - x_0 x' + j(w) r' + j(r) w'$

$\frac{1}{2}R = -j(v) v' + j(w) w' + j(r) r' + j(x) x'$

$\frac{1}{2}X_0 = v_0 b' - b v_0' + w^t x' - x^t w'$

$\frac{1}{2}X = b' v - b v' - x_0' w + x_0 w' + j(r) x' + j(x) r'$

$\frac{1}{2}B = v_0 x_0' - v_0' x_0 + v^t x' - x^t v'$

Lie subalgebras

In $Cl(\mathbb{C}, 4), Cl(3,1)$: are Lie subalgebras

The Lie algebra : $\{0, 0, 0, W, R, 0, 0, 0\} = \imath(Z) = Z; (Z)^t = -Z$

The Lie algebra : $\{A, 0, 0, W, R, 0, 0, B\}$

The Lie algebra : $\{0, 0, 0, W, R, X_0, X, 0\}$

The Lie algebra : $\{(A, V_0, V, W, W, -V_0, -V, A)\}$

See more in Shirokov.

Left invariant vector fields

The tangent vector space to the Clifford algebra, as a manifold, is Cl itself. The tangent $T(\tau) \in Cl$ to a path : $Z : [0,\infty] \to Cl :: Z(\tau)$ is $T(\theta) = \frac{dZ}{d\tau}|_{\tau=\theta}$. It is left invariant if $T(\tau) = L'_Z 1(T(0)) = Z(\tau) \cdot T(0)$ which gives the differential equation :

$\frac{dZ}{d\tau} = Z(\tau) \cdot T(0), Z(0) = T(0)$

Equivalently if $T : Cl \to Cl$ is a vector field, the integral curve going from 1 to a point $Z(\tau) = \Phi_T(\tau, 1)$ is given by :

$\frac{\partial}{\partial\tau}\Phi_T(\tau,1)|_{\tau=\theta} = T(Z(\theta))$

and the vector field is left invariant if $T(Z(\theta)) = Z(\theta) \cdot T(1)$

A Clifford algebra is a Lie algebra, and its left invariant vector fields are then characterized by the differential equation :

$$\frac{dZ}{d\tau} = Z(\tau) \cdot T; Z(\tau) = 1$$

which holds whatever the element T.

Exponential map

The map π_L is an algebra morphism so the exponential in the Clifford algebra has for image the exponential in the algebra of linear maps on a Banach vector space :

$\pi_L(\exp T) = \exp \pi_L(T) \Leftrightarrow [\pi_L(\exp T)] = \sum_{p=0}^{\infty} \frac{1}{p!} [\pi_L(T)]^p = \pi_L\left(\sum_{p=0}^{\infty} \frac{1}{p!} T^p\right)$

$\Leftrightarrow \exp T = \sum_{p=0}^{\infty} \frac{1}{p!} T^p$

The exponential is well defined for any element of the Clifford algebra.

The exponential map is the map : $\exp : Cl \to Cl :: \exp T = \Phi_T(1,1)$ which represents the integral curve of a left invariant vector field T.

$$Z(\tau) = \exp \tau T \Leftrightarrow \frac{dZ}{d\tau} = Z(\tau) \cdot T$$

$Z(\tau)$ is the solution of the differential equation, which reads in coordinates :

$\left[\frac{dZ}{d\tau}\right] = [Z \cdot T] = [\pi_R(T)][Z(\tau)]; Z(0) = 1$

with a fixed matrix $[\pi_R(T)]$ so the solution is given by the exponential of a matrix :

$[Z] = [\exp[\pi_R(T)]][1] = [\exp[D_T][\pi_L(T^t)][D_T]][1] = [1 \cdot \exp T] = [\exp T]$

The exponential map is then defined over all the Clifford algebra and :

$$\exp : Cl \to Cl :: Z = \exp T = \sum_{p=0}^{\infty} \frac{1}{p!} T^p$$

$\frac{d}{d\tau} \exp \tau T = \exp \tau T \cdot T \Leftrightarrow (\exp \tau T)^{-1} \cdot (\exp \tau T) = T$ the left invariant vector fields are given by $Z(\tau) = \exp \tau T$.

The map $T \to \exp T$ is smooth, with derivative $\frac{d}{dT}\exp T|_{T=u} = \exp u$ considered as a linear map from u to $\exp u$, that is :

$\left[\frac{d}{dT}\exp T|_{T=u}\right] = [\pi_L(\exp u)]$

$\det[\pi_L(\exp u)] = \exp Tr(\pi_L(u))$

$Tr(\pi_L(u)) = \sum_\alpha [\pi_L(u)]_\alpha^\alpha = 2^n \langle T\rangle$

$\det[\pi_L(\exp u)] = \exp 2^n \langle T\rangle \neq 0$

thus, according to the constant rank theorem (Maths.1452) exp is a local diffeomorphism on the Clifford algebra (Maths.1452).

The map : $Z(\tau) = \exp(\tau T)$ defines a one parameter group with infinitesimal generator T : $Z(\tau + \tau') = Z(\tau)\cdot Z(\tau')$ and $Z(\tau)^{-1} = Z(-\tau)$.

The inverse map $(\exp)^{-1}$, similar to a logarithm, has for derivative

$[\pi_L(\exp u)]^{-1} = \left[\pi_L\left((\exp u)^{-1}\right)\right] = [\pi_L(\exp(-u))]$.

The set $GCl = \{\exp Z, Z \in Cl\}$ is the group of invertible elements of the Clifford algebra, with the Clifford algebra itself as Lie algebra.

Not all elements of a Clifford algebra can be written as an exponential. Ex : $Z \in Cl_R$ have no inverse because :

$\forall n > 0 : Z^n \in Cl_R$ but $1 \notin Cl_R$ so there is an exponential but $\exp Z \notin Cl_R$.

From the definition :

$\exp(T)^t = (\exp T)^t ; \imath(\exp T) = \exp(\imath(T))$

The exponential has well known general properties (Maths.1751) in particular :

$\forall T \in Cl : \exp(ad(T)) = Ad_{\exp T}$

$g \cdot \exp T \cdot g^{-1} = Ad_g \exp T = \exp(Ad_g T)$

For $Z(\tau) = Ad_{\exp \tau T} X$ with fixed T, X

$$\frac{d}{d\tau}(Ad_{\exp \tau T} X) = Ad_{\exp \tau T}[T, X]$$

Special values of the exponential map

In particular if $Z \cdot Z = \lambda \neq 0 \in \mathbb{C}$:

$\exp T = \sum_{p=0}^\infty \frac{1}{p!}T^p = \sum_{p=0}^\infty \frac{1}{(2p)!}T^{2p} + T\cdot\sum_{p=0}^\infty \frac{1}{(2p+1)!}T^{2p}$

$= \sum_{p=0}^\infty \frac{1}{(2p)!}\lambda^p + T\cdot\sum_{p=0}^\infty \frac{1}{(2p+1)!}\lambda^p$

Let us denote $\lambda = \mu^2$ with any square root μ of λ

$\exp T = \sum_{p=0}^\infty \frac{1}{(2p)!}\mu^{2p} + T\cdot\frac{1}{\mu}\sum_{p=0}^\infty \frac{1}{(2p+1)!}\lambda^{2p+1} = \cosh\mu + \frac{1}{\mu}(\sinh\mu)T$

Let us denote : $\lambda_T = \cosh\mu, \mu_T = \frac{1}{\mu}(\sinh\mu)$

$\lambda_T^2 - \mu_T^2\mu^2 = 1 = \lambda_T^2 - \mu_T^2(T\cdot T)$

that we can write :

$$T.T \in \mathbb{C} \Rightarrow \exp T = \cosh\mu + \frac{1}{\mu}(\sinh\mu)T; \mu^2 = T\cdot T$$

$\cosh\mu, \frac{1}{\mu}(\sinh\mu)$ are always real.

If $\lambda \in \mathbb{R}$:

$\lambda > 0 : \exp Z = \cosh\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\left(\sinh\sqrt{\lambda}\right)Z$

$\lambda < 0 : \exp Z = \cos\sqrt{-\lambda} + \frac{1}{\sqrt{-\lambda}}\left(\sin\sqrt{-\lambda}\right)Z$

$\lambda = 0 : \exp Z = 1 + Z$
$(\exp T)^{-1} = \exp(-T) = \cosh \mu + \frac{1}{\mu}(\sinh \mu) T$

Killing form

The Killing form is the bilinear map

$$B(Z, Z') = Tr\left(ad(Z) \circ ad(Z')\right)$$

It is preserved by all automorphisms on the Lie algebra.

Moreover :

$$B(X, [Y, Z]) = B([X, Y], Z)$$

The Killing form is degenerate : it is null on the radical, and non degenerate on Cl/rad.

In $Cl(\mathbb{C}, 4)$:

$B(Z, Z') = 32\left(v_0 v_0' + v^t v' - w^t w' - r^t r' - x_0 x_0' - x^t x' + bb'\right) = 32\left(\langle Z^t, Z'\rangle - aa'\right)$

2.1.8 Representation of Clifford algebras

Like any Lie algebra a Clifford algebra has the geometric representation on itself with the adjoint map : (Cl, ad).

Like any finite dimensional Lie algebra, Clifford algebras can be represented on algebras of matrices. Their dimension and field depend on the field, signature and dimension of the underlying vector space F. A fundamental representation (A, γ) of a Clifford algebra is then defined by a set of generators $\gamma_j = \gamma(\varepsilon_j)$ which are invertible and meet the conditions :

$\gamma_j \gamma_k + \gamma_k \gamma_j = 2\langle \varepsilon_j, \varepsilon_k\rangle I$

For any element Z of the Clifford algebra $\gamma(Z)$ is then deduced by linear combinations and products of the generators.

Each set of generators defines a faithful irreducible representation. Any set of matrices deduced by conjugation with a fixed matrix defines an equivalent representation. So that all the equivalent representations are given by a change of basis on the Clifford algebra :

$g \in Spin : \varepsilon_j \to \widetilde{\varepsilon}_j = Ad_g \varepsilon_i$
$E_\alpha \to \widetilde{E}_\alpha = Ad_g E_\alpha$
$\gamma_j \to \widetilde{\gamma}_j = \gamma(g) \gamma_j \gamma(g)^{-1}$
$\gamma(Z) \to \widetilde{\gamma}(Z) = \gamma(g) \gamma(Z) \gamma(g)^{-1}$

Complex Clifford algebras are represented on algebras of complex matrices :

$Cl(\mathbb{C}, 2n)$ on $L(\mathbb{C}, 2^n)$

$Cl(\mathbb{C}, 2n+1)$ on $L(\mathbb{C}, 2^n) \times L(\mathbb{C}, 2^n)$ or on $L\left(\mathbb{C}, 2^{n+1}\right)$ with 2 blocs $2^n \times 2^n$ matrices in diagonal.

$Cl(\mathbb{C}, 4)$ is represented on the algebra $L(\mathbb{C}, 4)$ of 4×4 matrices.

Real Clifford algebras are represented on algebras of complex, real or quaternionic matrices depending on the size and signature : . $Cl(3,1)$ is represented on 4×4 real matrices,$Cl\left(1,3\right)$ is represented on pairs of 2×2 quaternionc matrices.

An algebraic representation (A,γ) defines a geometric representation (E,ϑ) on any vector space E on the same field, with the adequate dimension : with any basis of E, the matrices $\gamma\left(Z\right)$ represent endomorphisms $\vartheta\left(g\right)$.

If γ_j is a set of generators, the conjugates γ_j^* or the transpose γ_j^t define another set of generators, and the contragredient representation (A,γ^*) which is usually not equivalent to (A,γ) .

Equivalence with the adjoint representation of $SL(\mathbb{C},n)$

$Cl\left(\mathbb{C},2n\right)$ has a faithful representation on the algebra $L\left(\mathbb{C},2^n\right)$ of $2^n\times 2^n$ complex matrices, which is itself the Lie algebra of the complex invertible matrices $GL\left(\mathbb{C},2^n\right)$. $L\left(\mathbb{C},2^n\right)$ is not semi-simple, its radical Z_L is the scalar matrices and $sl\left(\mathbb{C},2^n\right)=L\left(\mathbb{C},2^n\right)/Z_L$ is the semi simple Lie algebra of invertible matrices with determinant 1, it has the dimension $2^{2n}-1$ and the rank 2^n-1. So $Cl\left(\mathbb{C},2n\right)/\mathbb{C}$ is a semi simple Lie algebra of rank 2^n-1.

Let us consider the representation (T_1G, Ad) of a group G belonging to the Clifford algebra $Cl\left(\mathbb{C},2n\right)$ on the Clifford algebra itself with the adjoint map. The Lie algebra $T_1G \subset Cl\left(\mathbb{C},2n\right)$

Let us consider the action :

$\Theta : G \to \mathcal{L}\left(L\left(\mathbb{C},2^n\right);L\left(\mathbb{C},2^n\right)\right) :: \Theta\left(g\right)\left(M\right)=\left[\gamma\left(g\right)\right]\left[M\right]\left[\gamma\left(g\right)\right]^{-1}$

$\Theta\left(g\cdot g'\right)\left(M\right)=\left[\gamma\left(g\cdot g'\right)\right]\left[M\right]\left[\gamma\left(g\cdot g'\right)\right]^{-1}=\Theta\left(g\right)\circ\Theta\left(g'\right)\left(M\right)$

$\forall\left[M\right]\in L\left(\mathbb{C},2^n\right), \exists Z\in L\left(\mathbb{C},2^n\right):\left[M\right]=\left[\gamma\left(Z\right)\right]$

$\Theta\left(g\right)\left(\gamma\left(Z\right)\right) = \left[\gamma\left(g\right)\right]\left[\gamma\left(Z\right)\right]\left[\gamma\left(g\right)\right]^{-1} = \left[\gamma\left(g\cdot Z\cdot g^{-1}\right)\right] = \left[\gamma\left(Ad_gZ\right)\right] \Leftrightarrow \Theta\left(g\right)\circ\gamma=\gamma\circ Ad_g \Leftrightarrow \Theta\left(g\right)=\gamma\circ Ad_g\circ\gamma^{-1}$

We have the commuting diagram :

$$\begin{array}{ccccc}
Cl\left(\mathbb{C},2n\right) & & Ad_g & & Cl\left(\mathbb{C},2n\right) \\
Z & \to & \to & \to & Ad_g\left(Z\right) \\
\downarrow & & & & \downarrow \\
\gamma & & & & \gamma \\
\downarrow & & & & \downarrow \\
\gamma\left(Z\right) & \to & \to & \to & \Theta\left(g\right)\left(\gamma\left(Z\right)\right) \\
L\left(\mathbb{C},2^n\right) & & \Theta\left(g\right) & & L\left(\mathbb{C},2^n\right)
\end{array}$$

The representation $\left(Cl\left(\mathbb{C},2n\right),Ad\right)$ of G is equivalent to the representation $\left(L\left(\mathbb{C},2^n\right),\Theta\right)$ of G by $\Theta\left(g\right)=\gamma\circ Ad_g\circ\gamma^{-1}$ and the morphism is an isomorphism because γ is bijective. The action Θ is just the adjoint action on matrices and the representation $\left(L\left(\mathbb{C},2^n\right),\Theta\right)$ of G is a subrepresentation of the adjoint representation $\left(L\left(\mathbb{C},2^n\right),\Theta\right)$ of $GL\left(\mathbb{C},2n\right)$, as $\left(Cl\left(\mathbb{C},2n\right),Ad\right)$ is a subrepresentation of the group $GCl\left(\mathbb{C},2n\right)$ of invertible elements of $Cl\left(\mathbb{C},2n\right)$.

The 2^n matrices $\gamma\left(E_\alpha\right)$ are linearly independent because E_α are independent, thus they constitute a basis of $L\left(\mathbb{C},2^n\right)$. In this basis the matrix of $\Theta\left(g\right)$ is the same as Ad_g in the orthonormal basis of $Cl\left(\mathbb{C},2n\right)$:

$\Theta\left(g\right)\left(M\right)=\Theta\left(g\right)\left(\sum_\alpha\kappa^\alpha\left[\gamma\left(E_\alpha\right)\right]\right)=\sum_\alpha\kappa^\alpha\left[\gamma\left(g\right)\right]\left[\gamma\left(E_\alpha\right)\right]\left[\gamma\left(g\right)\right]^{-1}$

$=\sum_\alpha\kappa^\alpha\left[\gamma\left(Ad_g\left(E_\alpha\right)\right)\right]=\sum_\alpha\kappa^\alpha\left[\gamma\left(\sum_\beta\left[Ad_g\right]_\alpha^\beta E_\beta\right)\right]$

$= \sum_{\alpha,\beta} [Ad_g]^{\beta}_{\alpha} \kappa^{\alpha}\gamma (E_{\beta})$

Whenever the group G is defined by a condition on the matrix Ad_g the same condition applies on the representation $(L(\mathbb{C}, 2^n), \Theta)$.

The map γ depends on a choice of generators but it is faithful. To each $2^n \times 2^n$ matrix representing $[\Theta(g)]$ corresponds a unique matrix Ad_g and thus a unique g, up to the product by a constant.

Cartan algebras

These results are useful to find the Cartan algebras of a Clifford algebra.

$Cl(\mathbb{C}, 2n)/Z_C$ is a semi simple, complex Lie algebra. A Cartan algebra can be obtained through the equivalence of the representations $(Cl(\mathbb{C}, 2n), Ad)$ of $GCl(\mathbb{C}, 2n)$ and $(L(\mathbb{C}, 2^n), \Theta)$ of $SL(\mathbb{C}, 2^n)$, whose derivative is the adjoint representation $(sl(\mathbb{C}, 2^n), ad)$ of $sl(\mathbb{C}, 2^n)$.

The root spaces decomposition of the representation $(sl(\mathbb{C}, 2^n), ad)$ is based on the Cartan algebra of diagonal matrices, and the Cartan algebra of $Cl(\mathbb{C}, 2n)$ is given by the $2^n - 1$ elements E_α of the basis which are represented by diagonal matrices.

2.2 Identification of the group U

The basic assumption is that the group U belongs to $Cl(\mathbb{C}, 4)$ and acts on $Cl(\mathbb{C}, 4)$ in a representation $(Cl(\mathbb{C}, 4), Ad)$. The only mathematical structure which has a physical meaning is the tetrad, and from there the Clifford algebra $Cl(3,1)$, and we need to define a way to go from $Cl(3,1)$ to $Cl(\mathbb{C}, 4)$ which has also a physical meaning. By doing this we will define a real structure on $Cl(\mathbb{C}, 4)$, different from the usual one, and from there a Hermitian scalar product. The group U is then the real part of the unitary group acting by the adjoint map on $Cl(\mathbb{C}, 4)$. The main tool to identify the group is by looking at the structures given by the set of vectors $(e^a, \nu^a, u^a, d^a), (e_L, \nu_L, u_L, d_L), (e_R, \nu_R, u_R, d_R),$ (q_r, q_b, q_g), that is invariant vector spaces and orthogonal projections.

2.2.1 Morphism from $Cl(3,1)$ to $Cl(\mathbb{C}, 4)$

One of the main features of $Cl(3,1)$ is the special role of the time vector ε_0.. In $Cl(\mathbb{C}, 4)$ all the vectors $(\varepsilon_j)_{j=0}^{3}$ are on the same footing. In order to keep this physical feature in $Cl(\mathbb{C}, 4)$ we define a morphism, based on ε_0.

Definition

The map C, from a real 4 dimensional vector space F endowed with a bilinear form of signature $(3,1)$ to $\mathbb{C}^4$ endowed with its canonical bilinear form, defined by :

$C : F \to \mathbb{C}^4 :: C(\varepsilon_0) = i\varepsilon_0, j = 1, 2, 3 : C(\varepsilon_j) = \varepsilon_j$

can be extended to a Clifford morphism (injective but not onto) :

$$C : Cl(3,1) \to Cl(\mathbb{C}, 4) :: C([a, v_0, v, w, r, x_0, x, b]) = (a, iv_0, v, iw, r, x_0, ix, ib)$$

On the left hand side the components[3] $[a, v_0, v, w, r, x_0, x, b]$ which are necessarily real, are in the orthonormal basis of $Cl(3,1)$, and on the right hand side the components $(a, iv_0, v, iw, r, x_0, ix, ib)$ are expressed in the orthonormal basis of $Cl(\mathbb{C},4)$ and are either real or pure imaginary.

There is a similar procedure for the signature $(1,3)$: $C' : F \to \mathbb{C}^4 :: C(\varepsilon_0) = \varepsilon_0, j = 1,2,3 : C(\varepsilon_j) = i\varepsilon_j$

The proof, based on the universal property of Clifford algebras, holds for any dimension or signature, and can be found in Maths.750.

The image

$$Cl_R = \{(a, iv_0, v, iw, r, x_0, ix, ib), a, v_0, v, w, r, x_0, x, b \in \mathbb{R}\} \subset Cl(\mathbb{C},4)$$

is a real Clifford subalgebra of $Cl(\mathbb{C},4)$, Clifford isomorphic to $Cl(3,1)$ and C is a real Clifford morphism :

$$\begin{gathered}\forall \alpha, \beta \in \mathbb{R} : C(\alpha Z + \beta Z') = \alpha C(Z) + \beta C(Z') \\ C(Z \cdot Z') = C(Z) \cdot C(Z') \\ C([X,Y]) = [C(X), C(Y)] \\ C(\exp Z) = \exp C(Z)\end{gathered}$$

As a consequence, if $L \subset Cl(3,1)$ is a Lie algebra, then $C(L)$ is a real Lie algebra in $Cl(\mathbb{C},4)$, and if $G \subset Cl(3,1)$ is a group then $C(G)$ is a group in $Cl(\mathbb{C},4)$.

C commutes with transposition :

$$C(Z)^t = C(Z^t)$$

C preserves the scalar product in the meaning :

$$\langle C(Z), C(Z') \rangle_{Cl(\mathbb{C},4)} = \langle Z, Z' \rangle_{Cl(3,1)}$$

The matrix of the map $C : Cl(3,1) \to Cl(\mathbb{C},4) :: [C(Z)] = [C][Z]$ is a diagonal 16×16 matrix which is a "square root" of the matrix $[\eta]$ of the scalar product on $Cl(3,1)$: $[C]^2 = [\eta], [C]\overline{[C]} = I, \overline{[C]} = [\eta][C]$

Real structure on $Cl(\mathbb{C},4)$

The Clifford algebra $Cl(\mathbb{C},4)$ splits, as a vector space, in two real vector subspaces :

$Cl(\mathbb{C},4) = Cl_R \oplus iCl_R$ with $Cl_R = C(Cl(3,1))$

which are isomorphic to $Cl(3,1)$. $Cl_R \oplus iCl_R$ is a real form of $Cl(\mathbb{C},4)$.

The 16 dimensional complex vector space $Cl(\mathbb{C},4)$ becomes a 32 real dimensional vector space. From a mathematical point of view this is a real structure, different but similar to the usual one based on the components in a basis.

[3]We stick to the notation used previoulsy : [] denotes components in Cl(3,1) and () denotes components in Cl(C,4), with the same conventions for a,v,w,...

Any element of $Cl(\mathbb{C},4)$ has a unique decomposition in a real and an imaginary part, in accordance with the specificities of the physical geometry : $Z = \operatorname{Re} Z + i \operatorname{Im} Z$ where

$$\operatorname{Re}(a, v_0, v, w, r, x_0, x, b) = (\operatorname{Re} a, i \operatorname{Im} v_0, \operatorname{Re} v, i \operatorname{Im} w, \operatorname{Re} r, \operatorname{Re} x_0, i \operatorname{Im} x, i \operatorname{Im} b)$$
$$\operatorname{Im}(a, v_0, v, w, r, x_0, x, b) = (\operatorname{Im} a, -i \operatorname{Re} v_0, \operatorname{Im} v, -i \operatorname{Re} w, \operatorname{Im} r, \operatorname{Im} x_0, -i \operatorname{Re} x, -i \operatorname{Re} b)$$

are vectors of $Cl(\mathbb{C},4)$ with complex components with respect to the orthonormal basis of $Cl(\mathbb{C},4)$.

Complex conjugate

A real structure in a complex vector space defines a complex conjugation. The new complex structure defines a new complex conjugation, denoted $CC(Z)$, and we will keep $\overline{[Z]}$ to denote the usual conjugation, based on the components.

The real and imaginary part of a vector $Z \in Cl(\mathbb{C},4)$ in the real structure built from C are then denoted and defined by :

$$\operatorname{Re} Z = \frac{1}{2}\left([Z] + [\eta]\,\overline{[Z]}\right); \operatorname{Im} Z = \frac{1}{2i}\left([Z] - [\eta]\,\overline{[Z]}\right)$$

We have a real bijective map :

$\Im : Cl(3,1) \times Cl(3,1) \to Cl(\mathbb{C},4) :: \Im(X,Y) = C(X) + iC(Y)$

$\Im^{-1} : Cl(\mathbb{C},4) \to Cl(3,1) \times Cl(3,1) :: \Im^{-1}(Z) = \left(C^{-1}(\operatorname{Re} Z), C^{-1}(\operatorname{Im} Z)\right)$

With this new real structure we can define a complex conjugation in $Cl(\mathbb{C},4)$:

$$CC : Cl(\mathbb{C},4) \to Cl(\mathbb{C},4) :: CC(Z) = \operatorname{Re} Z - i \operatorname{Im} Z$$

The operation is antilinear and $CC(CC(Z)) = Z$. This is an involution on the vector space and it commutes with transposition and the graded involution.

$Cl(3,1)_R = \{Z \in Cl(\mathbb{C},4) :: CC(Z) = Z\}$

$CC(Z \cdot Z') = CC(Z) \cdot CC(Z')$

The relation with the usual complex conjugation is : $CC(Z) = [\eta]\,\overline{[Z]}$ with the matrix $[\eta]$ of the scalar product in $Cl(3,1)$.

$$CC(a, v_0, v, w, r, x_0, x, b) = \left(\overline{(a)}, -\overline{(v_0)}, \overline{(v)}, -\overline{(w)}, \overline{(r)}, \overline{(x_0)}, -\overline{(x)}, -\overline{(b)}\right)$$

The adjoint is then defined as : $Z^* = CC(Z^t)$

$$(a, v_0, v, w, r, x_0, x, b)^* = \left(\overline{(a)}, -\overline{(v_0)}, \overline{(v)}, \overline{(w)}, -\overline{(r)}, -\overline{(x_0)}, \overline{(x)}, -\overline{(b)}\right)$$

Real map

The complex conjugate of the linear map $F \in \mathcal{L}(Cl(\mathbb{C},4); Cl(\mathbb{C},4))$ is the map :

$$CC(F) \in \mathcal{L}(Cl(\mathbb{C},4); Cl(\mathbb{C},4)) : CC(F)(Z) = CC(F(CC(Z)))$$

The map is real if $CC(F) = F$: it maps real vectors to real vectors,and imaginary vectors to imaginary vectors. If $CC(F) = -F$ then it inverses the structures. With $F = Ad_g, g \in Cl(\mathbb{C}, 4)$:

$CC(Ad_g)(Z) = CC(Ad_g CC(Z)) = CC\left(g \cdot CC(Z) \cdot g^{-1}\right)$

$= CC(g) \cdot Z \cdot CC\left(g^{-1}\right) = Ad_{CC(g)} Z$

The map Ad_g is real if $g \in Cl_R$ or $g \in iCl_R$, with $Ad_{CC(g)} Z = Ad_{-CC(g)} Z$.

Hermitian scalar product

We define the Hermitian form in $Cl(\mathbb{C}, 4)$:

$$\langle X, Y\rangle_H = \langle CC(X), Y\rangle_{Cl(\mathbb{C},4)} = \left\langle CC\left(X^t\right) \cdot Y\right\rangle$$

It is linear in the second argument and antilinear in the first, it is Hermitian.

$\langle (a, v_0, v, w, r, x_0, x, b), (a', v_0', v', w', r', x_0', x', b')\rangle_H$

$= \overline{(a)}a' - \overline{(v_0)}v_0' + \overline{(v)}^t v' - \overline{(w)}^t w' + \overline{(r)}^t r' + \overline{(x_0)}x_0' - \overline{(x)}^t x' - \overline{(b)}b'$

Because C is a Clifford algebra morphism we still have :

$\forall u, v \in F = Span(\varepsilon_j)_{j=0}^3 : 2\langle u, v\rangle_H = 2\langle CC(u), v\rangle_{Cl} = CC(u) \cdot v + v \cdot CC(u) = CC(u^t) \cdot v + v \cdot CC(u^t) = u^* \cdot v + v \cdot u^*$

$\langle X_1 \cdot X_2 ... X_p, Y_1 \cdot Y_2 ... Y_q\rangle_H = \langle CC(X_1) \cdot CC(X_2) ... CC(X_p), Y_1 \cdot Y_2 ... Y_q\rangle_{Cl} = \delta_{pq}\langle CC(X_1), Y_1\rangle_{Cl} ... \left\langle CC\overline{X_p}, Y_p\right\rangle_{Cl} = \delta_{pq}\langle X_1, Y_1\rangle_H ... \langle X_p, Y_p\rangle_H$

But : $\forall u \in F : u \cdot u^* = (\langle u, u\rangle_H, , 0, 0, 2i \operatorname{Im}(v_0 v), j(\operatorname{Re} v)\operatorname{Im} v)$ so we have $u \cdot u^* = \langle u, u\rangle_H$ only if $u \in Cl_R$ or $u \in iCl_R$.

The matrix of the form in the usual orthonormal basis is $[\eta]$:

$\langle X, Y\rangle_H = [CC(X)]^t [Y] = \left[[\eta]\overline{[X]}\right]^t [Y] = \overline{[X]}^t [\eta][Y]$

$\langle X, Y\rangle_H$ has the signature $(8; 8)$.

$\langle X, Y\rangle_H = \langle \operatorname{Re} X - i \operatorname{Im} X, \operatorname{Re} Y + i \operatorname{Im} Y\rangle_{Cl(\mathbb{C},4)}$

$= \langle \operatorname{Re} X, \operatorname{Re} Y\rangle_{Cl(\mathbb{C},4)} + \langle \operatorname{Im} X, \operatorname{Im} Y\rangle_{Cl(\mathbb{C},4)} - i\langle \operatorname{Im} X, \operatorname{Re} Y\rangle_{Cl(\mathbb{C},4)} + i\langle \operatorname{Re} X, \operatorname{Im} Y\rangle_{Cl(\mathbb{C},4)}$

$\langle X, Y\rangle_H$ is real on $Cl(3,1)_R, iCl(3,1)_R$.

The usual basis $(E_\alpha)_{\alpha=0}^{16}$ is orthonormal : $\langle E_\alpha, E_\beta\rangle_H = \eta_{\alpha\beta} = \pm 1$.

The Hermitian product is preserved by the graded involution and by transpose. It is preserved by a map F if :

$\langle X, Y\rangle_H = \langle F(X), F(Y)\rangle_H = \langle CC(F(X)), F(Y)\rangle_{Cl(\mathbb{C},4)}$

$= \langle CCF(CC(X)), F(Y)\rangle_{Cl(\mathbb{C},4)} = [CC(X)]^t [CCF]^t [F][Y] = [CC(X)]^t [Y]$

$[CCF]^t [F] = I$

With $F = Ad_g$ if $[CC(Ad_g)]^t [Ad_g] = \left[Ad_{CC(g)}\right]^t [Ad_g] = \left[Ad_{CC(g^t)}\right][Ad_g] = \left[Ad_{CC(g^t) \cdot g}\right] = I \Leftrightarrow CC(g^t) \cdot g \in \mathbb{C}$

In particular it will happen if $g = C(s)$ or if $g = iC(s)$ with $s^t \cdot s \in \mathbb{R}$.

We have an extension of the theorem about reflections.

2.2.2 Reflections

On a n dimensional complex vector space F, endowed with a bilinear symmetric form and a real structure, one can define a Hermitian product. A linear map which preserves the Hermitian product is represented by a unitary matrix, with the appropriate signature. Such a map is also an orthogonal map on the $2n$ dimensional real vector space. Indeed $U(n,p,q) \subset O(2n,p,q) \cap GL(\mathbb{C},n)$. Then it can be expressed as the product of at most $2n$ real reflections.

On $Cl(\mathbb{C},4)$ a real reflection is a map :

$R(u) : Cl_R \to Cl_R :: R(u)z = z - 2\frac{\langle u,z\rangle_{Cl}}{\langle u,u\rangle} u$ where u,z are vectors of the real part of $F = Span(\varepsilon_i)_{i=0}^3$

Writing $u = C(u_1), z = C(z_1)$:

$R(u)z = C(z_1) - 2\frac{\langle C(u_1),C(z_1)\rangle_{Cl}}{\langle C(u_1),C(u_1)\rangle_{Cl}} C(u_1) = C(z_1) - 2\frac{\langle u_1,z_1\rangle_{Cl(3,1)}}{\langle u_1,u_1\rangle_{Cl(3,1)}} C(u_1) = C\left(z_1 - 2\frac{\langle u_1,z_1\rangle_{Cl(3,1)}}{\langle u_1,u_1\rangle_{Cl(3,1)}} u_1\right) = C(R(u_1)z_1)$

and :

$R(u_1)z_1 = -Ad_{u_1}z_1$

$R(u)z = -C(Ad_{u_1}z_1) = -Ad_{C(u_1)}C(z_1)$

As $Ad_{ig} \sim Ad_g$ the vectors u can belong to $\operatorname{Re}(F)$ or $i\operatorname{Re}(F)$.

Then $Ad_{u_1...u_p}$ preserves the Hermitian product :

$\left\langle Ad_{u_1...u_p}Z, Ad_{u_1...u_p}Z'\right\rangle_H = \left\langle Ad_{u_1...u_p}CC(Z), Ad_{u_1...u_p}Z'\right\rangle_{Cl(\mathbb{C},4)}$

$= \langle CC(Z), Z'\rangle_{Cl(\mathbb{C},4)} = \langle Z, Z'\rangle_H$

Any map on F can be extended over the Clifford algebra by

$[Ad_g](E_\alpha) = [Ad_g]\left(\varepsilon_{j_1}\cdot .... \cdot \varepsilon_{j_q}\right) = [Ad_g](\varepsilon_{j_1})\cdot ... \cdot [Ad_g]\left(\varepsilon_{j_q}\right)$

So any map which preserves both the Hermitian product and the vector space F is necessarily of the form $Ad_{u_1...u_p}$ where u_j are at most 8 vectors of $\operatorname{Re}(F)$ or $i\operatorname{Re}(F)$.

Then it preserves the vector subspace :

$(0,0,0,w,r,0,0,0)$:

$Z = w_1\varepsilon_0\cdot\varepsilon_1 + w_2\varepsilon_0\cdot\varepsilon_2 + w_3\varepsilon_0\cdot\varepsilon_3 + r_1\varepsilon_3\cdot\varepsilon_2 + r_2\varepsilon_1\cdot\varepsilon_3 + r_3\varepsilon_2\cdot\varepsilon_1$

because

$Ad_g(w_1\varepsilon_0\cdot\varepsilon_1 + w_2\varepsilon_0\cdot\varepsilon_2 + w_3\varepsilon_0\cdot\varepsilon_3 + r_1\varepsilon_3\cdot\varepsilon_2 + r_2\varepsilon_1\cdot\varepsilon_3 + r_3\varepsilon_2\cdot\varepsilon_1)$

$= Ad_g\varepsilon_0\cdot Ad_g(w_1\varepsilon_1 + w_2\varepsilon_2 + w_3\varepsilon_3) + r_1 Ad_g\varepsilon_3\cdot Ad_g\varepsilon_2 + r_2 Ad_g\varepsilon_1\cdot Ad_g\varepsilon_3 + r_3 Ad_g\varepsilon_2\cdot Ad_g\varepsilon_1$

$= \sum_{j=0}^3 [Ad_g]_0^j \varepsilon_j \cdot \left(\sum_{k=0}^3 \left(w_1[Ad_g]_1^k + w_2[Ad_g]_2^k + w_3[Ad_g]_3^k\right)\varepsilon_k\right)$

$+ \sum_{j,k=0}^3 \left(r_1[Ad_g]_3^j[Ad_g]_2^k + r_2[Ad_g]_1^j[Ad_g]_3^k + r_3[Ad_g]_2^j[Ad_g]_1^k\right)\varepsilon_j\cdot\varepsilon_k$

$= \sum_{\{j,k\}=0}^3 \{w_1\left([Ad_g]_0^j[Ad_g]_1^k - [Ad_g]_1^j[Ad_g]_0^k\right) + w_2\left([Ad_g]_0^j[Ad_g]_2^k - [Ad_g]_0^k[Ad_g]_2^j\right) + w_3\left([Ad_g]_0^j[Ad_g]_3^k - [Ad_g]_0^k[Ad_g]_3^j\right)$

$+ r_1\left([Ad_g]_3^j[Ad_g]_2^k - [Ad_g]_2^j[Ad_g]_3^k\right) + r_2\left([Ad_g]_1^j[Ad_g]_3^k - [Ad_g]_3^k[Ad_g]_1^j\right) + r_3\left([Ad_g]_2^j[Ad_g]_1^k - [Ad_g]_1^k[Ad_g]_2^j\right)\}\varepsilon_j\cdot\varepsilon_k$

$= W_1\varepsilon_0\cdot\varepsilon_1 + W_2\varepsilon_0\cdot\varepsilon_2 + W_3\varepsilon_0\cdot\varepsilon_3 + R_1\varepsilon_3\cdot\varepsilon_2 + R_2\varepsilon_1\cdot\varepsilon_3 + R_3\varepsilon_2\cdot\varepsilon_1$

2.2.3 Unitary group

The sets G of vectors of $Cl(\mathbb{C},4)$ which can be written as the product of p vectors of $F = Span(\varepsilon_j)_{j=0}^3$ take the form :

- for $p=1$ they constitute a group only if

$G=\left\{\lambda\left(0,v_0,v,0,0,0,0,0,0,0\right),\lambda\in\mathbb{C},v_0^2+v^tv=0\right\}$ with a fixed vector

- for $p=3$ $G=\left\{\left(0,v_0,v,0,0,x_0,x,0\right),v_0^2+v^tv+x_0^2+x^tx=1,v_0x_0+v^tx=0\right\}$ but they never constitute a group if p is odd as can be checked with the graded involution.

- for p even we have a group, which is different from the Spin group $Spin\left(\mathbb{C},4\right)$, for which the vectors must be such that $\langle u,u\rangle_{Cl}=1$.

Definition

For any $g\in Cl\left(\mathbb{C},4\right):\langle g,g\rangle_H=\langle CC\left(g\right),g\rangle_{Cl(\mathbb{C},4)}=\langle CC\left(g^t\right)\cdot g\rangle=CC\left(g^t\right)\cdot g$ and we define the unitary group by :

$$U=\left\{g\in Cl\left(\mathbb{C},4\right):CC\left(g^t\right)\cdot g=\langle g,g\rangle_H\in\mathbb{C}\right\}$$

It can equivalently be defined as the direct product : $U=\mathbb{C}\times U_0$ where U_0 is the Lie group :

$$U=\mathbb{C}\times U_0:U_0=\left\{g\in Cl\left(\mathbb{C},4\right):CC\left(g^t\right)\cdot g=1\right\}$$

whose Lie algebra is :

$$T_1U_0=\left\{T\in Cl\left(\mathbb{C},4\right):CC\left(T^t\right)+T=0\right\}$$

$$T_1U_0=\left\{\left(iA,V_0,iV,iW,R,X_0,iX,B\right),A,V_0,V,W,R,X_0,X,B\in\mathbb{R}\right\}$$

U_0 is a 16 real dimensional Lie group and U is a 18 real dimensional Lie group whose Lie algebra is the direct product $\mathbb{C}\times T_1U$ with bracket : $[z.T,z\cdot T']=[[z,z']\cdot[T,T']]=0$.

Properties

Its adjoint map preserves the Hermitian product.

The inverse reads :

$$g\in U:g^{-1}=CC\left(g^t\right)/\langle g,g\rangle_H=\left(\overline{(a)},-\overline{(v_0)},\overline{(v)},\overline{(w)},-\overline{(r)},-\overline{(x_0)},\overline{(x)},-\overline{(b)}\right)/\langle g,g\rangle_H$$

The computation of $CC\left(g^t\right)\cdot g$ give necessary relations between the components :

$$U=\{g=(a,v_0,v,w,r,x_0,x,b)\in Cl(\mathbb{C},4):CC(g^t)\cdot g=\langle g,g\rangle_H\}$$
$$\operatorname{Im}(av_0+bx_0+v^tw+x^tr)=0$$
$$\operatorname{Im}(ax_0+bv_0+v^tr+x^tw)=0$$
$$\operatorname{Im}(ab+v_0x_0+v^tx+w^tr)=0$$
$$\operatorname{Im}(x_0v+bw+ar+v_0x)+j(\operatorname{Re}v)\operatorname{Im}v-j(\operatorname{Re}w)\operatorname{Im}w+j(\operatorname{Re}r)\operatorname{Im}r-j(\operatorname{Re}x)\operatorname{Im}x=0$$
$$\operatorname{Re}(av+v_0w-x_0r-bx+j(x)w-j(v)r)=0$$
$$\operatorname{Re}(v_0v+aw-br-x_0x+j(r)w-j(v)x)=0$$
$$\operatorname{Re}(-bv+x_0w-v_0r+ax+j(v)w-j(x)r)=0$$
$$a\overline{(a)}-v_0\overline{(v_0)}+v^t\overline{(v)}-w^t\overline{(w)}+r^t\overline{(r)}+x_0\overline{(x_0)}-x^t\overline{(x)}-b\overline{(b)}=\langle g,g\rangle_H$$

The only additional condition for U_0 is $\langle g,g\rangle_H=1$.

Real part

The elements of U have a real and an imaginary part. Only the real part is a subgroup.

$g\in\operatorname{Re}U\subset Cl_R, g=ks, k\in\mathbb{C}, s\in U_0\Leftrightarrow CC(g)=CC(k)\,CC(s)$

either : $k\in\mathbb{R}, s\in\operatorname{Re}(U_0)$ or $k\in i\mathbb{R}, s\in\operatorname{Im}(U_0)$ but only $\operatorname{Re}(U_0)$ is a group so $\operatorname{Re}U$ can be equivalently defined as the product of the groups $\mathbb{R}\times\operatorname{Re}(U_0)$.

The real part of $g\in U$ reads :

$$\operatorname{Re}U=\{g=(a,iv_0,v,iw,r,x_0,ix,ib)\in Cl_R:CC(g)=g,g^t\cdot g=\langle g,g\rangle_H\}$$
$$av_0+bx_0+v^tw+x^tr=0$$
$$ab+v_0x_0+v^tx+w^tr=0$$
$$av-v_0w-x_0r+bx-j(x)w-j(v)r=0$$
$$a^2-v_0^2+v^tv-w^tw+r^tr+x_0x_0-x^tx-b^2=\langle g,g\rangle_H=\langle g,g\rangle_{Cl}$$

Adjoint map

Using the identity $CC(g^t)/\langle g,g\rangle_H=g^{-1}$ one can compute the adjoint map.

For $g\in Cl_R\Leftrightarrow g=(a,iv_0,v,iw,r,x_0,ix,ib), a,v_0,x_0,b\in\mathbb{R}, v,w,r,x\in\mathbb{R}^3$ the adjoint map reads :

$Ad_g(A,V_0,V,W,R,X_0,X,B)=\left(\widetilde{A},\widetilde{V_0},\widetilde{V},\widetilde{W},\widetilde{R},\widetilde{X_0},\widetilde{X},\widetilde{B}\right)/\langle g,g\rangle_H$

$\widetilde{A}=\langle g,g\rangle_H A+2(av+v_0w-x_0r-bx-j(r)v-j(x)w)^tV$

$\widetilde{V_0}=\left(a^2-v_0^2-v^tv+w^tw+r^tr-x_0^2-x^tx+b^2\right)V_0$

$+2i(v_0v+aw+br+x_0x-j(x)v-j(r)w)^tV+2(ax_0+bv_0+w^tx+v^tr)B$

$\widetilde{V}=2i(v_0v+aw+br+x_0x+j(w)r+j(x)v)V_0+\left(a^2+v_0^2+v^tv+w^tw+r^tr+x_0^2+x^tx+b^2\right)V$

$+2(x_0j(v)-bj(w)+aj(r)-v_0j(x)+j(r)j(r)+j(v)j(v)-j(x)j(x)-j(w)j(w))V$

$+2i(bv+x_0w+v_0r+ax+j(w)v+j(r)x)B$

$\widetilde{W}=\left(a^2+v_0^2-v^tv-w^tw+r^tr-x_0^2+x^tx-b^2\right)W$

$+2(-x_0j(v)+bj(w)+aj(r)-v_0j(x)+j(r)j(r)+j(w)j(w)-j(v)j(v)-j(x)j(x))W$

$+2i(v_0j(v)+aj(w)+bj(r)+x_0j(x)+j(x)j(v)+j(r)j(w)+j(w)j(r)+j(v)j(x))R$

$+2i(bv+x_0w+v_0r+ax+j(w)v+j(r)x)X_0$

$-2\{ax_0+bv_0+w^tx+r^tv+aj(v)+v_0j(w)+x_0j(r)+bj(x)+j(x)j(w)$
$+j(w)j(x)+j(v)j(r)+j(r)j(v)\}X$
$\widetilde{R}=2i(v_0j(v)+aj(w)+bj(r)+x_0j(x)+j(w)j(r)+j(r)j(w)+j(x)j(v)+j(v)j(x))W$
$+\langle g,g\rangle_H R+2(x_0j(v)+bj(w)+aj(r)+v_0j(x)+j(v)j(v)+j(w)j(w)+j(r)j(r)+j(x)j(x))R$
$+2(rx_0-va+xb-wv_0+j(v)r-j(x)w)X_0$
$+2i(bj(v)+x_0j(w)+v_0j(r)+aj(x)+j(v)j(w)+j(w)j(v)+j(x)j(r)+j(r)j(x))X$
$\widetilde{X_0}=2i(bv+x_0w+v_0r+ax-j(x)r-j(v)w)^tW+2(x_0r-av+v_0w-bx-j(v)r-j(x)w)^tR$
$+\left(a^2+v_0^2+v^tv+w^tw+r^tr+x_0^2+x^tx+b^2\right)X_0+2i(v_0v+aw+br+x_0x-j(r)w-j(v)x)^tX$
$\widetilde{X}=2\{ax_0+bv_0+x^tw+v^tr+aj(v)+v_0j(w)+x_0j(r)+bj(x)+j(w)j(x)$
$+j(x)j(w)+j(v)j(r)+j(r)j(v)\}W$
$+2i(bj(v)+x_0j(w)+v_0j(r)+aj(x)+j(w)j(v)+j(v)j(w)+j(x)j(r)+j(r)j(x))R$
$+2i(v_0v+aw+br+x_0x+j(r)w+j(v)x)X_0+$
$+\left(a^2-v_0^2-v^tv+w^tw+r^tr-x_0^2-x^tx+b^2\right)X$
$+2(-x_0j(v)-bj(w)+aj(r)+v_0j(x)-j(w)j(w)-j(v)j(v)+j(x)j(x)+j(r)j(r))X$
$\widetilde{B}=2(-ax_0+bv_0+x^tw-v^tr)V_0+2i(ax+bv+x_0w+v_0r+j(v)w-j(x)r)^tV$
$+\left(a^2+v_0^2-v^tv-w^tw+r^tr-x_0^2+x^tx-b^2\right)B$

Ad_g maps the vector subspaces :
$(A,0,0,0,0,0,0,B)\to\left(\widetilde{A},\widetilde{V_0},\widetilde{V},0,0,0,0,\widetilde{B}\right)$
$(0,V_0,V,0,0,0,0,0)\to\left(\widetilde{A},\widetilde{V_0},\widetilde{V},0,0,0,0,\widetilde{B}\right)$
$(0,0,0,W,R,0,0,0)\to\left(0,0,0,\widetilde{W},\widetilde{R},\widetilde{X_0},\widetilde{X},0\right)$
$(0,0,0,0,0,X_0,X,0)\to\left(0,0,0,\widetilde{W},\widetilde{R},\widetilde{X_0},\widetilde{X},0\right)$

Subgroups of U

The elements of the group such that the adjoint map preserves both the Hermitian product and the vector space F can be written as the product of an even number of vectors of $\operatorname{Re}F$ or $i\operatorname{Re}F$. With the graded involution we have necessarily $\imath(g)=g$ which implies : $g=(a,0,0,w,r,0,0,b)$. They belong to U and meet the general conditions :

$\operatorname{Im}(ab+w^tr)=0$
$\operatorname{Im}(bw+ar)-j(\operatorname{Re}w)\operatorname{Im}w+j(\operatorname{Re}r)\operatorname{Im}r=0$
$\operatorname{Re}(aw-br+j(r)w)=0$
which sum sup, for $g=(a,0,0,iw,r,0,0,ib)\in Cl_R$, to $ab+w^tr=0$.
So the elements of $Cl(\mathbb{C},4)$ which read

$$U_7=\left\{g=(a,0,0,iw,r,0,0,ib),ab+w^tr=0,a,b\in\mathbb{R},w,r\in\mathbb{R}^3\right\}$$

constitute a 7 real dimensional Lie subgroup U_7 of U.
$\langle g,g\rangle_H=\langle g,g\rangle_{Cl}=a^2-w^tw+r^tr-b^2$

The matrix $[Ad_g]$ of the adjoint map, expressed in the orthonormal basis with our usual notation takes the form :

$$[Ad_g]_{16\times16} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & [N(v_0,v)] & 0 & 0 & 0 \\ 0 & 0 & [M(w,r)] & 0 & 0 \\ 0 & 0 & 0 & [N(x_0,x)] & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $[N(v_0,v)], [N(x_0,x)], [M(w,r)]$ are the matrices :

$[N(v_0,v)]_{4\times4} = [N(x_0,x)]_{4\times4} = \frac{1}{\langle g,g\rangle_H}\{(a^2 + w^t w + r^t r + b^2) I_4$

$+2\begin{bmatrix} 1 & i(aw + br - j(r)w)^t \\ i(aw + br + j(w)r) & -bj(w) + aj(r) - j(w)j(w) + j(r)j(r) \end{bmatrix}\}$

$[M(w,r)]_{6\times6} = \frac{1}{\langle g,g\rangle_H}\{I_6$

$+2\begin{bmatrix} bj(w) + aj(r) + j(r)j(r) + j(w)j(w) & i(aj(w) + bj(r) + j(r)j(w) + j(w)j(r)) \\ i(aj(w) + bj(r) + j(w)j(r) + j(r)j(w)) & bj(w) + aj(r) + j(w)j(w) + j(r)j(r) \end{bmatrix}\}$

It maps the vector subspaces :

$(0, V_0, V, 0, 0, 0, 0, 0) \to \left(0, \widetilde{V_0}, \widetilde{V}, 0, 0, 0, 0, 0\right), (0, 0, 0, 0, 0, X_0, X, 0) \to \left(0, 0, 0, 0, 0, \widetilde{X_0}, \widetilde{X}, 0\right)$

with the same matrix, and maps

$(A, 0, 0, 0, 0, 0, 0, B) \to \left(\widetilde{A}, 0, 0, 0, 0, 0, 0, \widetilde{B}\right)$

$(0, 0, 0, W, R, 0, 0, 0) \to \left(0, 0, 0, \widetilde{W}, \widetilde{R}, 0, 0, 0\right)$

Group Spin$(\mathbb{C}, 4)$

The group $Spin(\mathbb{C}, 4)$ has for Lie algebra :

$T_1 Spin(\mathbb{C}, 4) = \left\{T = (0, 0, 0, W, R, 0, 0, 0), W, R \in \mathbb{C}^3\right\} \Rightarrow T^t + T = 0$ and $g^t \cdot g = 1$

Its elements read : $g = (a, 0, 0, w, r, 0, 0, b)$ and $g^{-1} = g^t = (a, 0, 0, -w, -r, 0, 0, b)$. From the equation $g^t \cdot g = 1$ we get the relations between the components :

$w^t r = -ab$

$a^2 + b^2 + w^t w + r^t r = 1$

They can also be computed through the exponential.

$T_1 Spin(\mathbb{C}, 3) = \left\{T_r = (0, 0, 0, 0, R, 0, 0, 0), R \in \mathbb{C}^3\right\}$ is the Lie algebra of the Lie group $Spin(\mathbb{C}, 3)$

$T_r \cdot T_r = -R^t R$ thus the elements of the group read :

$\exp T_r = \cosh \mu_r + \frac{\sinh \mu_r}{\mu_r}(T_r)$ with $\mu_r^2 = -R^t R = T_r \cdot T_r$

The vector space $\left\{T_w = (0, 0, 0, W, 0, 0, 0, 0), W \in \mathbb{C}^3\right\}$ is not a Lie algebra but :

$T_w \cdot T_w = -W^t W$

$\exp T_w = \cosh \mu_w + \frac{\sinh \mu_w}{\mu_w}(T_w)$ with $\mu_w^2 = -W^t W = T_w \cdot T_w$

One can define as chart of the manifold $Spin(\mathbb{C}, 4)$ the map :

$Spin(\mathbb{C}, 3) \times T_w \to Spin(\mathbb{C}, 4) :: g = s \cdot \exp T_w = \exp T_r \cdot \exp T_w$

then, using $T_w \cdot T_r = (0, 0, 0, j(W)R, 0, 0, 0, -W^t R)$ one gets :

$a = \cosh \mu_w \cosh \mu_r$

$w = \frac{\sinh \mu_w}{\mu_w}\left(\cosh \mu_r - \frac{\sinh \mu_r}{\mu_r} j(R)\right) W$

$r = \cosh \mu_w \frac{\sinh \mu_r}{\mu_r} R$

$b = -\frac{\sinh \mu_w}{\mu_w}\frac{\sinh \mu_r}{\mu_r}(W^t R)$

$g^{-1} = (a, 0, 0, -w, -r, 0, 0, b)$
$g = (\operatorname{Re} a, 0, 0, i \operatorname{Im} w, \operatorname{Re} r, 0, 0, i \operatorname{Im} b) + i (\operatorname{Im} a, 0, 0, -i \operatorname{Re} w, \operatorname{Im} r, 0, 0, -i \operatorname{Re} b)$
$(\operatorname{Re} w)^t \operatorname{Re} r - (\operatorname{Im} w)^t \operatorname{Im} r = \operatorname{Im}(a) \operatorname{Im}(b) - \operatorname{Re}(a) \operatorname{Re}(b)$
$(\operatorname{Re} w)^t \operatorname{Im} r + (\operatorname{Im} w)^t \operatorname{Re} r = -\operatorname{Im}(a) \operatorname{Re}(b) - \operatorname{Im}(b) \operatorname{Re}(a)$
$-\operatorname{Im}^2(a) - \operatorname{Im}^2(b) + \operatorname{Re}^2(a) + \operatorname{Re}^2(b) + (\operatorname{Re} w)^t \operatorname{Re} w - (\operatorname{Im} w)^t \operatorname{Im} w + (\operatorname{Re} r)^t \operatorname{Re} r - (\operatorname{Im} r)^t \operatorname{Im} r = 1$
$2 \operatorname{Im}(a) \operatorname{Re}(a) + 2 \operatorname{Im}(b) \operatorname{Re}(b) + (\operatorname{Re} w)^t \operatorname{Im} w + (\operatorname{Im} w)^t \operatorname{Re} w + (\operatorname{Re} r)^t \operatorname{Im} r + (\operatorname{Im} r)^t \operatorname{Re} r = 0$

The matrix $[Ad_g]$, for $g \in Spin(\mathbb{C}, 4)$, expressed in the orthonormal basis, takes a form similar as above for U_7. It leaves globally invariant and acts with the same matrix on the vector spaces $(0, V_0, V, 0, 0, 0, 0, 0)$, $(0, 0, 0, 0, 0, X_0, X, 0)$ and leaves globally invariant $(0, 0, 0, W, R, 0, 0, 0)$.

The group $Spin(\mathbb{C}, 4)$ has a real and imaginary parts, which are copies of $Spin(3, 1)$:
$Spin(\mathbb{C}, 4) = C(Spin(3, 1)) \oplus iC(Spin(3, 1))$
$= (a, 0, 0, iw, r, 0, 0, ib) \oplus (ia', 0, 0, w', ir', 0, 0, b')$
$C(Spin(3, 1))$ is a real form of $Spin(\mathbb{C}, 4)$. $C(T_1 Spin(3, 1))$ is a subalgebra of $T_1 Spin(\mathbb{C}, 4)$ and $C(Spin(3, 1))$ is a subgroup of $Spin(\mathbb{C}, 4)$.
Because for $g \in \operatorname{Re} Spin(\mathbb{C}, 4) : g^t \cdot g = 1 = \langle g, g \rangle_{Cl}, CC(g) = g$
$\Rightarrow g \in \operatorname{Re} U_0 \Rightarrow \operatorname{Re} Spin(\mathbb{C}, 4) \subset \operatorname{Re} U_0 \subset \operatorname{Re} U$
In $Cl(3, 1)$ the Spin group has for Lie algebra :
$T_1 Spin(3, 1) = \left\{ T = [0, 0, 0, W, R, 0, 0, 0], W, R \in \mathbb{R}^3 \right\}$
It is 6 real dimensional and isomorphic to $Spin(1, 3)$ and to $SL(\mathbb{C}, 2)$. The set : $\left\{ T = [0, 0, 0, 0, R, 0, 0, 0], W, R \in \mathbb{R}^3 \right\}$ is the Lie algebra of the compact Lie group $Spin(3)$, isomorphic to $SU(2)$, as well as its image $C(Spin(3))$.

2.3 Gravitational Field

The gravitational field is characterized by its action on vectors (e^a, ν^a, u^a, d^a), $a = 1, 2, 3$ (the index a corresponds to the generations) bearing the kinematic characteristics of elementary particles. It leaves globally invariant
$E_G = Span(e^a, \nu^a, u^a, d^a)$. Its action on E_G is equivalent to the action by a group isomorphic to $Spin(3, 1)$.

From the previous results we understand better the importance and significance of the Spin group. The motion of a particle, its most significant feature, is not measured in the tetrad, but in the Clifford algebra $Cl(3, 1)$. The group is then the group which defines the change of basis, and this group is necessarily the Spin group, to respect the specificities of the basis of a Clifford algebra.

In the experimental process of revealing the characteristics of matter and fields, the first layer is the interpretation of the trajectories, and the first, and most common, representation is by spinors, using the Clifford algebra $Cl(3, 1)$ with $Spin(3, 1)$. The adjoint map with the Spin group maps vectors of the tetrad to vectors of the tetrad, so the vectors (e^a, ν^a, u^a, d^a) corresponding to the kinetic characteristics of the particles should be found in the vectors of its underlying vector space F of the Clifford algebra, and in $Cl(\mathbb{C}, 4)$ in $\operatorname{Re}(F)$.

Moreover $C(Spin(3,1))$ maps real vectors to real vectors and imaginary vectors to imaginary vectors.

We can then make the following assumptions :

Proposition 1 *The kinematic characteristics of the particles are defined by 12 vectors* $\psi_G^a = (e^a, \nu^a, u^a, d^a)_{a=1,2,3}$ *belonging to* $\mathrm{Re}(F)$ *:*

$$\psi_G^a = iv_0^a\varepsilon_0 + v_1^a\varepsilon_1 + v_2^a\varepsilon_2 + v_3^a\varepsilon_3 = iv_0^a\varepsilon_0 + v^a, v_0^a \in \mathbb{R}, v^a \in \mathbb{R}^3$$

Proposition 2 *The gravitational field is defined by the subgroup of* $\mathrm{Re}\,U$:

$$U_G = C(Spin(3,1)) = \left\{g = (a,0,0,iw,r,0,0,ib), a,b \in \mathbb{R}, w,r \in \mathbb{R}^3\right\}$$
$$T_1U_G = \left\{(0,0,0,iW,R,0,0,0), W,R \in \mathbb{R}^3\right\}$$

with

$\mu_w^2 = W^tW, \mu_r^2 = -R^tR$

$a = \cosh\sqrt{W^tW}\cos\sqrt{R^tR}$

$w = \frac{\sinh\sqrt{W^tW}}{\sqrt{W^tW}}\left(\cos\sqrt{R^tR} - \frac{\sin\sqrt{R^tR}}{\sqrt{R^tR}}j(R)\right)W$

$r = \cosh\sqrt{W^tW}\frac{\sin\sqrt{R^tR}}{\sqrt{R^tR}}R$

$b = -\frac{\sinh\sqrt{W^tW}}{\sqrt{W^tW}}\frac{\sin\sqrt{R^tR}}{\sqrt{R^tR}}(W^tR)$

$g^{-1} = (a,0,0,-iw,-r,0,0,ib)$

and :

$w^tr = -ab$

$a^2 - b^2 - w^tw + r^tr = 1$

The action of the gravitational field is :

$\psi_0 \to \vartheta(g)(\psi_G^a) = Ad_g\psi_G^a$

with $[Ad_g] = \left(a^2 + b^2 + r^tr + w^tw\right)I_4$

$+2\begin{bmatrix} 1 & i(aw - br + j(w)r)^t \\ i(-aw + br + j(w)r) & aj(r) + bj(w) + j(r)j(r) + j(w)j(w) \end{bmatrix}$

and it preserves the Hermitian scalar product.

2.4 Weak field

2.4.1 Characteristics

It involves chirality, which is defined in even dimensional complex Clifford algebras by a volume element. This is in $Cl(\mathbb{C},4) : \omega = \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$. It splits $Cl(\mathbb{C},4)$ in two subsets :

$Cl^R = \left\{Z = \frac{1}{2}(Z + \omega \cdot Z)\right\} = \{Z : \omega \cdot Z = Z\} = \{Z : (a, v_0, v, w, w, -v_0, -v, a)\}$

$Cl^L = \left\{Z = \frac{1}{2}(Z - \omega \cdot Z)\right\} = \{Z : \omega \cdot Z = -Z\} = \{Z : (a, v_0, v, w, -w, v_0, v, -a)\}$

Cl^R is a 8 dimensional Lie subalgebra.

The subspaces Cl^R, Cl^L are orthogonal for $\langle\rangle_{Cl}$

Let us denote

$V_L = Span_{\mathbb{C}}(e_L, \nu_L, u_L, d_L) \subset Cl^L, V_R = Span_{\mathbb{C}}(e_R, \nu_R, u_R, d_R) \subset Cl^R$

V_L is globally invariant by the action of a group U_W isomorphic to $SU(2)$.

V_R is invariant by the action of a group isomorphic to $SU(2)$.

We assume that e_L, ν_L, u_L, d_L are linearly independent, and V_L is a 4 dimensional vector space.

2.4.2 Identification of the Group

The scalars are invariant in any action, thus if V_L is globally invariant then $\{V_L, 1\}$ is globally invariant and one can find a basis such that the scalar component of the vectors is null. Let $(e_j)_{j=1}^{4}$ be an orthonormal basis of V_L, completed by $(e_j)_{j=1}^{16}$ to have an orthonormal basis (for the Hermitian scalar product) of $Cl(\mathbb{C}, 4)$. The vectors e_j have for components H_j^α in the basis $(E_\alpha)_{\alpha=1}^{16}$ so :

$e_j = \sum_{\alpha=1}^{16} H_j^\alpha E_\alpha$

$\langle e_j, e_k \rangle_H = \eta_{jk} = \left\langle \sum_{\alpha=1}^{16} H_j^\alpha E_\alpha, \sum_{\beta=1}^{16} H_k^\beta E_\beta \right\rangle = \sum_{\alpha=1}^{16} \overline{(H_j^\alpha)} (H_k^\alpha) \eta_{\alpha\alpha}$

$[H]^* [\eta] [H] = [\eta]$

There is always an endomorphism J such that : $J(e_j) = E_j$: its matrix is such that

$[J][e_j] = \sum_{\alpha=1}^{16} J_\beta^\alpha H_j^\beta E_\alpha = E_j \Leftrightarrow \sum_{\alpha=1}^{16} J_\beta^\alpha H_j^\beta = \delta_j^\alpha$

and it suffices to take : $J_\beta^\alpha = \overline{(H_j^\alpha)} \eta_{\alpha\alpha}$.

We can assume that the scalar component of $(e_j)_{j=1}^{4}$ is null. So that for $j = 1...4 : J(e_j) = \varepsilon_{j-1}$.

V_L is mapped to $F = Span(\varepsilon_j)_{j=0}^{3}$.

If V_L is globally invariant by $Ad_g, g \in U_W$ then $J \circ Ad_g$ leaves invariant $F = Span(\varepsilon_j)_{j=0}^{3}$. And conversely for any map $\vartheta : F \to F$ the map $J^{-1} \circ \vartheta \circ J$ leaves invariant V_L. Then, using the previous result, $J^{-1} \circ \vartheta \circ J$ must take the form $Ad_{u_1 \ldots u_{2p}}$ where $(u_j)_{j=1}^{2p}$ are vectors of $\operatorname{Re} F$ or $i \operatorname{Re} F$. We assume that $U_W \subset Cl_R$ then it is necessarily a subgroup of U_7. Using the matrix form given previously it is easy to check that for U_7 the adjoint map preserves both the right handed and left handed parts Cl^L, Cl^R.

U_W is isomorphic to $SU(2)$ and $C(Spin(3))$ is isomorphic to $SU(2)$ and is a subgroup of U_7.

So we make the assumption :

Proposition 3 *The weak field can be represented by the subgroup $C(Spin(3))$ of $\operatorname{Re} U$:*

$$U_W = \left\{ g = (a, 0, 0, 0, r, 0, 0, ib) \in Cl(\mathbb{C}, 4), ab = 0, a^2 - b^2 = 1 - r^t r, a, b \in \mathbb{R}, r \in \mathbb{R}^3 \right\}$$

2.4.3 The vector spaces V_L, V_R

The vectors (e_L, ν_L, u_L, d_L) must be of the form : $(a, v_0, v, w, -w, v_0, v, -a)$. The basis of the vector space representing the states ψ is arbitrary. For the gravitational field we have seen that $\psi_G \in F = Span\left(\varepsilon_j\right)_{j=0..3}$. As $\psi = \psi_G + \psi_W + \psi_S$ it is logical to take the same basis, which sums up to assume that $J = Id$. With the assumption that the 4 vectors are linearly independent and belong to iCl_R the simplest solution is :

Proposition 4 *The vectors (e_L, ν_L, u_L, d_L) defining the characteristics of the left handed particles with respect to the weak field have the format :*

$$\begin{array}{c} e_L = \left(0, v_0^e, iv^e, 0, 0, v_0^e, iv^e, 0\right), v_0^e \in \mathbb{R}, v^e \in \mathbb{R}^3 \\ \nu_L = \left(0, v_0^e, iv^\nu, 0, 0, v_0^\nu, iv^\nu, 0\right), v_0^\nu \in \mathbb{R}, v^\nu \in \mathbb{R}^3 \\ u_L = \left(0, v_0^u, iv^u, 0, 0, v_0^u, iv^u, 0\right), v_0^u \in \mathbb{R}, v^u \in \mathbb{R}^3 \\ d_L = \left(0, v_0^d, iv^d, 0, 0, v_0^d, iv^d, 0\right), v_0^d \in \mathbb{R}, v^d \in \mathbb{R}^3 \end{array}$$

The group U_W acts on these vectors with the same matrix for the components $(v_0, v), (x_0, x)$:

$$[M_W]_{4\times 4} = \left(1 + 2b^2\right) I_4 + 2 \begin{bmatrix} 1 & -ibr^t \\ ibr & aj(r) + j(r)j(r) \end{bmatrix}$$

The vectors (e_R, ν_R, u_R, d_R) must be of the form : $(a, v_0, v, w, w, -v_0, -v, a)$ and be invariant under the action of U_W. It will happen only if $\psi_R = (a, 0, 0, 0, 0, 0, 0, a)$. If $\psi_R \in iCl_R$ all the charges are null, so the only solution is :

Proposition 5 *The vectors (e_R, ν_R, u_R, d_R) defining the characteristics of the right handed particles with respect to the weak field are :*

$$\begin{array}{c} e_R = a^{er} + a^{er}\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3, a^{er} \in \mathbb{R} \\ \nu_R = a^{\nu r} + a^{\nu r}\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3, a^{\nu r} \in \mathbb{R} \\ u_R = a^{ur} + a^{ur}\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3, a^{ur} \in \mathbb{R} \\ d_R = a^{dr} + a^{dr}\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3, a^{dr} \in \mathbb{R} \end{array}$$

2.5 Strong Field

We have 3 vectors (q_r, q_b, q_r). We assume that the 3 vectors are linearly independent, belong to iCl_R and span a 3 real dimensional vector space E_s. The challenge is to find a subgroup U_w of $\operatorname{Re} U$, isomorphic to $SU(3)$ for which the adjoint map keeps E_s globally invariant.

2.5.1 Identification of the group

Let $(e_j)_{j=1}^3$ be an orthonormal basis of E_s and assume that $\langle e_j, e_j\rangle_H = 1$. Because $E_S \subset iCl_R$ then $\langle e_j, e_j\rangle_H = \langle e_j, e_j\rangle_{Cl} = 1$. The restriction, that we will denote by h, to E_s of the scalar product on $Cl(\mathbb{C}, 4)$ can be defined through the orthonormal basis $(e_j)_{j=1}^3$ and is definite positive.

The Clifford algebra $Cl(3)$ is a sub Clifford algebra of $Cl(3,1)$, generated by the vectors $(\varepsilon_j)_{j=1}^3$, with the bilinear symmetric form of signature $(+;+;+)$. Its image $C(Cl(3)) = \operatorname{Re} Cl(\mathbb{C},3)$ is generated by the vectors $(\varepsilon_j)_{j=1}^3$ of $Cl(\mathbb{C},4)$ and is a real Clifford subalgebra.

Define the linear map : $f : E_s \to \operatorname{Re} Cl(\mathbb{C},3) :: f(e_j) = \varepsilon_j$.

Then for any vectors $u, v \in E_s$:

$f(u) \cdot f(v) + f(v) \cdot f(u) = f\left(\sum_{j=1}^3 u^j e_j\right) \cdot f\left(\sum_{j=1}^3 v^j e_j\right) + f\left(\sum_{j=1}^3 v^j e_j\right) \cdot f\left(\sum_{j=1}^3 u^j e_j\right)$

$= \left(\sum_{j=1}^3 u^j \varepsilon_j\right) \cdot \left(\sum_{j=1}^3 v^j \varepsilon_j\right) + \left(\sum_{j=1}^3 v^j \varepsilon_j\right) \cdot \left(\sum_{j=1}^3 u^j \varepsilon_j\right)$

$= 2\left\langle \sum_{j=1}^3 u^j \varepsilon_j, \sum_{j=1}^3 v^j \varepsilon_j \right\rangle_{Cl} = 2\left\langle \sum_{j=1}^3 u^j e_j, \sum_{j=1}^3 v^j e_j \right\rangle_{E_s}$

By the universal property of Clifford algebras, there is a unique Clifford algebra morphism $\varphi : Cl(E_s, h) \to \operatorname{Re} Cl(\mathbb{C},3)$ such that $f = \varphi \circ \jmath$ where $\jmath$ is the canonical injection $\jmath : E_s \to Cl(E_s, h)$.

The adjoint map Ad_g preserves globally E_s and then $Cl(E_s, h)$ which is a 8 real dimensional Clifford algebra. Its value is fully defined by its value on E_s, that is by a 3×3 matrix, which is necessarily unitary because $U_w \subset U$. So U_w is isomorphic to $SU(3)$.

The reasoning can be understood more intuitively by noticing that, because for the 3 vectors $e_j : \langle e_j, e_j \rangle_H = 1$, their components must be null for the vectors E_α of the basis of $Cl(\mathbb{C},4)$ whose product comprises ε_0. It sums up to eliminate ε_0 and indeed the vector subspace of T_1U_0, when ε_0 is eliminated, reads :

$T_1U_8 = \{T \in Cl(\mathbb{C},3) : CC(T^t) + T = 0\}$

$= \{(iA, 0, iV, 0, R, X_0, 0, 0), A, V, R, X_0 \in \mathbb{R}\}$

It is a 8 real dimensional Lie subalgebra, of the subgroup of $\operatorname{Re} U$:

$U_8 = \{g = (a, 0, v, 0, r, x_0, 0, 0) \in Cl_R : CC(g) = g, g^t \cdot g = \langle g, g \rangle_H\}$

The condition : $ab + v_0 x_0 + v^t x + w^t r = 0$ is removed, because it comes from the last component (B) of the product $g \cdot CC(g)^t = 1$, so the 8 components a, v, r, x_0 are not related.

Proposition 6 *The strong field can be represented by the subgroup U_w of $\operatorname{Re} U$ whose elements read in $Cl(\mathbb{C},4)$:*

$$U_w = \{g = (a, 0, v, 0, r, x_0, 0, 0) \in Cl_R\}$$

2.5.2 Identification of the vectors (q_r, q_b, q_r)

From the previous results their components must be null for the vectors E_α of the basis of $Cl(\mathbb{C},4)$ whose product comprises ε_0. As for the gravitational and the weak fields we can assume that (q_r, q_b, q_r) are multiple of some vectors of the basis (E_α).

Proposition 7 *The vectors* (q_r, q_b, q_g) *defining the characteristics of the quarks with respect to the strong field have the format :*

$$q_r = ir_1^r \varepsilon_3 \cdot \varepsilon_2 + ir_2^r \varepsilon_1 \cdot \varepsilon_3 + ir_3^r \varepsilon_2 \cdot \varepsilon_1 + ix_0^r \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3, x_0^r, r_1^r, r_2^r, r_3^r \in \mathbb{R}$$
$$q_b = ir_1^b \varepsilon_3 \cdot \varepsilon_2 + ir_2^b \varepsilon_1 \cdot \varepsilon_3 + ir_3^b \varepsilon_2 \cdot \varepsilon_1 + ir_0^b \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3, x_0^b, r_1^b, r_2^b, r_3^b \in \mathbb{R}$$
$$q_g = ir_1^g \varepsilon_3 \cdot \varepsilon_2 + ir_2^g \varepsilon_1 \cdot \varepsilon_3 + ir_g^3 \varepsilon_2 \cdot \varepsilon_1 + ix_0^g \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3, x_0^g, r_1^g, r_2^g, r_3^g \in \mathbb{R}$$

2.6 Definition of the group $\widehat{U}$ and its action

2.6.1 Definition of $\widehat{U}$

For the gravitational, weak and strong fields this is the real part of U which is involved. We have seen previously that the representation of the EM field by the group $U(1)$ can be done through the action $\vartheta(\phi, U)\psi = (\exp i\phi) Ad_g \psi$. It preserves the Hermitian product. Then the group $\widehat{U}$, representing all the fields, could be defined as $\widehat{U} = U(1) \times \operatorname{Re} U$. But there is the issue of the Lie algebra. Because $\operatorname{Re} U$ can be defined equivalently as the product $\mathbb{R} \times \operatorname{Re}(U_0)$ we could consider $T_1 U_0$. However $T_1 U_0 = (iA, V_0, iV, iW, R, X_0, iX, B)$ has a real and an imaginary part :

$T_1 U_0 = (0, 0, 0, iW, R, X_0, iX, 0) \oplus (iA, V_0, iV, 0, 0, 0, 0, B)$

and, as we can see for the strong field, we need both parts : $T_1 U_8$ is a real 8 dimensional Lie algebra, which does not belong to the real part of $T_1 U_0$.

$T_1 U_0$ is a Lie algebra and gives the group by the exponential. The computation of the coefficients of $g = (a, v_0, v, w, r, x_0, x, b) \in U_0$ can be done from T by the exponential using the decomposition :

$T_1 U_0 = (iA, 0, 0, 0, 0, 0, 0, 0) \oplus (0, 0, 0, iW, R, 0, 0, 0) \oplus (0, 0, 0, 0, 0, X_0, iX, 0) \oplus (0, V_0, iV, 0, 0, 0, 0, B)$

i) The center of $T_1 U_0$ gives the abelian group $U(1) = \{(\exp iA, 0, 0, 0, 0, 0, 0, 0)\}$

ii) $(0, 0, 0, iW, R, 0, 0, 0)$ is the Lie algebra of the Real part of $Spin(\mathbb{C}, 4)$ and we have seen that its elements can be written :

$s = (a, 0, 0, iw, r, 0, 0, ib)$ with

$\exp T_r = \exp(0, 0, 0, 0, R, 0, 0, 0) = \cos \mu_r + \frac{\sin \mu_r}{\mu_r}(T_r)$ with $\mu_r^2 = R^t R = -T_r \cdot T_r$

$\exp T_w = \exp(0, 0, 0, iW, 0, 0, 0, 0) = \cosh \mu_w + \frac{\sinh \mu_w}{\mu_w}(T_w)$ with $\mu_w^2 = W^t W = T_w \cdot T_w$

$a = \cosh \mu_w \cos \mu_r$

$w = \frac{\sinh \mu_w}{\mu_w}\left(\cos \mu_r - \frac{\sin \mu_r}{\mu_r} j(R)\right) W$

$r = \cosh \mu_w \frac{\sin \mu_r}{\mu_r} R$

$b = -\frac{\sinh \mu_w}{\mu_w} \frac{\sin \mu_r}{\mu_r}(W^t R)$

Then a chart of $\operatorname{Re} Spin(\mathbb{C}, 4)$ is :

$(0, 0, 0, iW, R, 0, 0, 0) \to \operatorname{Re} Spin(\mathbb{C}, 4) :: g = \exp T_r \cdot \exp T_w$

iii) $(0, 0, 0, iW, R, 0, 0, 0) \oplus (0, 0, 0, 0, 0, X_0, iX, 0)$ is the real part of $T_1 U_0$, this is a Lie algebra. The exponential on the vector space $T_1 U_0 / T_1 Spin(3,1) = \{T_x = (0, 0, 0, 0, 0, X_0, iX, 0)\}$ is well defined.

$T_x \cdot T_x = \left(-X_0^2 + X^t X, 0, 0, 0, 0, 0, 0, 0\right)$

thus : $\exp T_x = \cosh \mu_x + \frac{\sinh \mu_x}{\mu_x} T_x$ with $\mu_x^2 = -X_0^2 - X^t X$ and one can check that $(\exp T_x)^t \cdot \exp T_x = 1$

iv) The exponential on the vector space $(0, V_0, iV, 0, 0, 0, 0, B) = T_1 U_0 / (T_1 U(1) + \operatorname{Re} T_1 U_0)$ reads :

$T_v \cdot T_v = (0, V_0, iV, 0, 0, 0, 0, B) \cdot (0, V_0, iV, 0, 0, 0, 0, B) = \left(V_0^2 - V^t V + B^2, 0, 0, 0, 0, 0, 0, 0\right) \in Cl(3,1)_R$

$\exp T_v = \cosh \mu_v + \frac{\sinh \mu_v}{\mu_v} T_v$ with $\mu_v^2 = T_v \cdot T_v = V_0^2 - V^t V + B^2$

If $\mu_v^2 = V_0^2 - V^t V + B^2 > 0$ then $\exp T_v = \cosh \mu_v + \frac{\sinh \mu_v}{\mu_v} T_v$

If $\mu_v^2 = V_0^2 - V^t V + B^2 < 0$ then $\exp T_v = \cos \mu_v + \frac{\sin \mu_v}{\mu_v} T_v$

and in both cases $\cosh \mu_v, \frac{\sinh \mu_v}{\mu_v} \in \mathbb{R}$.

And we can check that $\exp T_V \in U_0$.

Because $T_v = iC([0, -V_0, V, 0, 0, 0, 0, -B]) \in iCl(3,1)_R$ the exponential $\exp T_v$ has a real and an imaginary part.

v) And a chart of U_0 is :

$g = (a, v_0, v, w, r, x_0, x, b)$

$= e^{iA}(\cosh \mu_w \cos \mu_r, 0, 0, i\frac{\sinh \mu_w}{\mu_w} \left(\cos \mu_r - \frac{\sin \mu_r}{\mu_r} j(R)\right) W, \cosh \mu_w \frac{\sin \mu_r}{\mu_r} R, 0, 0,$

$-i\frac{\sinh \mu_w}{\mu_w} \frac{\sin \mu_r}{\mu_r} (W^t R)) \cdot \left(\cosh \mu_x, 0, 0, 0, 0, \frac{\sinh \mu_x}{\mu_x} X_0, i\frac{\sinh \mu_x}{\mu_x} X, 0\right)$

$\cdot \left(\cosh \mu_v + \frac{\sinh \mu_v}{\mu_v} T_v\right)$

A full computation shows that $g = e^{iA}(a, v_0, v, w, r, x_0, x, b)$ where $a, v_0, v, w, r, x_0, x, b$ are complex, but do not depend on A. So that $g \cdot g' = e^{i(A+A')}(a, v_0, v, w, r, x_0, x, b) \cdot (a', v_0', v', w', r', x_0', x', b')$. As the EM field must be part of the unitary field we need to keep the whole of $T_1 U_0$.

Proposition 8 *The force fields have the unified representation with the real 16 real dimensional Lie group $\widehat{U}$ whose Lie algebra is :*

$$T_1 \widehat{U} = \left\{T \in Cl(\mathbb{C}, 4) : CC\left(T^t\right) + T = 0\right\}$$

Proposition 9 Proposition 10

$$T_1 \widehat{U} = \left\{(iA, V_0, iV, iW, R, X_0, iX, B), A, V_0, X_0, B \in \mathbb{R}, V, W, R, X \in \mathbb{R}^3\right\}$$

Then the elements of the group read :

$\widehat{U} = \{g \in Cl(\mathbb{C}, 4) : CC(g^t) \cdot g = 1\}$

$= \left\{e^{iA}(a, v_0, v, w, r, x_0, x, b), A, a, v_0, x_0, b \in \mathbb{C}, v, w, r, x \in \mathbb{C}^3\right\}$

Proposition 11 *The state of elementary particles are represented in the Clifford algebra $Cl(\mathbb{C}, 4)$ and the action of $\widehat{U}$ on $Cl(\mathbb{C}, 4)$ is :*

$$\vartheta : \widehat{U} \to \mathcal{L}(Cl(\mathbb{C}, 4); Cl(\mathbb{C}, 4)) :: \vartheta(g)\psi = e^{iA} Ad_g \psi$$

2.6.2 Fiber bundles

The present model is slightly different from the general picture, it brings mostly simplifications. The physical principal bundle is $P_G(M, Spin(3,1), \pi_G)$: the observer chooses a spatial orthonormal basis, which is completed for the time vector ε_0 by the direction of his velocity, and altogether we have a tetrad $(\varepsilon_j)_{j=0}^3$, which defines the gauge. A change of tetrad is given by the action of $Spin(3,1)$. The Clifford bundle is built from the tetrad and is a vector bundle $P_G[Cl(3,1), Ad]$. The morphism C, which can be defined with any orthonormal basis, provides a copy of $Cl(\mathbb{C},4)$ at each point, which changes in a change of gauge by the adjoint map with $Spin(3,1)$. This is equivalent to define a vector bundle $P_{Cl} = P_G[Cl(\mathbb{C},4), Ad]$. A point p of P_{Cl} is a basis of $Cl(\mathbb{C},4)$ located at $m \in M$.

There is no principal bundle with group $\widehat{U}$: an element $g \in \widehat{U}$ located at m is just a vector of the vector bundle P_{Cl}, which changes as such in a change of gauge given by $Spin(3,1)$. The force field is measured with respect to the standard $p(m) = 1$ by a value $g \in \widehat{U}$ and changes in a change of gauge as $Cl(\mathbb{C},4)$ by the adjoint action of $Spin(3,1)$. The field has a value at each point of a given area Ω of the universe represented by the manifold M. This is equivalent to assume that there is a section $\mathbf{U} \in \mathfrak{X}(P_{Cl})$:

$\mathbf{U} : \Omega \to P_{Cl} :: \mathbf{U}(m) = g = \sum_{\alpha=1}^{16} g(m)^\alpha E_\alpha(m)$

In a given environment, to each type of elementary particle one can associate a section $\psi \in \mathfrak{X}(P_{Cl})$:

$\psi : \Omega \to P_{Cl} :: \psi(m) = \vartheta(g(m))\psi_0$

then the state of the particle along its trajectory $q(t)$ is given by $\psi(t) = \psi(q(t))$

The value of the field varies from one point to another. Its variation along a curve with tangent V is $\frac{dg}{d\tau} = \sum_{\alpha=0}^3 V^\alpha \partial_\alpha g$. The variation is measured with respect to its starting value, that is by $g^{-1}\frac{dg}{d\tau}$ which belongs to the Lie algebra $T_1\widehat{U}$. So one assumes that there is, at each point m, a map

$$T : TM \to T_1\widehat{U} :: \sum_{\alpha=1}^{16}\sum_{\beta=0}^{3} T_\beta^\alpha E_\alpha \otimes d\xi^\beta$$

which gives the value of $g^{-1}\frac{dg}{d\tau}$ along any direction.

A particle is not immobile and along its world line with a velocity $\sum_{\alpha=0}^3 V^\alpha \partial\xi_\alpha \in TM$ its state changes as :

$\frac{d\psi}{dt} = \frac{d}{dt}\vartheta(g(m))\psi_0 = \vartheta'\left(\frac{dg}{dt}\right) = \vartheta'\left(g \cdot \left(\sum_{\beta=0}^3 V^\beta T_\beta\right)\right) = \vartheta'(g \cdot T) = \vartheta(g) \cdot \vartheta'(1)(T)$

using the identities : $\vartheta'\left(\frac{dg}{dt}\right) = \vartheta(g)\vartheta'(1) L_{g^{-1}} g\left(\frac{dg}{dt}\right) = \vartheta(g)\vartheta'(1) g^{-1} \cdot \frac{dg}{dt} = \vartheta(g) \circ \vartheta'(1)(T)$

We get back the general model of a connection acting on the state of a particle along its trajectory. In this picture the potential of the field is the quantity $T \in \Lambda_1\left(M; T_1\widehat{U}\right)$.

The variation of the field or of the state are not necessarily continuous, the general mathematical representation is through the 1st jet bundle $J^1P_{Cl} = \{m, g, \delta_\alpha Z, \alpha = 0...3\}$ where $\delta_\alpha Z = \sum_{\beta=1}^{16} \delta Z_\alpha^\beta E_\alpha \in Cl\,(\mathbb{C}, 4)$. In a continuous process $\delta_\alpha Z = \partial_\alpha Z$ is the partial derivative of Z in the direction $\alpha = 0...3$. The possible variation of the field is then $\delta g = (m, g, \delta_\alpha T, \alpha = 0...3)$ and the possible variation of the state is $\delta\psi = (m, \psi, \delta_\alpha \psi, \alpha = 0...3)$.

All the quantities measured in $Cl\,(\mathbb{C}, 4)$ change in a change of gauge, that is of tetrad, for instance when one goes from an observer to another at the same location, by a change of basis $(\varepsilon_j)_{j=0}^3$. The potentiaI changes with an affine law.

So a unique principal bundle, with a clear physical and geometric meaning, is necessary in the model.

2.7 Characteristics of elementary particles

2.7.1 State vectors of the elementary particles

The fundamental state vectors are the sum of the gravitational, weak and strong components.

For each generation $a = 1, 2, 3$:

Left handed Leptons :

$e_L = (0, v_0^e + iv_0^{ae}, iv^e + v^{ae}, 0, 0, v_0^e, iv^e, 0) = (0, iv_0^{ae}, v^{ae}, 0, 0, v_0^e, iv^e, 0) \oplus (0, v_0^e, iv^e, 0, 0, 0, 0, 0)$

$\nu_L = (0, v_0^e + iv_0^{a\nu}, v^{a\nu} + iv^\nu, 0, 0, v_0^\nu, iv^\nu, 0)$

Right handed Leptons :

$e_R = (a^{er}, iv_0^{ae}, v^{ae}, 0, 0, 0, 0, a^{er}) = (a^{er}, iv_0^{ae}, v^{ae}, 0, 0, 0, 0, 0) \oplus (0, 0, 0, 0, 0, 0, 0, a^{er})$

$\nu_R = (a^{\nu r}, iv_0^{a\nu}, v^{a\nu}, 0, 0, 0, 0, a^{\nu r})$

Left handed Quarks :

$u_{Lr} = (0, v_0^u + iv_0^{au}, v^{au} + iv^u, 0, ir^r, v_0^u + ix_0^r, iv^u, 0) = (0, iv_0^{au}, v^{au}, 0, 0, v_0^u, iv^u, 0) \oplus (0, v_0^u, iv^u, 0, ir^r, ix_0^r, 0, 0)$

$u_{Lg} = (0, v_0^u + iv_0^{au}, v^{au} + iv^u, 0, ir^g, v_0^u + ix_0^g, iv^u, 0)$

$u_{Lb} = \left(0, v_0^u + iv_0^{au}, v^{au} + iv^u, 0, ir^b, v_0^u + ix_0^b, iv^u, 0\right)$

$d_{Lr} = \left(0, v_0^d + iv_0^{ad}, v^{ad} + iv^d, 0, ir^r, v_0^d + ix_0^r, iv^d, 0\right)$

$d_{Lg} = \left(0, v_0^d + iv_0^{ad}, v^{ad} + iv^d, 0, ir^g, v_0^d + ix_0^g, iv^d, 0\right)$

$d_{Lb} = \left(0, v_0^d + iv_0^{ad}, v^{ad} + iv^d, 0, ir^b, v_0^d + ix_0^b, iv^d, 0\right)$

Right handed quarks :

$u_{Rr} = (a^{ur}, iv_0^{au}, v^{au}, 0, ir^r, ix_0^r, 0, a^{ur}) = (a^{ur}, iv_0^{au}, v^{au}, 0, 0, 0, 0, 0) \oplus (0, 0, 0, 0, ir^r, ix_0^r, 0, a^{ur})$

$u_{Rg} = (a^{ur}, iv_0^{au}, v^{au}, 0, ir^g, ix_0^g, 0, a^{ur})$

$u_{Rb} = \left(a^{ur}, iv_0^{au}, v^{au}, 0, ir^b, ix_0^b, iv^u, a^{ur}\right)$

$d_{Rr} = \left(a^{dr}, iv_0^{ad}, v^{ad}, 0, ir^r, ix_0^r, iv^d, a^{dr}\right)$

$d_{Rg} = \left(a^{dr}, iv_0^{ad}, v^{ad}, 0, ir^g, ix_0^g, iv^d, a^{dr}\right)$

$d_{Rb} = \left(a^{dr}, iv_0^{ad}, v^{ad}, 0, ir^b, ix_0^b, iv^d, a^{dr}\right)$

2.7.2 Antiparticles

The distinction particles / antiparticles is equivalent to the choice of a representation or its contragredient. The Lie algebra of the field, as well as the fundamental vectors ψ_0 have both a real and an imaginary part. The contragredient representation sums up to swap the real and the imaginary parts : $\psi^c = \operatorname{Im}\psi + i\operatorname{Re}\psi = iCC(\psi)$. As we can see from the previous table no fermion is its own antiparticle.

In the process C' replace C and in the Hermitian form $-\eta$ replace η. Or, in other words :

Proposition 12 *The distinction between particles / antiparticles is equivalent to the choice of a signature* $(3,1)$ *or* $(1,3)$ *for the metric.*

2.7.3 Lagrangian

Along the trajectory $q(t)$ of an elementary particle the Hermitian product is constant :

$\langle \psi(q(t)), \psi(q(t)) \rangle_H = \langle \vartheta(g(q(t)))\psi_0, \vartheta(g(q(t)))\psi_0 \rangle_H = \langle \psi_0, \psi_0 \rangle_H$

and by derivation along the trajectory :

$\left\langle \frac{d}{dt}\psi(q(t)), \psi(q(t)) \right\rangle_H + \left\langle \psi(q(t)), \frac{d}{dt}\psi(q(t)) \right\rangle = 0 \Rightarrow \left\langle \psi(q(t)), \frac{d}{dt}\psi(q(t)) \right\rangle \in i\mathbb{R}$

$\left\langle \vartheta(g(m))\psi_0, \vartheta\left(\frac{dg}{dt}\right)\psi_0 \right\rangle_H = \langle \vartheta(g)\psi_0, \vartheta(g)\circ\vartheta'(1)(T)(\psi_0) \rangle_H$

$= \langle \psi_0, \vartheta'(1)(T)\psi_0 \rangle_H$

where $T = \sum_{\beta=0}^{3} V^\beta T_\beta$ involves the velocity of the particle and the potential of the field.

Then we can take for the Lagrangian representing the interactions fields / particles :

$L_p = \frac{1}{i}\langle \psi_0, \vartheta'(1)(T)(\psi_0) \rangle_H$

$\vartheta'(1)(T)(\psi_0) = iA\psi_0 + ad(T)(\psi_0)$

$$L_p = \frac{1}{i}\langle \psi_0, iA\psi_0 + ad(T)(\psi_0) \rangle_H$$

2.7.4 Charges

Denoting $\psi_0 = (a, v_0, v, w, r, x_0, x, b)$ with complex components,

$T = (iA, V_0, iV, iW, R, X_0, iX, B)$ the computation gives :

$$L_p = \frac{1}{i}\langle \psi_0, \vartheta'(1)(T)(\psi_0) \rangle_H = Q_A A + Q_{V_0} V_0 + Q_V^t V + Q_W^t W + Q_R^t R + Q_{X_0} X_0 + Q_X^t X + Q_B B$$

with the charges, which are represented by a vector Q of $Cl(\mathbb{C}, 4)$:

$$Q_A = \langle \psi_0, \psi_0 \rangle_H$$
$$Q_{V_0} = 4\,\mathrm{Im}\,(v^t w + b x_0)$$
$$Q_V = 4\,\mathrm{Re}\,(v_0 w - b x + j\,(r)\,v)$$
$$Q_W = 4\,\mathrm{Re}\,(-v_0 v + x_0 x + j\,(r)\,w)$$
$$Q_R = 4\,(-j\,(\mathrm{Re}\,v)\,\mathrm{Im}\,v + j\,(\mathrm{Re}\,w)\,\mathrm{Im}\,w - j\,(\mathrm{Re}\,r)\,\mathrm{Im}\,r + j\,(\mathrm{Re}\,x)\,\mathrm{Im}\,x)$$
$$Q_{X_0} = 4\,\mathrm{Im}\,(b v_0 + x^t w)$$
$$Q_X = 4\,\mathrm{Re}\,(b v - x_0 w + j\,(x)\,r)$$
$$Q_B = 4\,\mathrm{Im}\,(v_0 x_0 + v^t x)$$

The use of the opposite signature sums up to change $\eta \to -\eta$ and the Hermitian product takes the opposite sign. As we can see all the charges take the opposite value for the corresponding antiparticle.

Using the value of ψ_0 for the elementary particles we get :

Left handed leptons :

$e_L : Q_A = (v^{ae})^t\,(v^{ae}) - (v_0^{ae})^2, Q_{V_0} = 0, Q_V = 0, Q_W = 4\,(v_0^{ae} v^e - v_0^e v^{ae}), Q_R = 4j\,(v^e)\,v^{ae}, Q_{X_0} = 0, Q_X = 0, Q_B = 4\left(v_0^{ae} v_0^e + (v^{ae})^t\,v^e\right)$

Right handed Leptons :

$e_R : Q_A = (v^{ae})^t\,v^{ae} - (v_0^{ae})^2, Q_{V_0} = 0, Q_V = 0, Q_W = 0, Q_R = 0, Q_{X_0} = 4a^{er} v_0^{ae}, Q_X = 4a^{er} v^{ae}, Q_B = 0$

Left handed Quarks :

$u_{Lr} : Q_A = (v^{au})^t\,v^{au} + (r^r)^t\,r^r + (x_0^r)^2 - (v_0^{au})^2, Q_{V_0} = 0, Q_V = 4j\,(v^u)\,r^r, Q_W = 4\,(v_0^{au} v^u - v_0^u v^{au} - x_0^r v^u), Q_R = 4j\,(v^u)\,v^{au}, Q_{X_0} = 0,$

$Q_X = 4j\,(r^r)\,v^u, Q_B = 4\left(v_0^u x_0^r + v_0^{au} v_0^u + (v^{au})^t\,v^u\right)$

Right handed Quarks :

$u_{Rr} : Q_A = (v^{au})^t\,v^{au} + (r^r)^t\,r^r + (x_0^r)^2 - (v_0^{au})^2, Q_{V_0} = 4a^{ur} x_0^r, Q_V = 0, Q_W = 0, Q_R = 0, Q_{X_0} = 4a^{ur} v_0^{au}, Q_X = 4a^{ur} v^{au}, Q_B = 0$

and similar values for the other particles.

The EM charge is the same for right handed and left handed particles as it should be expected.

2.7.5 Momentum and energy

The motion of a particle is part of its state, and is represented through the component of $Spin\,(3,1)$ in g, and its variation then belongs to $T_1 Spin\,(3,1)$. It has a translational (corresponding to W) and a rotational (corresponding to R) component. In a representation based on $Spin\,(3,1)$ the usual "spin" (spin up or down) is directly accounted for in the component R, because $Spin\,(3,1)$ distinguishes the two rotations at the difference with $SO\,(3,1)$.

Outside Particles Physics the momentum can be represented by a spinor, using the representation of $Cl\,(\mathbb{C},4)$ on 4×4 complex matrices, and then the spinor is a 4 complex dimensional vector, which has a right and a left handed part, usually related. This representation is very convenient as it can be easily extended to deformable solids in the RG context. The dynamic part of the Lagrangian can then be expressed through a 3 dimensional vector which, with the mass, sums up the kinematic characteristics of the body. This is the equivalent of a "charge vector".

In a GUT all the components of the fields are assumed to act simultaneously, so there is no need for a specific concept of momentum. It is replaced by a tensor :

$\mathcal{M} = \sum_{\alpha=1}^{16} \sum_{\beta=0}^{3} \psi^\alpha V^\beta E_\alpha \otimes \partial\xi_\beta \in P_{Cl} \otimes TM$

and there is a differential operator :

$P_{Cl} \otimes TM \to P_{Cl} \otimes TM :: \mathcal{M}(\psi \otimes V) = \vartheta(u)(\psi_0) \otimes \sum_{\beta=0}^{3} V^\beta E_\alpha \otimes \partial\xi_\beta$

The difference in the strength of the fields should be reflected in the value of the charges. The inertial features are accounted for in the gravitational charges. The gravitational field involves the components W and R of the Lie algebra $T_1\widehat{U}$. $Q_W = 0$ for the right handed particles and the component W is not involved in the other fields, but the rotational component R is, and this raises some intriguing questions. We can assume that elementary particles have a rotational motion, which is represented by the component R in $T_1 Spin(3,1)$. At the atomic scale the only assumption that we can make is that the rotational motion is at a constant speed (its change entails a variation of kinetic energy) and the component R of the field acts on the axis of rotation. $Q_R = 0$ for the right handed particles, as could be expected, but non null for the left handed particles. We know actually little about the gravitational field, which is weak and varies very slowly in space and time. But it could be involved in the only genuine known random process, that is the spontaneous decay with the weak field.

For the same reasons in a GUT the inertial concepts are replaced by the gravitational features, so there is no longer a specific concept of kinetic energy, which is replaced by the energy of the particle with respect to the fields, and more precisely with respect to its motion in the field, represented by the Lagrangian L_p.

Collisions

The generalization to a GUT model is then immediate : this is the sum of the momenta $\mathcal{M}_1 + \mathcal{M}_2$ which is conserved; for each component of ψ. If the fundamental states do not change the charges are constant, and the potential has the same value for each particle, then the sum of energies L_p is conserved.

2.7.6 Composite particles

Whenever several particles are associated to form a system, they constitute a system and, according to the theorems of quantization, their state can be represented by the tonsorial product of the individual states. Several processes are then at play.

Symmetries

In a system where several components are identical (elementary particles of the same type) they are indistinguishable and the tensor of the system must be either symmetric or antisymmetric (for the corresponding components). Because the states are valued in $Cl(\mathbb{C},4)$ we have convenient tools to study the possible configurations, with the involutions. They commute with the tensorial product so that we have for instance :

$(\psi_1 \otimes \psi_2)^t = \psi_1^t \otimes \psi_2^t, \imath(\psi_1 \otimes \psi_2) = \imath(\psi_1) \otimes \imath(\psi_2),$
$CC(\psi_1 \otimes \psi_2) = CC(\psi_1) \otimes CC(\psi_2)$

Decay

Composite particles can broke down in other composite or elementary particles, and this process is at the core of most experiments in Particle Physics. Its modelization is based on the decomposition of tensorial representations.

The orbit of a given elementary particle j is a vector subspace V_j of $Cl(\mathbb{C},4)$ globally invariant by $\widehat{U}$, and a composite particle is then represented in the tensorial product of representations (V_j,ϑ). Such tensorial products are equivalent to the sum of representations (V_k,ϑ) and the mathematical operation represents the decay. The study of all the possible cases is the bread and butter of Particles Physicists, and in this endeavour an essential tool is the knowledge of the fundamental weights.

In the present picture the operation is simpler : all the vectors subspaces belong to $Cl(\mathbb{C},4)$, there is a unique group and the derivative of the representation is nothing more than a representation of $T_1\widehat{U}$ on itself through ad. The Cartan algebra of $Cl(\mathbb{C},4)$ is 4 complex dimensional, given by the vectors :

$$T_1\mathfrak{T} = \{A + W_1\varepsilon_0 \cdot \varepsilon_1 + R_1\varepsilon_3 \cdot \varepsilon_2 + B\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3, A, W_1, R_1, B \in \mathbb{C}\}$$

We have a similar result by selecting the components w_2, r_2 or w_3, r_3.
Thus the Cartan algebra of $T_1\widehat{U}$ is
$\{iA + iW_1\varepsilon_0 \cdot \varepsilon_1 + R_1\varepsilon_3 \cdot \varepsilon_2 + B\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3, A, W_1, R_1, B \in \mathbb{R}\}$
The 12 root vectors :

$$\begin{aligned}
Y_1(\epsilon_{11},\epsilon_{12}) &= \tfrac{1}{2}\{i(\varepsilon_0) + \epsilon_{11}(\varepsilon_1) + i\epsilon_{12}(\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3) + \epsilon_{11}\epsilon_{12}(\varepsilon_0 \cdot \varepsilon_3 \cdot \varepsilon_2)\}, \epsilon_{ij} = \pm 1\\
Y_3(\epsilon_{31},\epsilon_{32}) &= \tfrac{1}{2}\{i(\varepsilon_0 \cdot \varepsilon_2) + \epsilon_{31}(\varepsilon_0 \cdot \varepsilon_3) + i\epsilon_{32}(\varepsilon_1 \cdot \varepsilon_3) + \epsilon_{31}\epsilon_{32}(\varepsilon_2 \cdot \varepsilon_1)\}, \epsilon_{ij} = \pm 1\\
Y_2(\epsilon_{21},\epsilon_{22}) &= \tfrac{1}{2}\{i(\varepsilon_2) + \epsilon_{21}(\varepsilon_3) + i\epsilon_{22}(\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_3) + \epsilon_{21}\epsilon_{22}(\varepsilon_0 \cdot \varepsilon_2 \cdot \varepsilon_1)\}, \epsilon_{ij} = \pm 1
\end{aligned}$$

are orthogonal (for the Hermitian product) and, with $T_1\mathfrak{T}$, span $Cl(\mathbb{C},4)$.
The value of the adjoint map for any vector T of $T_1\mathfrak{T}$ is given by :

$$\begin{aligned}
ad(T)(Y_1(\epsilon_{11},\epsilon_{12})) &= -(iW_1\epsilon_{11} + B\epsilon_{12})Y_1(\epsilon_{11},\epsilon_{12})\\
ad(T)(Y_3(\epsilon_{31},\epsilon_{32})) &= i(R_1\epsilon_{31} + W_1\epsilon_{31}\epsilon_{32})Y_3(\epsilon_{31},\epsilon_{32})\\
ad(T)(Y_2(\epsilon_{21},\epsilon_{22})) &= (iR_1\epsilon_{21} - B\epsilon_{22})Y_2(\epsilon_{21},\epsilon_{22})
\end{aligned}$$

To any vector subspace V_j it is then possible to associate a fundamental weight Γ_j, and if $(V_1, \vartheta), (V_2, \vartheta)$ are the representations associated to the particles 1,2 the tensorial product $(V_1 \otimes V_2, \vartheta \times \vartheta)$ of a composite particle is an irreducible representation with the weight $\Gamma_1 + \Gamma_2$. And conversely the composite particle can decay in two elementary particles.

Of course the decay occurs only if the balance of energy is favorable, which can involve an supply from the fields.

3 THE PROPAGATION OF THE FIELD AND THE BOSONS

3.1 The implementation of the Principle of Least Action

The force field propagates in the vacuum (that is where there is no particle) by interacting with itself. This phenomenon is modeled by implementing the Principle of Least Action and involves a special derivative of the potential, the strength of the field $\mathcal{F}$. It is useful to remind its definition as it is at the core of the representation of bosons.

3.1.1 The strength of the field

The potential is a tensor $T : TM \to T_1\widehat{U} :: \sum_{\alpha=1}^{16}\sum_{\beta=0}^{3} T_\beta^\alpha E_\alpha \otimes d\xi^\beta$ which is a one form on M valued in the vector bundle P_{Cl}. Its derivative is computed through a Lie derivative, along a curve $q : \mathbb{R} \to M :: q(\tau)$ with tangent V. The basic idea is to define the derivative at $q(\tau)$ as the limit of $\frac{1}{h}(T(q(\tau+h)) - T(\tau-h))$ when $h \to 0$. Because the referential are not the same at $q(\tau+h)$ and $T(\tau-h)$, to compute the difference it is necessary to "transport" the tensor along the curve, using its flow Φ_V, by two operations called "push-forward" $\Phi_V(q(\tau-h), h)^* T(q(\tau-h))$ and "pull-back"

$\Phi_V(q(\tau+h), -h)^* T(q(\tau+h))$ which provide two "copies" of the tensor located at the same point $q(\tau)$, expressed in the same referential, so that the quantity

$\frac{1}{h}\Delta T = \frac{1}{h}\left(\Phi_V(q(\tau+h), -h)^* T(q(\tau+h)) - \Phi_V(q(\tau-h), h)^* T(q(\tau-h))\right)$

can be computed. Then the derivative, called the Lie derivative, is just $\pounds_V T = \lim_{h\to 0}\frac{1}{h}\Delta T$. There are some complications because the curve is actually on a vector bundle, but the principle is the same (see Th.Physics p.216,334). The result is a 2 form on M valued in the vector bundle

$\mathcal{F} = \sum_{\{\alpha,\beta\}=0}^{3}\sum_{\gamma=1}^{16}\mathcal{F}_{\alpha\beta}^{\gamma} d\xi^\alpha \wedge d\xi^\beta \otimes E_\gamma \in \Lambda_2(M; P_{Cl})$

which has the property that, in a change of gauge, it transforms by the linear map $Ad_g, g \in Spin(3,1)$ and not by an affine map as the potential. As a consequence it can figure explicitly in a Lagrangian.

$\mathcal{F}_{\alpha\beta}^{\gamma} = \partial_\alpha T_\beta^\gamma - \partial_\beta T_\alpha^\gamma + [T_\alpha, T_\beta]^\gamma$

The bracket comes from the fact that $\frac{dg}{d\tau} = g \cdot T$.

For the scalar component representing the EM field one gets back a scalar 2 form.

In the procedure above it is possible that the quantity ΔT does not converge to 0 when $\tau \to 0$. Then we have a discontinuity.

3.1.2 The model of particles and field interacting

It requires the definition of a Lagrangian for the field, which takes the form of a scalar product $\langle \mathcal{F}, \mathcal{F} \rangle$. This is done in 2 steps.

i) On a manifold endowed with a metric there is a scalar product of scalar valued 2 forms : $\langle \mathcal{F}, \mathcal{K} \rangle = \sum_{\{\alpha,\beta\}} \mathcal{F}^{\alpha\beta} \mathcal{K}_{\alpha\beta}$ where the indices are raised with the metric.

ii) On the Lie algebra one can use the Killing form, which is preserved by the adjoint map with $Spin(3,1)$, but here it is better to use the Hermitian product because it accounts for the scalar component.

Then the scalar product is :

$\langle \mathcal{F}, \mathcal{K} \rangle = \sum_{\{\alpha,\beta\}} \left\langle \mathcal{F}^{\alpha\beta}, \mathcal{K}_{\alpha\beta} \right\rangle_H$

For a system with N particles one can define a section $\psi \in \mathfrak{X}(P_{Cl})$ for each type of particles, then the full Lagrangian over an area Ω followed by the observer from $t = t_1$ to $t = t_2$ is :

$L = \int_\Omega \langle \mathcal{F}, \mathcal{F} \rangle \varpi_0 + \sum_j \int_{t_1}^{t_2} \sum_{\beta=1}^{16} Q_j^\beta T_\alpha^\beta (q_j(t)) V_j^\alpha (q_j(t)) dt$

with the velocity V_j (which is a variable), the charge vector $Q_j \in Cl(\mathbb{C}, 4)$ of each particle and a volume density ϖ_0 computed from the metric.

The condition for an equilibrium is that L is stationary. A rigorous solution can be found by using the method of variational derivatives (see Maths.7.6.1) which can be extended to fiber bundles. The equations introduce the currents for the field $\phi = \sum_{\alpha,\beta=0}^{3} \left[\mathcal{F}^{\alpha\beta}, T_\beta\right] \partial\xi_\alpha \in TM \otimes P_{Cl}$ and for the particles $J = V \otimes Q \in TM \otimes P_{Cl}$ and the equations sum up to : $J = \phi$ and $d(*\mathcal{F}) = 0$ with the Hodge dual $*\mathcal{F}$.

The model assumes that there is no collision, and we have seen how to deal with them.

When there are particles the solution for the field are actually "distributions" : they are maps T acting on sections $\psi \in \mathfrak{X}_c(P_{Cl})$ with compact support such that they give the expected result. So they can be seen as operators acting on a Hilbert space as it is usual in QTF, but the result comes from a rigorous implementation of mathematical theorems, and not from a physical assumption on the nature of fields.

However the model does not answer all the questions.

i) The Principle of Least Action is based on the balance of energy, at equilibrium, between the components of the system. It provides - complicated - partial differential equations with initial conditions which should be known on a spatial hypersurface, which is generally impossible. Anyway the purpose of the Physicist is not to know the value of the field as it can emanate from far away sources, but to forecast the interactions between definite components.

ii) It is obvious that the interactions between a field, assumed to exist everywhere and continuous, and point-wise particles, introduce discontinuities. The state of the particles changes, usually continuously, and they carry this change over their trajectory. For the field this is more complicated : the change must be smeared out by propagation, which is not an instantaneous process.

And this leads to the next subsection.

3.2 The propagation of the field in the vacuum

The Principle of Least Action provides a single equation $d(*\mathcal{F}) = 0$ (or equivalent in any theory such as the Maxwell's equations) for the propagation of the field in the vacuum. Because the equations are complicated usually one uses special solutions such as "plane waves", but an attentive look at the case of the EM field, the only one for which we can proceed to reliable experiments, tells us that this not all the story.

3.2.1 The speed of light

One of the strongest result of Physics is that "light propagates a constant spatial speed c". But on what experiments is it based ?

Their principle is that a small variation of the field (a signal) occurs at some point, it is detected at different points with some delay, from which one can compute an apparent spatial speed. The conclusion comes from the facts :

i) the signal can be acknowledged : it can be attenuated, or distorted (by the Doppler effect for instance), but it is recognizable.

ii) the signal reaches different points, so it follows different 4 dimensional curves, whose spatial length can be computed.

iii) there is a constant relation between the spatial length and the time delay, even when the observers are in motion : the velocity $V = \frac{dq}{d\tau}$ of the propagation of the signal on the curve is such that the Lorentz scalar product is null : $\langle V, V \rangle = 0$.

These results are not at all obvious, and they are certainly not a consequence of the previous model. There should be a unique value of the field at a given point, and there is no reason why it should keep the "memory" of some signal in the past. A far away star does not give us any favor in dispensing its energy, but we can precisely guess its EM field. Many readers will jump to the classic answer : these results come from the propagation of a photon. But this is to answer the question by a riddle as we do not know what a photon is. When Bob calls Alice on her mobile phone, they do not exchange photons between their mobiles, but a signal whose propagation has been engineered using well known fields equations. And the First Principle of Optics says that "light propagates in straight lines". In General Relativity it is usually assumed that light propagates along geodesics, that is curves such that the covariant derivative of its tangent, using the connection of the gravitational field, is null. This is seen as the generalization of the idea that "light propagates along curves of shortest length", but this holds only if the connection is special (the Levy-Civita connection), and anyway in a GUT there is no reason to privilege the gravitational field.

3.2.2 The lines of propagation of the field

The physical part of the Geometry of the Universe is the metric. It is assumed to be defined everywhere and change with the location. But, for any manifold endowed with a metric, there are special curves of tangent V along which the metric g is transported by their flow : the Lie derivative $\pounds_V g = 0$. Through any point there are infinitely many such curves, which are integral curves of special vector fields, Killing vector fields, which have many properties (they preserve the scalar product of vectors, then the tetrads, and the volume form). They are the infinitesimal generators of isometries (maps whose derivative preserve the metric) which themselves constitute a Lie group, representing the physical symmetries of the Universe, which are, locally, at most 12. Because the standards are fixed through the principal bundle P_G and the tetrad, they are naturally preserved along a Killing curve, that is in the propagation of the field.

So, my assumption is the following :

Proposition 13 *The force field propagates along Killing curves.*

It has some important practical consequences. Any chart that we can conceive to locate a point (such as in Astrophysics) is based on the propagation of light. The fact that the spatial axes are also Killing curves gives a special form to the metric expressed in the chart, from which we have differential equations for the metric. And in common models, such as the study of the field in the environment of a particle or a material body, we can assume a rotational symmetry, the propagation lines are radial, as well as the Killing curves.

In a full model encompassing the metric, it appears that the metric itself depends only on the value of field at each point, which makes sense as they are the only quantities defined everywhere. This supports the proposition above.

The proposition does not answer the question of the speed of propagation. The condition for Killing curves $\pounds_V g = 0$ provides, for a given metric, 10 linear PDE, the condition for a speed c provides an additional equation $\langle V, V \rangle = 0$, with 20 parameters for the initial conditions, there is still plenty of room for the definition of lines of propagation. The propagation of the EM field can be a special case. It is generally assumed that the gravitational field propagates also at c (this is essential in cosmological models), so it seems sensible to extend the conclusion to a GUT.

Because there are infinitely many lines of propagation going through a point, the propagation of a signal occurs on a 4 dimensional cone, future oriented, with apex the source, and the energy is dissipated on 2 dimensional surfaces. As a consequence it decreases as the inverse of the square to the spatial distance from the source, as it is well known for the EM and the gravitational fields. The weak and strong fields seem to follow very different laws but they do not come from experimental measures. They appear in some experiments, and in a GUT the density of energy, which should, over all, stay constant on a propagation line, is a complicated mix of all the components of $\mathcal{F}$ in $T_1\widehat{U}$. So this issue is open.

3.3 Bosons

An analogy will help to understand the model : shock waves in a continuous medium. They can be easily modeled in General Relativity. A continuous medium is represented by material points traveling on integral curves of a common vector field V, their location is given by the value of the parameter τ on the curve and their initial position x. Some process, such as a vibration, occurs at each material point and is represented by a quantity $Y(\tau)$ in some vector bundle. A shock wave is a discontinuity in the derivative $\frac{dY}{d\tau}$. Its propagation can be modelled by a function $\vartheta(x) = \tau$ which gives, for each material point, the time at which it has been reached by the shock wave. It is then located at the points $\omega(t) : t = \vartheta(x)$, that is on a 3 dimensional hypersurface $\Omega_3(t)$. The discontinuity appears as a quantity which is added to the regular derivative, and is located on $\Omega_3(t)$, that is by a "Dirac's function".

3.3.1 Definition of a boson

When the field interacts with a particle a discontinuity appears. It is usually smeared out, and anyway it is in part artificial, a consequence of a model with point-wise particles. The possibility to detect the discontinuity depends on the scale of the observation, on its magnitude and on the speed of propagation, but one can expect that in some cases it is "too big to be smeared out". This happens notably when the interaction itself involves a significant amount of energy, such as in the photo-voltaic effect (an electron is extracted from its shell) or the break down of a particle. In a continuous medium, built around a vector field, a shock wave propagates on a 3 dimensional hypersurface. For a force field, defined everywhere, the propagation occurs on one dimensional lines. And we can assume naturally that they are the same as the usual lines of propagation of the field. They behave similarly to particles : they are the bosons.

Their representation comes from the definition of $\mathcal{F}$: the field is continuous but its derivative is not, and the discontinuity is represented by some $\Delta T \in TM^* \otimes T_1\widehat{U}$ propagating along a Killing curve and which is added to the regular derivative $\mathcal{F}$. At any point there is a unique Killing curve with a given tangent V, V is part of the definition of the boson, which is a one form at the difference of $\mathcal{F}$. The attenuation of a signal is due to its propagation on 3 dimensional hypersurfaces, we have nothing similar for bosons, which propagate without attenuation : the boson is transported along the curve with the law $\pounds_V \Delta T = 0$.

Proposition 14 *Bosons are discontinuities in the derivative of the field. They can be represented as one form $\Delta T \in \Lambda_1\left(M; T_1\widehat{U}\right)$ with support a Killing curve, and they propagate along this curve at the same speed as the field, by transport such that $\pounds_V \Delta T = 0$.*

3.3.2 Characteristics of bosons

As a consequence of the law $\pounds_V \Delta T = 0$ the quantity

$B_T = \sum_{\alpha}^{3} \Delta T_{\alpha}^{\beta} V^{\alpha} E_{\beta} \in T_1\widehat{U}$

is constant, this is a vector of P_{Cl} which is similar to the fundamental state ψ_0 of a particle.

Bosons can be quantized in a way similar to particles. The observable is B_T. In a change of gauge in P_{Cl} with $Spin(3,1)$ it transforms as a vector of $Cl(\mathbb{C}, 4)$

by a linear map, and not an affine map as the potential, because it is defined by a difference. So this is an element of P_{Cl}, belonging at each point m to the Lie algebra $T_1\widehat{U}$. At a difference with particles, it is not assumed that there is an action of $\widehat{U}$ on the fundamental state B_T. However the different types of particles and fields appear, layer after layer, in experiments which involve parts of the unified field $\widehat{U}$. Similarly bosons manifest themselves when these parts of the fields are involved, so one can associate bosons to each type of field, such that B_T belongs to the corresponding subalgebra of $T_1\widehat{U}$.

Bosons are discontinuities of the derivative $\mathcal{F}$ of the potential, so we can define the energy of a boson in a similar way :

$\Delta T = \sum_{\alpha=1}^{16} \sum_{\beta=0}^{3} \Delta T_\beta^\alpha d\xi^\beta \otimes E_\alpha$

We compute first the Lorentz scalar product of the one form with the metric for $\alpha = 1...16$: $\left\langle \sum_{\beta=0}^{3} \Delta T_\beta^\alpha d\xi^\beta, \sum_{\beta=0}^{3} \Delta T_\beta^\alpha d\xi^\beta \right\rangle_{TM} = \sum_{\beta=0}^{3} \Delta T_\beta^\alpha \Delta T^{\alpha,\beta}$

then the scalar product with the Hermitian form on $Cl(\mathbb{C},4)$:

$\langle Z, Z\rangle_H$ with $Z = \sum_{\alpha=1}^{16} \left(\sum_{\beta=0}^{3} \Delta T_\beta^\alpha \Delta T^{\alpha,\beta} \right) E_\alpha \in Cl(\mathbb{C},4)$

Because the boson propagates on a Killing curve, on which the Lorentz scalar product is constant, the first result is constant. Then the second scalar product, with the fixed Hermitian form, is also constant : the energy of a boson is constant, as was expected. In the quantization, the bosons which are observed correspond to definite levels of energy.

Because bosons appear with a definite level of energy, and the quantity $B_T \in Cl(\mathbb{C},4)$ is constant, one could assign charges to a boson, by taking each component of B_T. But this is just by similarity to particles. Bosons are not sources of the field. They interact with particles, but in a more complicated way, because they have a precise trajectory : their interaction is more similar to a collision between particles.

One can assign a momentum to a boson : $\mathcal{M}_B = B_T \otimes V$. Then its interaction with a particle is modelled as a collision : there is conservation of the total momentum and energy. Usually the particle keeps its fundamental state, and the boson disappears, as in the photo-electric effect. But the boson can survive, as in the Comton effect, then its fundamental state and its energy are modified.

4 CONCLUSION

In the model several choices have been made, somewhat arbitrary. Within the same framework other solutions could be proposed. It is still necessary to confront the results, expressed here in a general format, with the precise data available for the fermions. The rules for the process of decay are known, it manifests itself in a great variety of cases, which can be explored using the tools given here, and this opens a large field of research which is necessary for the validation of the model.

Some issues have a beginning of an answer, but claim a specific attention.

The propagation of the weak and strong fields is still not clear. One interest of a GUT is that it opens the possibility to explore the interactions between the components of the field. It is of paramount importance, practically and theoretically, to know if the gravitational field interacts in some way with the other components.

A full model incorporating the metric can be built using the proposed representation. It will be similar to the ones that I have presented in my book. The variation of the metric through the tetrad, using the relations given by the propagation along Killing vector fields, provides equations from which the metric can be computed from the field, without differential equations.

The concept of a boson as discontinuity in the field is robust. It provides the only sensible explanation which do not involve a new physical object. The model should be adjusted to the different known bosons. It is clear that the Higgs mechanism has no place here, but it could be useful to check what phenomenon its "discovery" means.

Overall the model presented helps to understand better the mechanisms involved at the level of elementary particles, it is relatively simple and computations are possible without additional hypotheses, from this point of view it is a great improvement from the Standard Model. This comes from the fact that the model is the extension of models, such the spinor and the Geometry of General Relativity, which work at any scale : the GUT is a nice prolongation of well proven theories. It is not a dramatic rupture with known concepts and Principles. But we must never forget that this is just a model, not the Reality itself.

5 BIBLIOGRAPHY

H.Araki *Mathematical theory of quantum fields* Oxford Science Publications (2000)

M.Le Bellac *Physique quantique* CNRS (2003)

A.Berlinet, C.Thomas-Agnan *Reproducing kernel, Hilbert spaces in probability and statistics* Springer (2004)

J.D.Bjorken,S.D.Drell *Relativistic quantum fields* Mc Graw Hill (1965)

N.Bogolubov,A.A.Logunov, A.I.Ossak, I.T.Todorov *General principles of quantum fields theory* Kluwer (1990)

O.Bratelli, D.W.Robinson *Operators algebras and quantum statistical mechanics* Springer (2002)

R.Coquereaux *Clifford algebra, spinors and fundamental interactions : twenty years after* arXiv:math-ph/0509040v1 (16 sep 2005)

W.N.Cottingham and D.A.Greenwood *An introduction to the standard model of particles physics 2nd edition* (2007) Cambridge University Press

P.Dechant *Clifford algebra is the natural framework for roots systems and Coxeter groups* arXiv:math-ph/1602-06003v1 (18 feb 2016)

P.A.M.Dirac *The principles of Quantum Mechanics* Oxford Science Publications (1958)

J.C.Dutailly *Mathematics for theoretical physics* (2016) Amazon E-book

J.C.Dutailly *Theoretical Physics* 2nd Edition (2017) Amazon book

R.P.Feynman, A.R.Hibbs *Quantum Mechanics and Path Integrals* Dover (2005)

Francis C.E.H *A construction of full QED using finite dimensional Hilbert space* EJTP 10 N°28 (2013)

Francis C.E.H *The Hilbert space of conditional clauses* arXiv:1205-4607 (2013)

G.Giachetta,L.Mangiarotti, G.Sardanashvily *Advanced classical field theory* World Scientific (2009)

M.Guidry *Gauge fields theories* Wiley (1991)

H.Halvorson *Algebraic quantum fields theory* arXiv:math-ph/0602036v1 14 feb 2006

S.S.Horuzhy *Introduction to algebraic quantum field theory* Riedel (1989)

J.M.Jauch *Foundations of Quantum Mechanics* AddisonWesley (1968)

O.Jefimenko *Causality, Electromagnetic Induction, and Gravitation* Electret Scientific, Star City, (2000)

A.W.Knapp *Lie groups beyond an introduction* Birkhäuser (2005)

I.Kolar, P.Michor, J.Slovak *Natural operations in differential geometry* Springer-Verlag (1991)

S.Kopeikin, B.Mashloon *Gravitomagnetic effects in the propagation of electromagnetic waves in variable gravitational fields of arbitrary moving and spinning bodies* arXiv / gr-qc / 0110101 v2 (25 oct 2001)

F.Laloë *Comprenons-nous vraiment la mécanique quantique ?* CNRS Editions (2011)

A.Garrett Lisi *An explicit embedding of gravity and the standard model* arXiv: gr-qc 1006498v1 (25 june 2010)

A.Garrett Lisi *An exceptionnaly simple theory of everything* arXiv:0711.0770v1 [hep-th] (6 nov 2007)

A.Garrett Lisi *Clifford bundle formulation of BF gravity generalized to the standard model* arXiv:gr-qc 0511120v2 (21 nov 2005)

R.Haag *Local quantum physics* 2nd Ed.Springer (1991)

G.Mackey *The mathematical fundations of Quantum Mechanics* W.A.Benjamin (1963)

M.A.Naimark *Linear representations of the Lorentz group* Pergamon (1964)

E.Negut *On intrinsic properties of relativistic motions* Revue Roumaine des Sciences Techniques (1990).

J.von Neumann *Mathematical Foundations of Quantum Mechanics*, Beyer, R. T., trans., Princeton Univ. Press. (1996 edition)

L.Nielsen *A Maxwell Analog Gravitation Theory* Niels Bohr Institute, Copenhagen, Gamma No. 9 (1972).

Y.Ohnuki *Unitary representations of the Poincaré group and relativistic wave equations* World Scientific (1988)

R.Omnès *The interpretation of quantum mechanics* Princeton (1994)

P.J.E.Peebles *Principles of physical cosmology* Princeton series in Physics (1993)

E.Poisson *An introduction to the Lorentz-Dirac equation* arXiv:gr-qc/ 9912045v1 (10 dec 1999)

S.Pokorski *Gauge fields theories* Cambridge (2000)

K.Popper *Quantum theory and the schism in physics* Routledge (1982)

R.Pöttgen *Search for dark matter with Atlas* (2016) Springer Theses

T.C.Quinn *Axiomatic approach to the radiation reaction of scalar point particles in curved space-time* arXiv:gr-qc/0005030v1/ (10 may 2000)

G.Röpke *Non equilibrium statistical physics* Wiley-vch (2013)

I.Schnaid *Wave function perturbations propagation in multi particles system* arXiv 1307.2510v1 [physics-gen.ph] (9 july 2013)

T.Schücker *Forces from Conne's geometry* arXiv:hep/th/0111236v3 (26 march 2007)

Mesgun Sehbatu *The standard model and beyond* (1992)

D.S.Shirokov *On some Lie groups containing spin groups in Clifford algebras* arXiv:math-ph:1607-07363v2 (19 Aug 2017)

A.Smola, A.Gretton, L.Song, B.Schölkop *A Hilbert Space Embedding for Distributions*

G.Svetlichny *Preparation for gauge theories* arXiv:math-ph 9902027 v3 (12 Mar 1999)

M.Tajmar, C.J. de Matos *Coupling of Gravitation and Electromagnetism in the Weak Field Approximation* arXiv:gr-qc/0003011 (2003)

H.Torunczyk *Characterizing Hilbert spaces topology* Fundamental mathematica (1981)

R.M.Wald *General Relativity* University of Chicago (1984)

S.Weinberg *The quantum theory of fields* Cambridge University Press (1995)

S.Weinberg *Dreams of a Final Theory* Pantheon Books (1992)
H.Weyl *The theory of groups and quantum mechanics* Dover (1931 / 1950)

6 ANNEX

Particles Physics relies heavily on mathematical concepts which are very technical, and rendered more difficult by the use of non conventional notations and definitions. So it is useful to remind some mathematical results.

6.1 Lie algebra and Lie groups

see Maths.5 for more.

6.1.1 Lie algebra

A Lie algebra is a vector space L endowed with a bilinear, antisymmetric, map, the bracket $[] : L \to L$ which satisfies the Jacobi identity : $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$. From which one defines the map $ad : L \to \mathcal{L}(L; L) :: ad(X)(Y) = [X, Y]$.

$ad_{[X,Y]} = [adX, adY]$

A Lie algebra is abelian if $[X, Y] = 0$.

A Lie subalgebra L_0 is a vector subspace which is also a Lie algebra : the bracket is closed : $\forall X, Y \in L_0, [X, Y] \in L_0$.

An ideal is a subalgebra L_0 such that : $\forall X \in L, Y \in L_0, [X, Y] \in L_0$. The center Z of L is the ideal : $\forall X \in Z, \forall Y \in L : [X, Y] = 0$.

The set $[L, L] = Span\{[X, Y], X, Y \in L\}$ is a Lie algebra and an ideal. For any subalgebra S the set $S^k = Span\left\{[X, Y], X, Y \in S^{k-1}\right\}$ defined by recursion, is a Lie subalgebra. It is said to be solvable if there is k such that $S^k = 0$. In any Lie algebra L there is an ideal, possibly null, which contains all the solvable ideals and the center of L, called the radical of L.

The Killing form is the 2 symmetric form $K(X, Y) = Tr(ad(X) \circ ad(Y))$. It is preserved by any automorphism.

If $(L_1, []_1), (L_2, []_2)$ are 2 Lie algebras, the sum $\left(L_1 \oplus L_2, []_{1+2}\right)$ is the Lie algebra defined as the vector space $L_1 \oplus L_2$ endowed with the bracket :

$[X_1 + X_2, Y_1 + Y_2]_{1+2} = [X_1, Y_1]_1 + [X_2, Y_2]_2$ and then L_1, L_2 are ideals in $\left(L_1 \oplus L_2, []_{1+2}\right)$. As a consequence, whenever a Lie algebra is a direct sum $L = \oplus_j L_j$, as vector space, and L_j are subalgebras of L with the common bracket, then $L = \oplus_j L_j$ as Lie algebra.

If L_0 is an ideal then L/L_0 is a subalgebra.

Any Lie algebra can be uniquely decomposed as the sum $L = L_0 \oplus L_1 \oplus L_2 ... \oplus L_n$ of subalgebras which are also ideal. L_0, is the radical. The quotient space $L/L_0 = L_1 \oplus L_2 ... \oplus L_n$ is a semi simple algebra. More generally a Lie algebra is semi-simple if its Killing form is non degenerate.

6.1.2 Lie group

A Lie group is a set which is a manifold, endowed with a continuous operation $\cdot$ for which it has the algebraic structure of a group. All finite dimensional Lie groups are smooth, and most of them (at least those encountered in Physics) are isomorphic to a group of matrices (but this is not a rule, at the difference with Lie algebras).

As any manifold a Lie group G has a tangent space TG and vector fields : $T : G \to TG$. The product defines a left and a right translation : $L_g : G \to G :: L_g(x) = g \cdot x, R_g : G \to G :: R_g(x) = x \cdot g$. and their derivatives which are linear map on TG.

A vector field $X : G \to TG$ is left invariant if $X(g) = L'_g(1)(X(1)) \Leftrightarrow X(g) = L'_g(1)(T); T \in T_1G$. The set of left invariant vector fields has the structure of a Lie algebra : any left invariant vector field can be defined from its value in the tangent space T_1G at $g = 1$ (which justifies our notation T_1G for the Lie algebra of a Lie group) and $[X,Y](g) = L'_g(1)[X(1), Y(1)]$

The adjoint map is the automorphism on the Lie algebra :

$Ad : G \to \mathcal{L}(T_1G; T_1G) ::$

$Ad_g T = L'_g(g^{-1}) \circ R'_{g^{-1}}(1)(T) = R'_{g^{-1}}(g) \circ L'_g(1)(T)$

and any continuous automorphism on the Lie algebra is of this type. It preserves the bracket on the Lie algebra.

6.1.3 Exponential

As for any manifold one can define on a Lie group the flow of a vector field, this is the family of maps :

$\Phi_V : \mathbb{R} \times G \to G :: \Phi_V(\tau, x)$ such that : $\frac{d}{d\tau}\Phi_V(\tau, x)|_{\tau=\theta} = V(\Phi_V(\theta, x))$

The exponential map on a Lie group is the flow of a left invariant vector field :

$\exp : \mathbb{R} \times T_1G \to G :: \exp \tau T = \Phi_X(\tau, 1)$

thus : $\frac{d}{d\tau} \exp \tau T|_{\tau=\theta} = L'_{\exp \theta T}(1) T = R'_{\exp \theta T}(1) T \Rightarrow \frac{d}{d\tau} \exp \tau T|_{\tau=0} = T$

The exponential is a diffeormorphism in the connected component of the identity, and the generator of one parameter groups. It is such that :

$\exp 0T = 1$

$\exp(\tau + \tau') T = \exp \tau T \cdot \exp \tau' T \Rightarrow (\exp \tau T)^{-1} = \exp(-\tau) T$

but $\exp(T + T') = \exp T \cdot \exp T'$ only if $[T, T'] = 0$.

$\exp Ad_g T = g \cdot \exp T \cdot g^{-1}$

$Ad_{\exp T} = \exp ad(T) \in \mathcal{L}(T_1G; T_1G)$

If f is a morphism between the groups G, H, then $f'(1)$ is a Lie algebra morphism between T_1G, T_1H and $\forall T \in T_1G : f(\exp_G T) = \exp_H(f'(1)T)$

The set generated by exp is the connected component of G. Conversely any Lie algebra is the Lie algebra of some Lie group. More generally, a Lie group is a manifold, thus it has charts. Let G be a connected Lie group such that ;

$T_1G = T_1H \oplus V$

where H is the Lie algebra of a subgroup H of G, and V any vector subspace of T_1G. Then the map :

$\varphi : H \times V \to G :: \varphi(h, v) = h \cdot \exp v$

is a smooth chart of G and a local diffeomorphism (Kolar p.45). $\exp v$ is understood as the restriction of exp : $T_1G \to G$ to the vector space V. The set G/H is a group if H is normal, and if not it is a submanifold called a homogeneous space.

6.1.4 Casimir operators

Casimir operators are actually elements of second order of the universal enveloping algebra of a Lie algebra, they are defined on the tensorial product of a Lie algebra. Their image in any representation (E, f) is given by a polynomial of $f(\kappa_j)$, they commute with $f(X), \forall X \in L$ and acts on E by a scalar, which depends only of the dimension of E. So they can be used to label the fundamental representations.

6.2 Representations of groups and algebras

6.2.1 Definition

An algebraic representation of a Lie algebra L is a couple (A, f) of an algebra A and a smooth morphism $f : L \to A$ such that :

$f([X, Y]) = f(X) f(Y) - f(Y) f(X)$

An algebraic representation of a Lie group G is a couple (A, F) of an algebra A and a smooth morphism $F : G \to A$ such that :

$F(X \cdot Y) = F(X) F(Y)$

If the algebra is an algebra of matrices, necessarily finite dimensional, then this is a matricial representation. Any Lie algebra has a representation on an algebra of matrices, this is not true for Lie groups.

A geometric representation on a vector space E is a representation where the algebra is the set of linear maps $\mathcal{L}(E; E)$.

A finite dimensional n representation on a set of matrices gives a geometric representation on $\mathbb{R}^n$ or $\mathbb{C}^n$ where the matrices act on column vectors the usual way. For Lie algebra or groups of matrices this is the standard representation. Conversely a geometric representation on a finite dimensional vector space E gives a matricial representation by fixing a basis.

A geometric representation (E, f) gives the contragredient representation $\left(E^*, f^{-1}\right)$ on the dual, and usually the two representations are not equivalent. Similarly with the conjugate transpose for complex structures.

A functional representation is a representation on an algebra of functions, where the Lie algebra or group acts on the arguments. It is finite dimensional on an algebra of polynomials.

Any Lie algebra has its adjoint representation (L, ad) on itself. Any Lie group has its adjoint representation $(T_1 G, Ad)$ on its Lie algebra.

Any representation (A, f) of a Lie algebra gives a representation (A, f_{L_0}) of a Lie subalgebra L_0 by restriction of f to L_0. Similarly for the representation of a subgroup.

Two representations $(A_1, f_1), (A_2, f_2)$ are equivalent if there is an isomorphism $\phi : A_1 \to A_2$ such that $\phi \circ f_1 = f_2 \circ \phi$.

A representation $(A, f), (A, F)$ is irreducible if the only subsets globally invariant by the action are 0 and A.

Any representation (A, F) of a Lie group gives a representation $(A, F'(1))$ of its Lie algebra, conversely a representation (A, f) of a Lie algebra gives a representation of a universal cover of the Lie group through the exponential by $F(\exp T) = \exp f(T)$.

6.2.2 Principal bundle

Any principal bundle $P_U(M, U, \pi_U)$ gives, with the geometric representation (E, ϑ) of the Lie group U an associated vector bundle, denoted $P_U[E, \vartheta]$ whose elements are a couple $(p(m), V)$ of a gauge at the point m and a vector V of E, with the equivalence relation in a change of gauge

$p(m) \to \widetilde{p}(m) = p(m) \cdot \varkappa(m)^{-1} \Rightarrow (p(m), V) \sim (\widetilde{p}(m), \vartheta(\varkappa(m))V)$

In particular for any principal bundle we have the adjoint bundle $P_U[T_1U, Ad]$.

6.2.3 Sum and product of representations

The sum of geometric representations (E_i, F_i) of a Lie group is given by the action on $\oplus_{k=1}^{l} E_k$

$(F_i \oplus F_j)(g)(u_i + u_j) = F_i(u_i) + F_j(u_j)$

Practically the sum of matricial representations $[M_i]$ is given by taking the block diagonal matrix : $[N_{i+j}] = \begin{bmatrix} [M_i] & 0 \\ 0 & [M_j] \end{bmatrix}$. And conversely a matricial representation is reducible if the matrices $[M]$ are similar to matrices of this form : $[N] = [Q][N_{i+j}][Q]^{-1}$ with a fixed matrix $[Q]$.

The tensorial product of geometric representations (E_i, F_i) of a Lie group is given by linear extension of the action on decomposable tensors of $\otimes_{k=1}^{l} E_k$:

$\Phi(g_1, ...g_k)(u_1 \otimes ... \otimes u_k) = F_1(u_1) \otimes ... \otimes F_k(u_k)$

and then it is convenient to use the Kronecker product of matrices : $[F_1] \otimes ... \otimes [F_k]$ (Maths.2.5.1) with dimensions the product of the dimensions.

Sum and tensorial products of representations of a Lie algebra are defined similarly by derivation of the action. The matrix of the representation of the tensorial product of Lie algebras representations is given by the Kronecker sum $[f_i] \oplus [f_j] = [f_i] \otimes I_{\dim f_j} + I_{\dim f_I} \otimes [f_j]$.

So representations are far from being unique.

An algebraic representation $(A, f), (A, F)$ of a Lie algebra or Lie group is said to be reducible if it is the sum of irreducible representations. In Quantum Physics it is common to say then that the representation degenerates and to call multiplets the components of the sum. All finite dimensional representations of complex semi-simple Lie algebras can be expressed as the sum of finite dimensional irreducible representations. Representations of Lie groups are in correspondence with the representations of their Lie algebras. So all representations are defined through irreducible representations of Lie algebras, which are the elementary bricks. One issue is then to identify and label the irreducible representations, and this is where the Cartan algebras are useful.

6.3 Root space decomposition of a Lie algebra

6.3.1 Cartan algebra

As any vector space a Lie algebra has a basis : $X = \sum_a X^a \overrightarrow{\kappa}_a$, and the components of the brackets, called structure coefficients C^a_{bc}, are expressed as : $[X, Y] = \sum_{a,b,c} C^a_{bc} X^b Y^c \overrightarrow{\kappa}_a$. Due to the Jacobi identity, they are related, but not linearly. In other words the elements of a Lie algebra can be expressed as a linear combinations of specific vectors, called generators, or their brackets. Their identification is done by a procedure called a root space decomposition and the existence, in any *semi-simple complex Lie algebra*, of a Cartan subalgebra H which has the following properties (more on this topic in Maths.5.1.3) :

i) H is abelian : $\forall h, h' \in H : [h, h'] = 0$.

ii) There is a set $(t_k)_{k=1}^{m-\dim H}$ of linearly independent vectors, which are common eigen vectors of $ad(h)$, the corresponding eigen spaces T_k are unidimensional, and the eigen values are linear functions $\alpha_k(h)$ of h :

$\forall k = 1...m - \dim H, h \in H : ad(h) t_k = [h, t_k] = \alpha_k(h) t_k$

The functions $\alpha_k \in H^*$, called the roots, do not depend on the choice of t_k - the eigen spaces T_k are unidimensional - and it is customary to label the eigen spaces T_k by α itself, and denote $\Delta(L) = \{\alpha \in H^*\}$. Then the vectors t_α, called the root vectors, are the dual of α, defined for instance through the Killing form K of L (which is non degenerate because L is semi simple) : $K(t_\alpha, X) = \alpha(X)$.

iv) $[T_\alpha, T_\beta] = T_{\alpha+\beta}$

iii) $L = H \oplus Span\left\{_{\alpha\in\Delta(L)} T_\alpha\right\}$

The set $\Delta(L)$ of roots is somewhat redundant, they are not necessarily independent, and complicated, but one shows (Maths.1714) that it can be ordered and reduced to a set of l simple roots $\Pi(L) = \{\alpha_k\}_{k=1}^l$ such that :

$\Pi(L)$ is a basis of H^*

$\forall \beta \in \Delta(L), \exists n_k \in \mathbb{N}$:

either $\beta = \sum_{k=1}^l n_k \alpha_k$ and one says that $\beta \in \Delta_+(L)$

or $\beta = -\sum_{k=1}^l n_k \alpha_k$ and one says that $\beta \in \Delta_-(L)$

$\beta \in \Delta_+(L) \Rightarrow -\beta \in \Delta_-(L)$

$\Delta(L) = \Delta_+(L) \oplus \Delta_-(L)$

By taking a basis $(h_i)_{i=1}^l$ of H and a set of root vectors

$\{t_{\alpha_k}, t_{-\alpha_k}, \alpha_k \in \Pi(L), K(t_{\alpha_k}, X) = \alpha_k(X)\}$ one gets a set of generators of L.

All Cartan algebras of a given Lie algebra L have the same dimension l, called the rank of L, and are isomorphic, so the procedure can be implemented with any one of them.

There are only 9 possible systems of roots :

- 4 of them correspond to families of algebras of matrices :

$sl(n+1, \mathbb{C}), so(2n+1, \mathbb{C}), sp(2n, \mathbb{C}), so(2n, \mathbb{C})$ $n > 1$. Their rank is then n.

- 5 to exceptional cases corresponding to unique Lie algebra.

They are tabulated (see Knapp) and the simple roots are expressed as linear combinations such as $\{e_i - e_j\}$ where the vectors $e_i \in \mathbb{R}^n$ figure as an orthonormal basis (with the Killing form) of the dual h^*.

If the complex, finite dimensional Lie algebra L is not semi-simple, it is the sum of its radical and a semi-simple Lie algebra. The radical is represented on itself by a set of triangular matrices, so the results still hold.

Actually the generators have little utility by themselves : it is by far easier to identify an element of a Lie algebra by its components in a basis, as any vector, than by the generators, which leads quickly to intractable computations. The interest of these results lies in the representation of Lie groups.

6.3.2 Fundamental representations of a Lie group

The construct with Cartan algebra is just a particular representation (L, ad) of L on itself. Its properties can be extended to any geometric representation (E, f) of L :

i) $\forall h \in H, f(h)$ acts diagonally on E. There are 1 forms $\lambda \in H^*$, called weights, and vector subspaces E_λ called weight spaces, such that : $\forall h \in H, \forall v \in E_\lambda : f(h)v = \lambda(h)v$. The weights λ, which belongs to H^* as the roots α, are real valued for any real linear combination of vectors of H. The set of weights λ is denoted $\Delta(E)$. Meanwhile the eigen spaces $T_\alpha \subset L$ are unidimensional, the weight spaces $E_\lambda \subset E$ can have any dimension, called the multiplicity of the weight.

ii) E is the direct sum of the weight spaces : $E = \oplus_{\lambda \in \Delta} E_\lambda$.

iii) $\forall \lambda \in \Delta(E), \forall \alpha \in \Delta(L) : f(t_\alpha) E_\lambda \sqsubseteq E_{\lambda+\alpha}$ with the root vector $t_\alpha \in L$ associated to the root $\alpha : ad(h)(t_\alpha) = \alpha(h) t_\alpha$

As a consequence E itself can be generated by successive applications of $f(t_\alpha)$: one gets increasing vector subspaces. For this, one needs a starting point, given by a special vector.

There is a unique, up to multiplication by a scalar, vector $V \in E$ with the properties :

$\forall h \in H : f(h)V = \mu(h)V \Leftrightarrow V \in E_\mu$

$\dim E_\mu = 1$

$\forall \alpha \in \Delta_+(L), t_\alpha \in \Delta_+(L)^* : f(t_\alpha)V = 0,$

all the weights are smaller than $\mu : \forall \lambda \in \Delta(E) : \exists n_k \in \mathbb{N} : \lambda = \mu - \sum_k n_k \alpha_k, \alpha_k \in \Pi(L)$ thus μ is called the highest weight, and it depends only on the root system $\Delta(L)$.

The weights of the contragredient representation of a given representation are the opposite $w_j \to -w_j$ (Knapp p.339).

So the classification of all irreducible representations of a complex, semi-simple Lie algebra is based on the following procedure :

- compute the roots $\Delta(L)$ and choose a simple system of roots $\Pi(L) = \{\alpha_k\}_{k=1}^l$ such that $\Delta(L) = \Delta_+(L) \oplus \Delta_-(L)$

- associate to each simple root a root vector $t_k \in T_{\alpha_k} \subset L : [h, t_k] = \alpha_k(h) t_k$

- the fundamental weights $(w_i)_{i=1}^l$ are normalized versions of the simple roots, by : $K^*(w_i, \alpha_j) = \frac{1}{2}\delta_{ij} K^*(\alpha_i, \alpha_j), j = 1...l$ where K^* is the dual of the

Killing form K on L (it is defined by the same coefficients as K) so that $(w_i)_{i=1}^l$ is an orthogonal basis of the dual H^*.

- any combination of l integers $(n_1, .n_k.., n_l)$, $n_k \in \mathbb{N}, n_1 \geq n_2.. \geq n_l$ defines a highest weight $W = \sum_{k=1}^l n_k w_k$. Then the other weights of the representation are $w = \sum_{i=1}^l m_i w_i, m_i \leq n_i \in \mathbb{N}$.

- for any $(n_1, .n_k.., n_l)$ there is a unique, up to isomorphism, irreducible representation.

- and the contragredient representation comes by taking the opposite combination $(-n_1, . - n_k.., -n_l)$

The procedure to built the representation associated to W is not automatic, there are several methods, and the dimension of E is fixed by the choice of W (it increases with W). One computes simultaneously the matrix of f and a basis ε_i of E, starting from a vector ε_1 taken as the vector V of highest weight W to which one applies successively $f(t_k), k < j$. For each application one gets a vector belonging to a different vector subspace E_{W-w_k}. And $E = Span\{f(t_1)...f(t_k)V\}$

The fundamental weights are tabulated with the roots for the 9 types of complex semi-simple Lie algebra, they are expressed as linear combination of the vectors e_i as the roots.

The l representations associated to the highest weight $W = w_j$ are called the fundamental representations[4]. The other weights of the representation are then : $w_1, ..w_k, ...w_{j-1}$. For the classical algebras of matrices the fundamental representations are alternate tensor products of the standard representation, with the exception of some representations of $so(n, \mathbb{C})$ which involve the spin group (Maths.5.4). The representations associated to a set $(n_1, ..., n_l)$ are defined by tensorial product of the fundamental representations, which can be on the alternate tensor products $\Lambda_j E$ or on the symmetric tensor products $\odot_j E$. All other representations of L are defined by sum or tensorial products of the fundamental representations. So, for classical algebras, the basic bricks are the standard representation or, for the orthogonal group, the spin representation, and their associated contragredient representation.

The tensorial product of irreducible representations is itself irreducible if the representations are unitary, but otherwise it can be reducible, that is equivalent to the sum of representations, and this is crucial in processes with composite particles, whose states are represented by tensorial products. If $(E_1, f_1), (E_2, f_2)$ are irreducible representations of the same Lie algebra, with highest weight W_1, W_2, the tensorial product $(E_1 \otimes E_2, f_1 \times f_2)$ is an irreducible representation with the weight $W_1 + W_2$.

One goes from the representation of the Lie algebra to the representation of the group by the exponential. The adjoint representation $(T_1 G, Ad)$ is irreducible iff there is no normal subgroup other than 1, then its dimension is the dimension of the Lie algebra, that is the dimension of the group itself, which can be different from the standard representation for group of matrices.

[4] In Particle Physics Fundamental representations = standard representations in Mathematics

The most important examples are $sl\left(n+1,\mathbb{C}\right)$:
roots : n+1 related vectors $e_1+...+e_{n+1}=0$
$\Delta_+=\left\{e_i-e_j,i<j\right\},\Pi=\left(e_1-e_2,...e_n-e_{n+1}\right)$
$w_j=e_1+e_2+...+e_j-\frac{1}{n+1}\left(e_1+..+e_{n+1}\right)$

The fundamental representations are on $\Lambda_j\mathbb{C}^{n+1}$ that is alternate tensor products of the standard representation.

The Casimir operator is $\left(\frac{1}{2}f\left(\kappa_1\right)^2+f\left(\kappa_1\right)+2f\left(\kappa_3\right)f\left(\kappa_2\right)\right)$ and acts as $\frac{1}{2}n\left(n-1\right)$

The complexified of $su\left(n\right)$ is $sl\left(n,\mathbb{C}\right)$ so the representations of $SU\left(n\right)$ are defined by restriction of the representations of $SL\left(n,\mathbb{C}\right)$.

$SL\left(2,\mathbb{C}\right)$ has a unique (up to isomorphism) representation of dimension n for each n, which is an alternate power of the standard representation, and no finite dimensional unitary representation.

$Spin\left(3,1\right)$ is isomorphic to $SL\left(\mathbb{C},2\right)$ and has the same representations.

$Spin\left(3\right)$ is isomorphic to $SU\left(2\right)$ and is the double cover of $SO\left(\mathbb{R},3\right)$, so the representations of $SO\left(\mathbb{R},3\right)$ are part of the representations of $SL\left(\mathbb{C},2\right)$ and of $Spin\left(3\right)$ (Clifford algebras representations).

More on this topic in Maths.5.4.

6.3.3 Compact real Lie groups

The previous results hold only for complex semi simple Lie algebras. There are similar results for real Lie groups, but they apply only when they are compact (see Knapp p.251 for more). A Lie group is compact if, as a manifold, it is compact. Any compact Lie group is isomorphic to a group of matrices. An abelian compact Lie group is called a torus. A compact, connected complex Lie group is necessarily a torus. In a compact Lie group G the exponential is onto, the Killing form is negative semi definite on T_1G and its kernel is the center Z of the group, then the Lie algebra T_1G/T_1Z is compact, semi-simple, and its Killing form is definite negative. The complexification (by allowing complex numbers) of a real Lie algebra T_1G gives a complex Lie algebra $\left(T_1G\right)_C=T_1G\oplus iT_1G$. If the group G is compact then, by the exponential, one gets a unique complex Lie group G_C which has $\left(T_1G\right)_C$ for Lie algebra and G is a subgroup of G_C. The group G is said to be a real form of G_C, and conversely any semi simple complex Lie group has a compact real form.

In any group G the map $J:G\to G::J\left(h\right)\left(g\right)=h\cdot g\cdot h^{-1}$ called conjugation defines an action of G on itself. It defines a relation of equivalence on G : $g\sim g'\Leftrightarrow\exists h\in G:g'=J\left(h\right)\left(g\right)=h\cdot g\cdot h^{-1}$ and the classes of conjugacy define a partition of G.

In any compact real Lie group there is a maximal abelian subalgebra (it is not contained in another abelian algebra) $T_1\Gamma$. By the exponential it gives a maximal torus Γ, which contains the center of G. If G is connected then : $\forall g\in G,\exists\gamma\in\Gamma,h\in G:g=h\cdot\gamma\cdot h^{-1}$. Such tori are not unique, they are conjugate from each others. The complexified $\left(T_1\Gamma\right)_C$ is also abelian : $\forall t,t'\in T_1\Gamma_C:\left[t,t'\right]=0$. and is a Cartan algebra of the Lie algebra $\left(T_1G\right)_C$. By the exponential $(T_1\Gamma)_C$ gives an abelian subgroup Γ_C of G_C. The adjoint map $Ad : G \to \mathcal{L}\left((T_1G)_C ; (T_1G)_C\right)$ is the exponential of the map ad on $(T_1G)_C$. Because the group is abelian the matrices $[Ad_\gamma] = [Ad_{\exp t}]$ commutant, so they have a common set of eigen vectors. There is a set (X_ξ) of linearly independent vectors of $(T_1G)_C$ which are common vectors of the matrices $[Ad_\gamma]$ with eigen values $\xi(\gamma) : \forall \gamma \in \Gamma_C : Ad_\gamma(X_\xi) = \xi(\gamma) X_\xi$ such that $(T_1G)_C = T_1\Gamma_C \oplus Span(X_\xi)$.

Each map $\xi(\gamma) : \Gamma_C \to \mathbb{C}$ is a morphism $\xi : \xi(\gamma \cdot \gamma') = \xi(\gamma)\xi(\gamma')$.

$\forall \gamma \in \Gamma_C, t' \in T_1\Gamma_C :: Ad_\gamma(t') = t'$

The maps $\xi(\gamma)$ are not linear, and cannot be constant $= 1$ but there can be γ, X_ξ such that $Ad_\gamma(X_\xi) = X_\xi$.

Let be $g \in G, \exists \gamma \in \Gamma, h \in G : g = h \cdot \gamma \cdot h^{-1}$:

Γ is a subgroup of Γ_C, it acts on $(T_1G)_C$ by the adjoint map and $\forall \gamma \in \Gamma : Ad_\gamma(X_\xi) = \xi(\gamma) X_\xi, Ad_\gamma(t') = t'$

$Ad_g(Ad_h X_\xi) = Ad_{g \cdot h} X_\xi = Ad_h Ad_\gamma X_\xi = \xi(\gamma) Ad_h X_\xi$

$Ad_g(Ad_h t') = Ad_{g \cdot h} t' = Ad_h Ad_\gamma t' = Ad_h t'$

The vectors $Ad_h X_\xi, Ad_h t'$ are linearly independent and span $(T_1G)_C$, they are the only eigen vectors of Ad_g with eigen values $\xi(\gamma), 1$.