Evolution of the ring of gravitating bodies with and without the central body and properties of their chaotic behavior

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The time dependence of distribution function for the system composed of material points having circular orbits at initial moment of time with and without massive central body is obtained. As a result of chaotization of orbits, a part of material points evaporates (acquires positive total energy). Dependence of the fraction of evaporated material points on the width of the disk, their mass and initial distance to the central body is obtained. Wherein, the maximum fraction of evaporated particles for the case with central body is equal to 0.4. Initial stage of evaporation of a ring of particles with a central body is a subdiffusion with the dependence of the mean square bias on time $\left\langle R^2 \right\rangle = K_\alpha t^\alpha$, where α = 0.27. For the case without a central body, the fraction of evaporated particles is obtained as a function of the number of particles in the ring and virial ratios. The dependence of the fraction of particle pairs leaving the system on the number of particles in the ring is obtained. The average fraction of pairs for the virial ratio $K = -U$ turned out to be 0.2. Power spectra characterizing the evaporation of particles, approximated by a power-law dependence on frequency, are obtained.

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Key words: distribution function, numerical calculations, gravitation forces, spectral power density, autocorrelation function

Introduction

Mutual influence of bodies interacting on the basis of gravitational field leads to the chaotization of their orbits (see, for example, [1], [2], [3]). According to the classical results of mechanics [4] in the planetary N-body problem, quasi-periodic orbits exist and thus the stability is ensured for sufficiently weak interaction. However, in the case when several orbits locate

relatively close from each other and the masses of the bodies are big enough, the chaos can cover significant areas of the phase space of the system.

On the other hand, the behavior of large number of bodies can be described in terms of distribution functions. For example, for the distribution of the stars in the galaxies the asymmetrical by velocity Maxwell distribution is often used [5]. However, the origin of such a dependence remains largely unclear. The distributions of comets and other bodies are also known. The observations of the motion of stars in several galaxies [6] lead to the conclusion that with time their motion becomes more regular. What can we say in this context about an arbitrary gravitating system?

It is well-known, that thermodynamics and statistical physics of the systems with long-range potentials greatly differs from the systems with short-range potentials (see, for example, [7-16]). In particular, it is seen in the fact that in such systems (for which the systems with gravitation refer to) the internal energy of the system is not an additive value as the interaction between farther areas of the system can't be neglected. As a consequence such systems can exist in long-living quasi-stationary states, and a concept of "local equilibrium" for such systems can't be introduced.

In a paper [17] the question of the applicability of thermodynamics to systems with gravity is considered in a broader context. It is shown that for such systems as material points interacting according to the law of gravity, dark matter, black holes, the universe as a whole thermodynamics is not applicable due to the lack of equilibrium (global or local). On the other hand, in systems in which gravity is not the only force, and along with it there are short-range forces, thermodynamics can be applied. Such systems include planets, stars, star clouds.

In papers [7, 8] quasi-stationary distribution of large number of bodies for the case of 3D gravity under the assumption of initial spherically symmetric distribution is obtained. As the basis of the particles dynamics in this case (in thermodynamical limit of large number of particles) the Vlasov equation for the one-particle distribution function performs:

$$\left(\frac{\partial}{\partial t} + \vec{p}\frac{\partial}{\partial \vec{q}} - \frac{\partial \Psi}{\partial \vec{q}}\frac{\partial}{\partial \vec{p}} \right) f\left(\vec{p},\vec{q},t\right) = 0 , \qquad (1)$$

where Ψ presents the gravitational potential produced by all the particles in the system.

However, for the non-symmetrical initial distributions the analytical solution can't be obtained. Wherein, the transitional evolutionary processes for such quasi-stationary states remain unclear.

The difference of particle distribution in a 3D cluster of particles with gravity and the Gauss distribution has also been repeatedly considered. For example in the paper [12] a "democratic" Gauss distribution was suggested to consider.

In paper [18], nonequilibrium fluctuations in the ring of bodies with gravity are considered. As a result, the power-law dependences of the mean-square relative fluctuations of the number of particles that left the system on the number of particles in it were obtained. It is shown that this dependence has a power-law form $\sim N^{\alpha}$, where α increases with virial ratio.

A system with an initial water-bag distribution is discussed in the paper [19]. The dependence of the fraction of evaporated particles from the cluster on time for different initial conditions is obtained. The fraction of these particles depends on the initial conditions so that for higher initial velocities, it is about 0.65, while for the lowest ones it is 0.45.

However, despite the results obtained, many questions of the temporal behavior of systems with gravity remain unresolved.

The modeling of mesoscopic system, which, on the one hand, demonstrates statistical properties, but on the other hand can be considered as dynamical system, is of interest. In such a system the correlations between particles, that can lead to chaotization of their movement (chaotization is connected with the concept of dynamical chaos which can be measured by the maximum Lyapunov exponent, which will be considered below) and as a consequence to evaporation (e.g. the gain of the particle of the positive total energy and as a consequence the escape to infinity of the system*)* of particles, are important.

The problem is that the evolution of distribution function for the system of bodies with gravity significantly depends on the initial conditions. For which initial conditions the material points don't leave a finite area of space at all? What can we say about the kinetics of the transitional processes in such a system? What is a relationship between the rate of evaporation of the particles and chaotization properties of their movement?

To answer these and other questions, in this paper the system representing a ring of interacting bodies rotating at initial moment of time on circular orbits with and without central

body is considered. The aim is to trace the orbits' evolution of system of bodies taking into account mutual gravitation influence of them.

1. The evolution of distribution function of material points in the system of particle ring with a central body

Consider the system consisting of massive central body and the bodies of lesser mass, at initial moment of time rotating around it on circular orbits (Fig. 1):

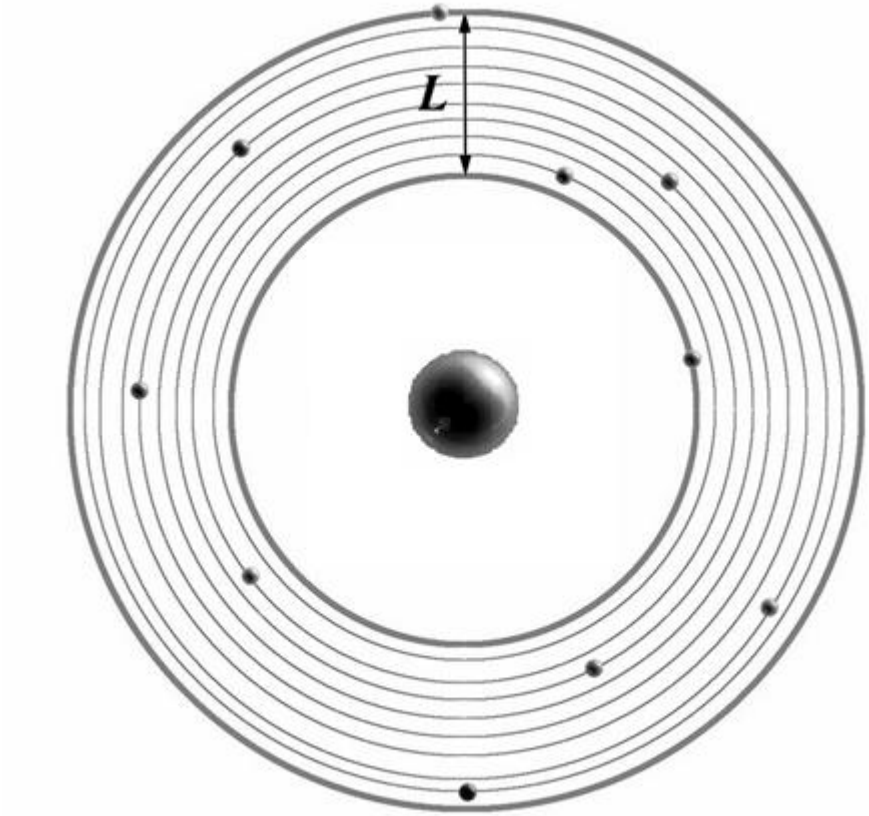

Fig.1.

All of the bodies are considered as the material points, which move only under the influence of gravitational forces in X-Y plane:

$$m\frac{d^2\vec{r}_i}{dt^2} = -G\frac{mM}{r_i^2}\frac{\vec{r}_i}{r_i} - \sum_k G\frac{m^2}{\left(\vec{r}_i - \vec{r}_k\right)^2}\frac{\vec{r}_i - \vec{r}_k}{\left|\vec{r}_i - \vec{r}_k\right|}, \qquad (2)$$

where m is the mass of the particle, M – is the mass of the central body, $\vec{r}_i$ - is the radius vector of the material point.

The velocities of the particles at initial moment of time are specified from the condition of stationary circular orbit with given initial radius and with the interaction of a particle only with the central body. The mass of the central body is equal to the mass of the Sun, the masses of the particles are measured by the mass of the Earth (6×10^{24} kg), and the distance is measured in AU (astronomical units, $1AU=1{,}5\times10^{11}$ m).

In series of papers [15, 16] the 3D gravity of particles moving along the ring is considered. This case turns out to be easy for theoretical consideration. However, in contrast with the case considered in this paper, in this case the particles can't leave the ring.

The dynamics of every particle occurs under the influence of classical gravitation force taking into account the interaction both with the central body and with the other particles. Numerical calculations were held with the step of 0.01 year. As a result (depending on the initial distributions of particles) the system partially chaotizates, which leads to the change of distribution function of velocities and coordinates.

To receive the distribution function of particles on velocities and coordinates the calculations were made for the ensemble of the following initial conditions: the particles are located on the orbits spaced apart from each other with equal distance, but their angular position is distributed randomly. The obtained result is averaged by 100 initial realizations. As a result, the distribution functions of the material points for velocity and radius-vector are obtained (Fig 2-5).

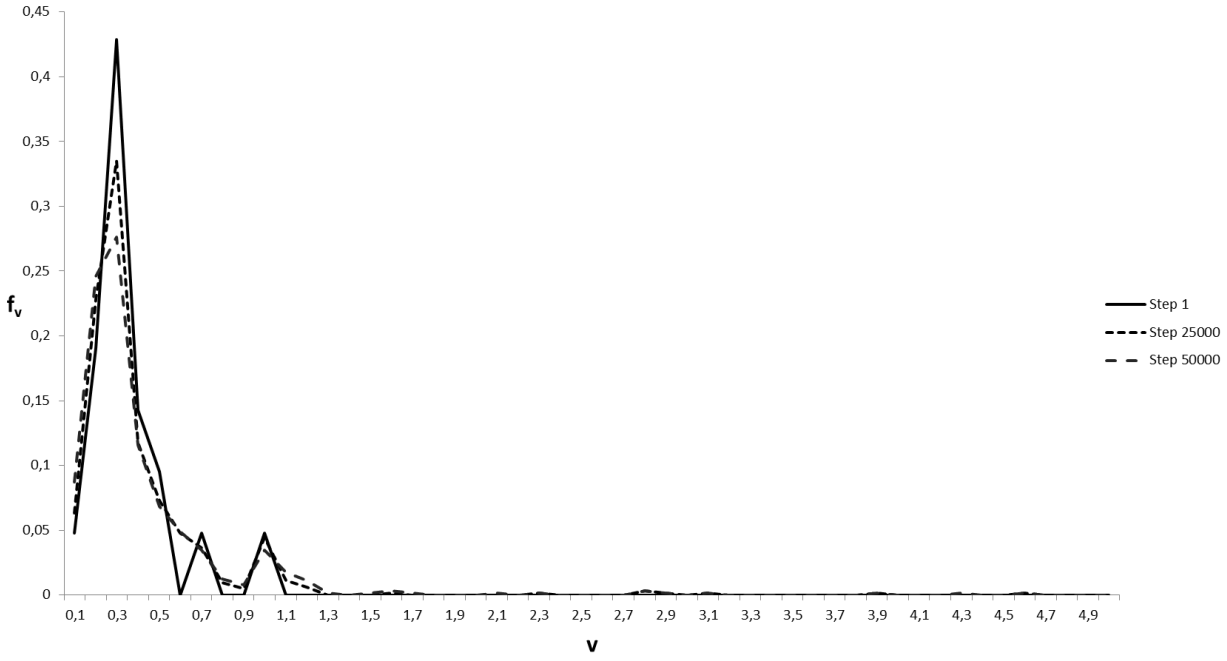
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Fig.2.

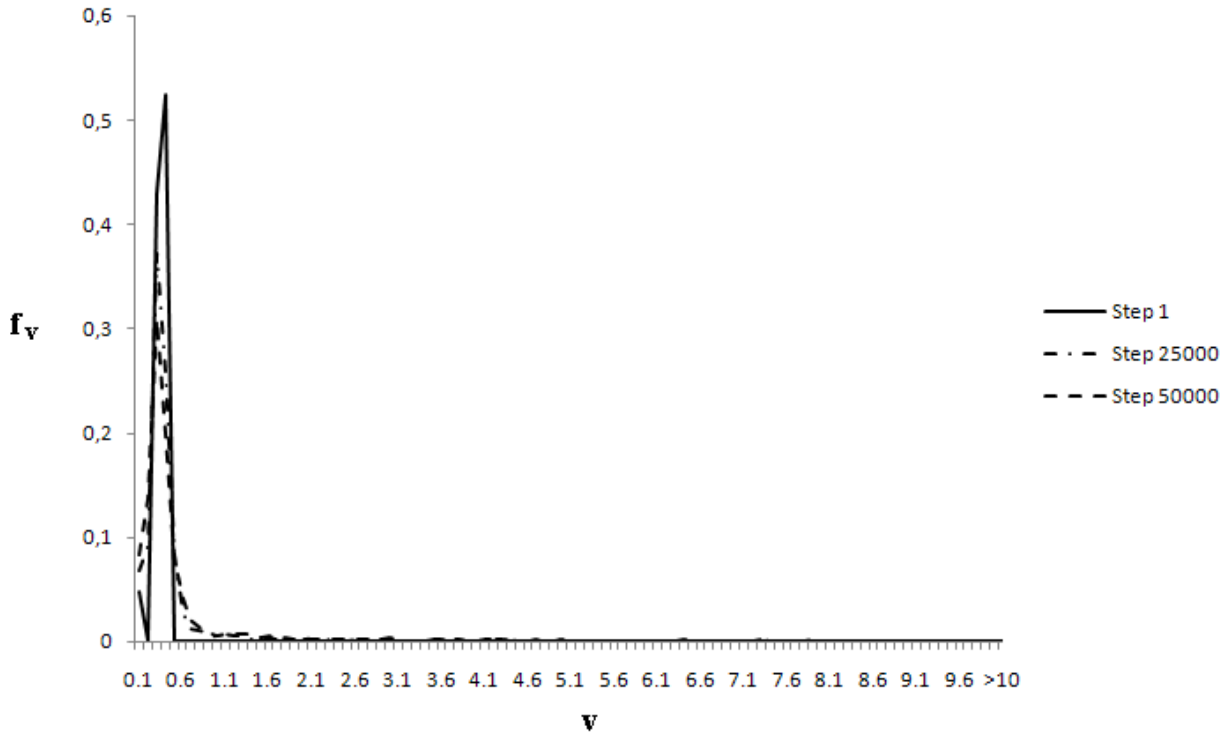
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Fig. 3.

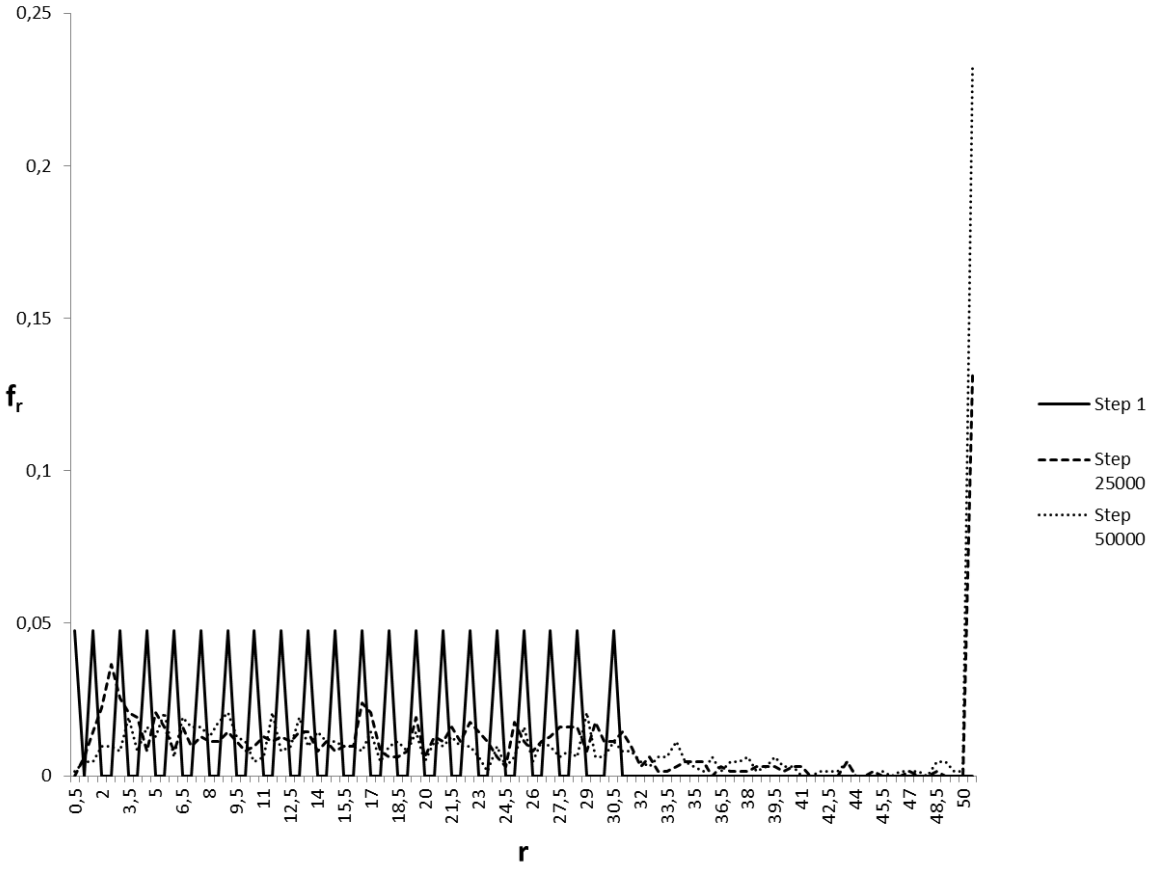
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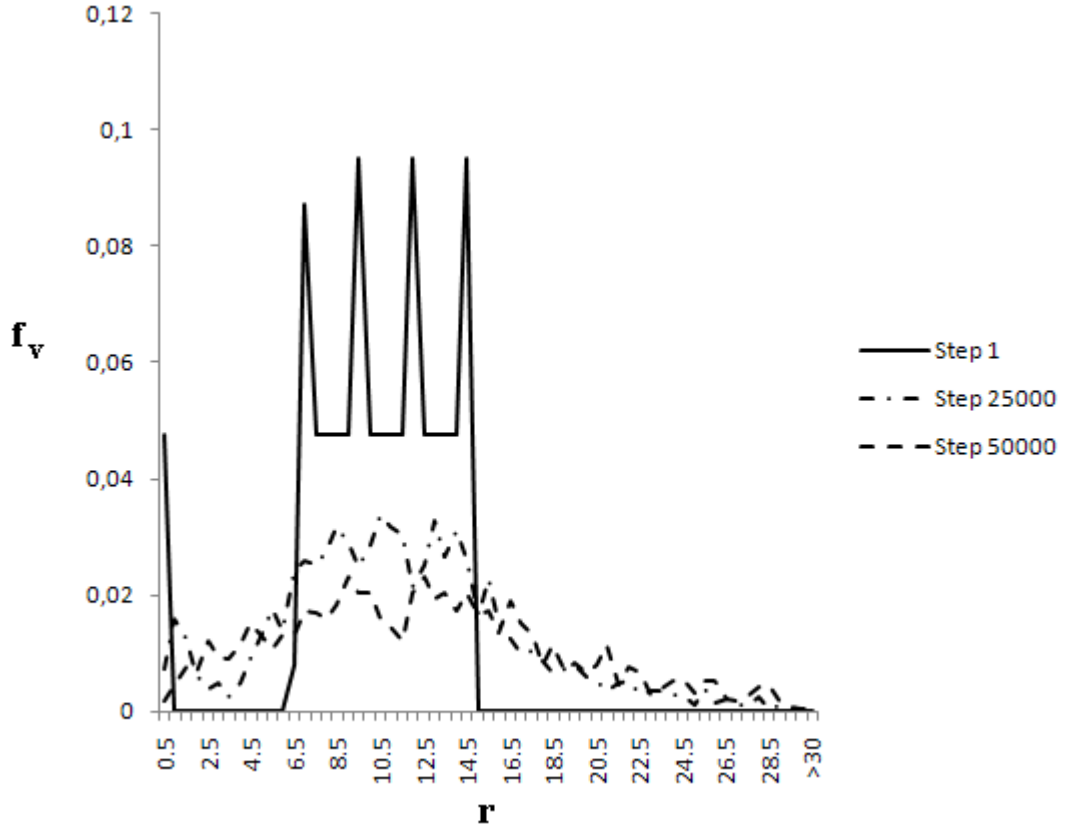
Fig. 4.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28.5
>30
r
Step 1
Step 25000
Step 50000

Fig. 5.

The calculations were made on the basis of the velocity form of Verlet algorithm (see, for example, [20, 21]). The peculiarity of the calculations for systems with gravity is the fact that the collisions between the particles are absent, though the enough close convergences of them are possible. Exactly as a result of such convergences the particle velocities change most strongly. For the providing of the correctness of the calculations the step of calculations was chosen so that the error induced by close convergences would be small compared with error induced by random specification of initial conditions.

From the Fig. 4 we can see that the distribution function being initially close to δ-function, changes with time, gradually occupying larger areas of the phase space. We determine how much obtained distribution functions differ from Gaussian function. To do this, we numerically find the dependence of the dispersion on time. For the Gaussian distribution such dependence should be linear according to the Einstein's equation, which for the 2-dimension case takes the form:

$$\left\langle R^2 \right\rangle = 4Dt, \tag{3}$$

where D is the diffusion coefficient of the particles. The obtained dependencies are depicted in Fig. 6.

Fig. 6.

As we can see from Fig. 7, both dependencies are not linear, and their average inclination decreases with time. Moreover, the dependence of dispersion of the velocity on time is stair-step, i.e. the velocity distribution of the particles is not Maxwell at initial stage of the evolution. Dependence obtained is characteristic for subdiffusion (see, for example [22]), for which the following equality is fulfilled:

$$\left\langle R^2 \right\rangle = K_\alpha t^\alpha .$$

Approximation by power function (Fig.7) gives index which is equal 0.27. For subdiffusion power distributions are characteristic, in contrast with Gaussian distribution for ordinary diffusion. Strictly speaking, for such a system we cannot introduce the diffusion coefficient because there is no local equilibrium in this system. We can use the term “effective diffusion coefficient”, which characterizes the degree of the chaotization of the system. The obtained results can be interpreted as follows: the degree of chaotization of the system decreases with time. The reason for this is that as a result of the destruction of the ring the average distances between the particles increase.

As it is known, the evolution of the isolated system of bodies where the total energy conserves is connected with the fulfillment of the Virial theorem:

$$2K + U = 0 \ , \tag{4}$$

where K and U are the total kinetic and potential system enegry.

As it was noted before [8], the fact that the system at the initial moment of time satisfies the Virial theorem does not mean that the system will stay further in the stationary state. For this it is necessary that the distribution function would be the stationary solution of Vlasov equation. The authors considered as an initial exactly that condition, which satisfies the Virial theorem since in this case the relaxation to the quasistationary state occurs by the mechanism mixing of particle trajectories. At the violation of conditions of the Virial theorem the mechanism will include excitation of parametric resonances and a nonlinear Landau damping.

In the present paper the initial conditions close to virial are chosen (initial orbits of the particles are circular, their radius corresponds to the interaction of the particles only with the central body). Such a choice is caused by the fact that for the enough rarefied system (in the limit of small masses of the material points or large distances between them) such initial state should preserve, e.g. it exactly corresponds to the concept of nonequilibrium quasistationary states (QSS) [8].

2. The evaporation of the ring of material points with a central body

As a result of chaotization of the motion of material points they will gain different energy. In particular, the part of the points can gain energy high enough for their escape to infinity (i.e. positive total energy). Such a process is largely similar to the evaporation of the liquid (solid) body for the systems with short-range potentials. In particular, the intensity of the evaporation in such systems increases with the growth of the temperature. However, in the case under consideration the systems, where the only force is gravity, we can't introduce the concept "temperature". Nevertheless, in the system under consideration with gravity a particle should also overcome a potential barrier.

The flow of the particles outside the orbit can be found on the basis of the distribution function.

We note that for the system consisting of more than two bodies the particle gain of the positive total energy, strictly speaking, is not enough for making a conclusion of that it will

escape to infinity. However, for the system with the ordered initial conditions (the circular orbits in our case) the probability of the fact that the particle having gained positive total energy in the future (until its distance to the central body will be high enough) will have time to get close with another particle and give energy to it (as a result of that the total energy will be negative) is small. This conclusion is confirmed by the numerical calculations.

We consider the dependence of the intensity of evaporation for the ring of interacting material points in the presence of the central body on the parameters of the systems. In Fig.7 the dependence of the fraction of evaporated particles on the mass of the particles and on the width of the ring is depicted.

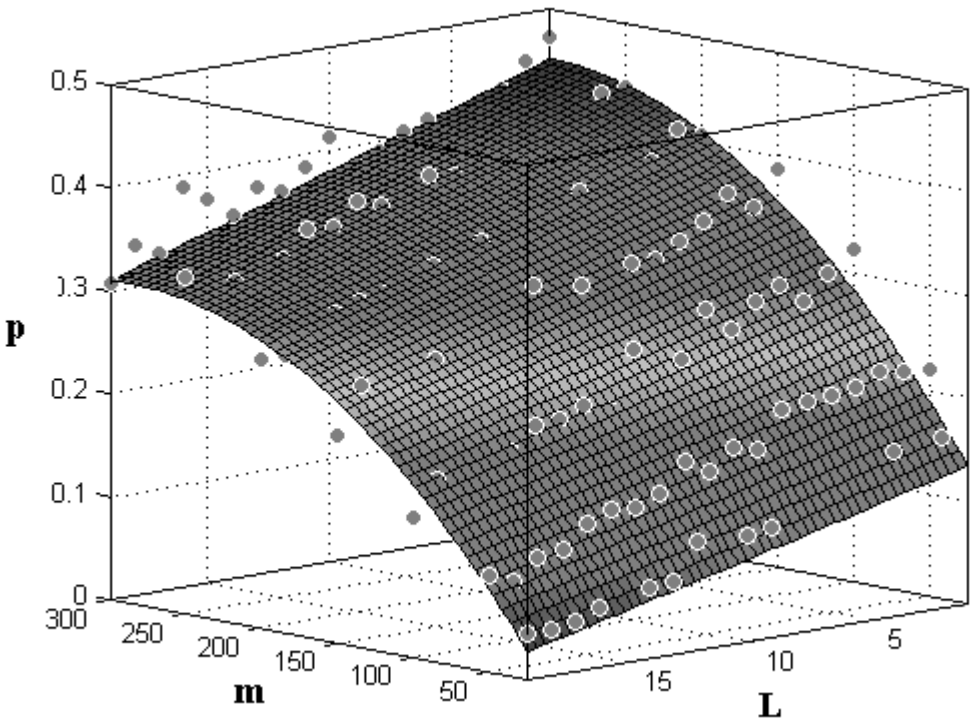

Fig. 7.

The points present the results of the numerical experiment.

The surface is approximated by the second-degree polynomial:

$$z = a + b_1 x + b_2 y + c_1 x^2 + c_2 xy + c_3 y^2 , \qquad (5)$$

where a = 0.08, b_1 = 0.002512, b_2 = -0.006061, c_1 = -4.145e-006, c_2 = -7.164e-006, c_3 = 1.385e-005.

Wherein the approximation is characterized by the following values*:* SSE (sum of squares due to error): 0.1494, R-square (coefficient of determination**)**: 0.9237, Adjusted R-square: 0.9207, RMSE (root mean squared error): 0.0343.

According to the results of the numerical experiments the following conclusions can be made:

1. The fraction of the evaporated particles with the increase of the masses of the particles and with the decrease of the radius of the ring tends to the certain value. In this case interaction between particles is the most intensive and can lead to the chaotization of their orbits and to evaporation. However, we should note that with the conservation of the total energy some particles can evaporate only at the expense of the others. Hence this fraction can't be equal to unit (if the central body is considered to be fixed). Such evaporation will lead to the formation of two phases: gas phase (where the particles have an infinite motion and weakly interact with each other with time) and "condensed" (ordered) phase, where the particles also increasingly weakly interact with each other, rotating on the orbits close to circular. The maximum fraction of the evaporated particles according to the calculations tends to 0.4.
2. The intersection of this surface with the XY plane gives the dependence of particles mass on the width of the ring which is a boundary of the "condensed" phase. This means that with smaller masses (or the larger width of the ring) no particle can leave the system. In Fig.7 the dependence of the ring width on the mass of a particle corresponding to the border of the condensed phase is depicted.

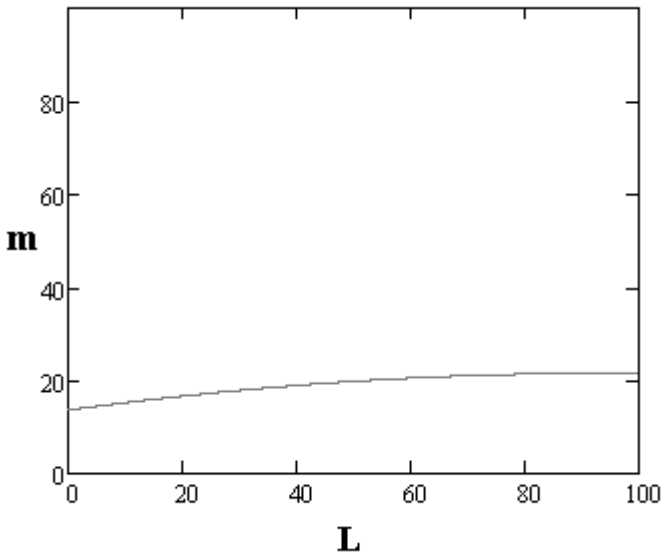

Fig. 8.

It is seen from the figure that with large masses of the particles the width of the ring corresponding to such a boundary also increases. With the mass of the particles tendency to zero, the width of the ring, corresponding to such a boundary must also tends to zero, however, such a property largely depends on limited accuracy of calculations.

Consider the dependence of the evaporated particles fraction on the average radius of the ring. Wherein, the smaller initial radius will correspond to the location of the particles in deeper potential well, and vice versa. In the Fig. 10 the dependence of the evaporated particles fraction on the initial distance to the central body is depicted.

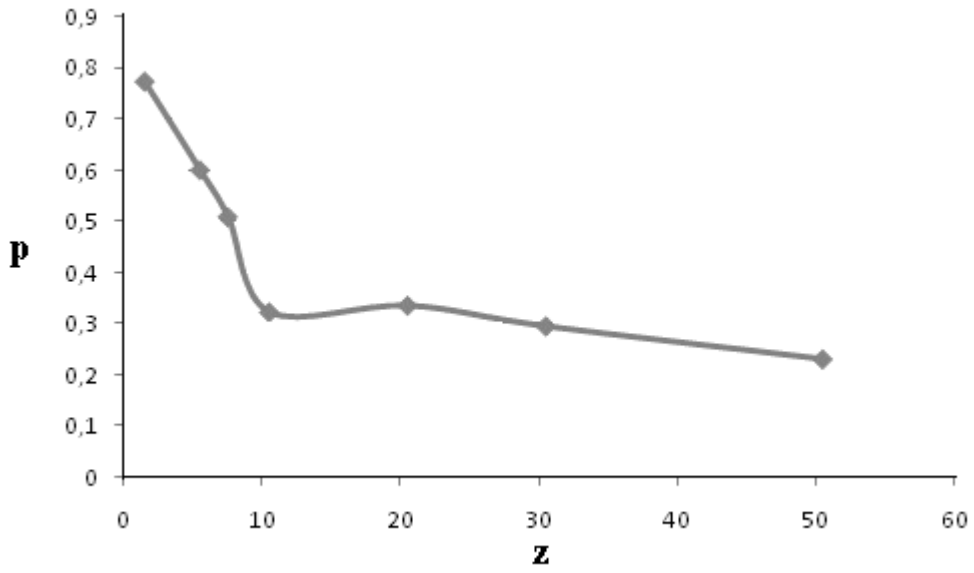

Fig. 9.

As we can see from the Fig. 9 this fraction increases with small distances, it also tends to some relatively constant value in remote distances. The form of the curve is obviously determined by the following effects having the opposite trends. On the one hand, with the specified width of the ring and with the specified number of the particles the frequency of their convergence on the close distance (when the significant change of the orbits occurs) decreases with increase of the ring width. On the other hand, with increase of the ring radius, particles are located on the edge of the potential well, i.e. the barrier which they must overcome to evaporate, is small.

3. The relationship between chaotization of particles trajectories and the rate of their evaporation

As correlations between the particles lead to their evaporation, it is important to trace the connection between the characteristics of the chaos in the system and systems parameters.

As a characteristic of the chaos in the system maximum Lyapunov exponent can act

$$h = lim_{\substack{d(0)\to 0 \\ t\to\infty}} \frac{1}{t} ln \frac{d(t)}{d(0)}, \quad (6)$$

where *d(0)* – the initial distance between the points in phase space. The value

$$\tau = 1 / h$$

represents specific time of the divergence of the trajectories. That is, the evolution of the close points is investigated and it is found to what extent their trajectories diverge with time. The measure of the divergence of the trajectories is the distance between the points *d(t).*

As a result of calculations for the case of 9 material points with masses equal to 1 the value τ is obtained approximately equal to 3 years.

For the planets of the Solar system the value of h is equal to 10^{-5} (e.g., [23]). Chaos has also been observed in the dynamics of exoplanets [24]. As the resonances and dynamical chaos, associated with them, are the reasons of particles evaporation, average time of particles evaporation should be of the order of the average time of the trajectories divergence.

It was obtained that maximum Lyapunov exponent depends weakly on the ring width and its radius.

4. Modeling the evolution of the ring of material points without a central body

Let us now consider a planar system with gravitation consisting of *N* material points without a central body interacting with each other according to the law of classical gravitation.

As an initial condition, a random distribution of points on the angle φ in the XY plane was chosen, wherein the points were in orbits spaced from each other at the same initial distance. The nearest point of the ring was located at a distance of 1 AU from the center, and the latter at a

distance of 30 AU. The velocity modulus was calculated from the corresponding ratio of the potential and kinetic energies. The number of particles varied from 20 to 50, respectively, the mass from 80 to 200 Earth masses. The total mass of all particles is constant and equal to 4000 Earth masses.

Two separate studies on the initial velocities were carried out. In the first case, the velocities were chosen to be directed along the tangent to the circular orbit. In the second one - at a certain angle to it.

The initial stages of the evolution of such a particle system (500 years for the case of various virial ratios with circular orbits and 1000 years in the study of pairs of material points flying out of the cluster) are investigated. The number of attempts for each ratio of masses and virial for the first case was 100, for the second case - 20. The fraction of evaporating particles was calculated for each case.

We note that, generally speaking, for systems with long-range interaction, the criterion for the fact that particle left the system is not obvious. Only in the limit of an infinitely long time interval we can state that a particle having a positive total energy leaves the system. In this case, it will be located at infinity. In the case of finite time, there is a finite probability for the particle to return back to the system, i.e. to acquire, generally speaking, an arbitrarily large negative energy.

When considering a system without a central body, the particle masses are large enough (in comparison with the case of the presence of a central body), accordingly, the probability that the particles will leave the system in pairs is high enough (this fact was confirmed by numerical experiments). In this case, the presence of positive energy for a particle does not yet mean that it has left the system. Therefore, for a case without a central body, a particle that has crossed a circular orbit with a radius equal to 3/2 from the maximum initial radius of the orbit of the outside particle was considered as evaporated. This criterion is also approximate.

4.1. A study of the total fraction of evaporated particles for circular orbits

The study of pairs of material points flying out of the cluster was investigated. The number of particles in this case varied from 20 to 40. The criteria for the pair were the following: the intersection of a circular orbit with a radius equal to 3/2 from the maximum initial radius of the particle orbit and the negativity of the total energy for two interconnected particles.

A simulation was carried out for three virial ratios: *K = -U, K = -0.8U, K = -0.75U,* where K is the kinetic energy of the system, and *U* is the potential energy of the system. For each ratio, two configurations of the directions of particle motion were investigated: the random direction of motion of the particles along a circular orbit and the same direction of motion in "domains" with 5 particles with a change in direction in each subsequent domain.

As a result, distribution functions were obtained for the fraction of evaporated particles (ξ). An example of such a function is shown in Fig. 10.

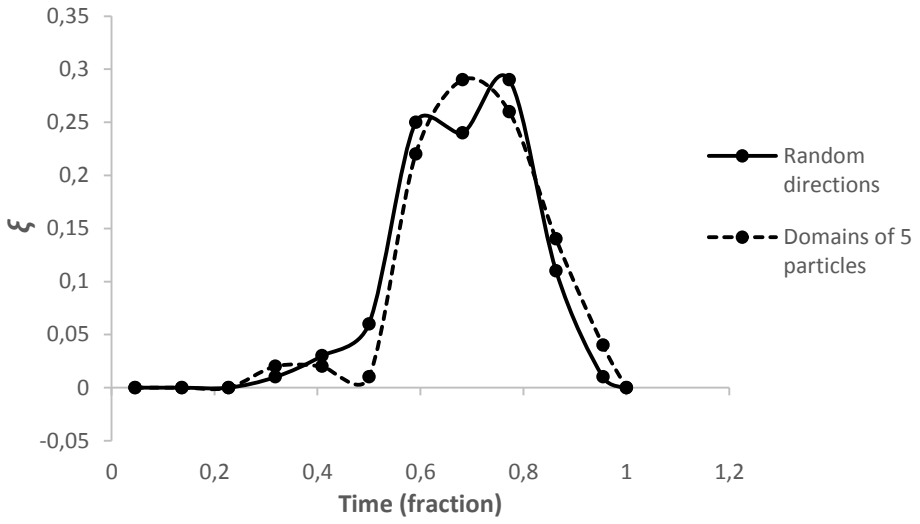

Figure 10.

This function can be approximated by a Gaussian:

$$f(x) = a_1 exp\left(-\left(\frac{x - b_1}{c_1} \right)^2 \right), \tag{4.1}$$

where $a_1 = 0.2968$, $b_1 = 0.6964$, $c_1 = 0.1748$.

It can be seen from these graphs that the probability of a fraction of evaporated particles almost does not depend on the configuration of the directions of particle motion (in this case these are configurations of random directions and directions of movements on domains).

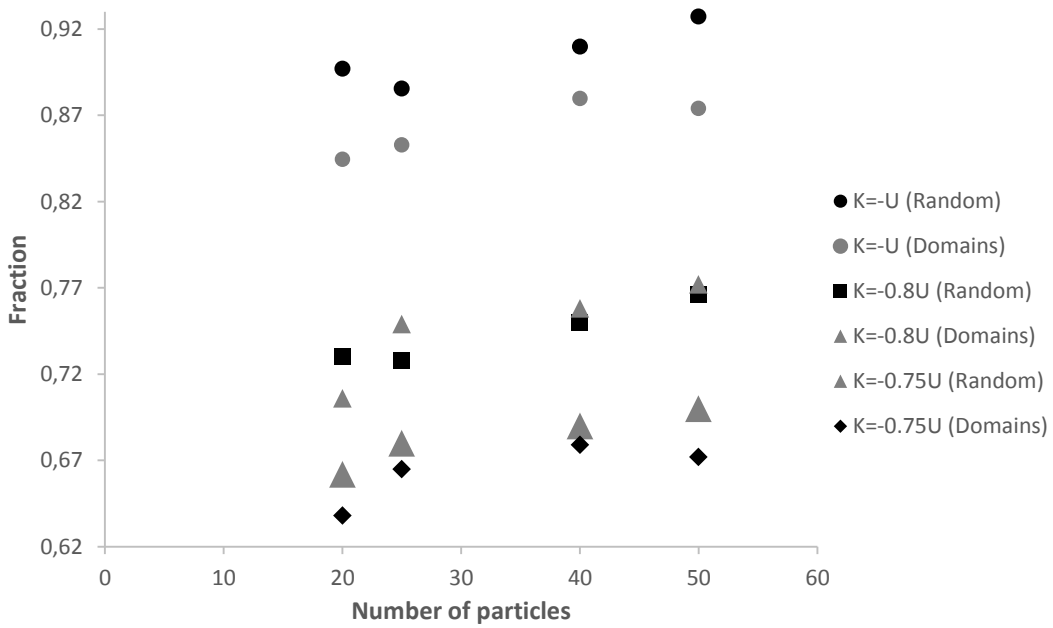

Figure 11.

Let us consider the dependence of the average fraction of evaporated particles on their total number, configurations, and virial ratio (Figure 11). As it can be seen from the figure, the fraction of evaporated particles remains approximately constant for any of the considered virial ratios. Moreover, the greater the kinetic energy, the greater the fraction of evaporated particles, and the initial configuration of the directions of particle motion almost does not affect the fraction of evaporated particles within the framework of a single virial ratio. With the growth of the virial ratio, the fraction of evaporated particles increases, which is natural.

Let us introduce the value N_E - the number of particles evaporated from the system during the time τ. We remind that a particle for a case without a central body is considered evaporated if it has crossed a circular orbit with a radius equal to 3/2 of the maximum initial radius of the particle orbit. The value of τ is random, for which we can introduce an average value:

$$\langle N_E \rangle = \frac{1}{n} \sum_{i=1}^{n} N_{Ei} \tag{4.2}$$

We also introduce the relative mean-square fluctuation of this quantity:

$$\frac{\left(\left\langle \left(\Delta N_E\right)^2 \right\rangle\right)^{1/2}}{\left\langle N_E \right\rangle} \tag{4.3}$$

and examine its dependence on the number of particles in the system. As a result of the processing of numerical data, it was found that there is an inverse polynomial dependence on the number of particles N, i.e $\vdots \frac{1}{N^{\alpha}}$ We consider it in logarithmic coordinates and approximate by a linear function with the method of least squares (Figure 12).

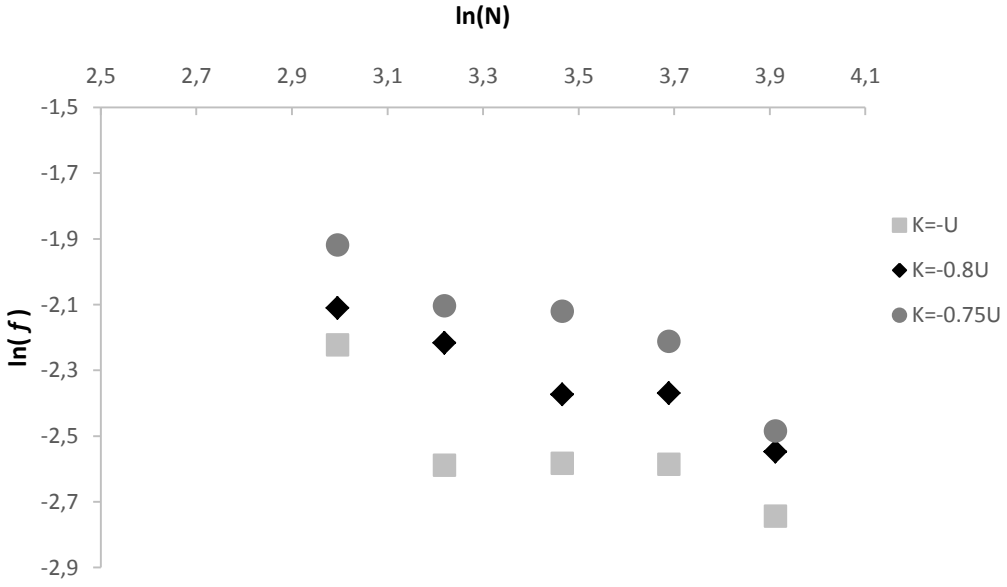

Fig. 12.

Where $K = -U$, $\alpha = 0.4498$; where $K = -0.8U$, $\alpha = 0.446$; where $K = -0.75U$, $\alpha = 0.535$.

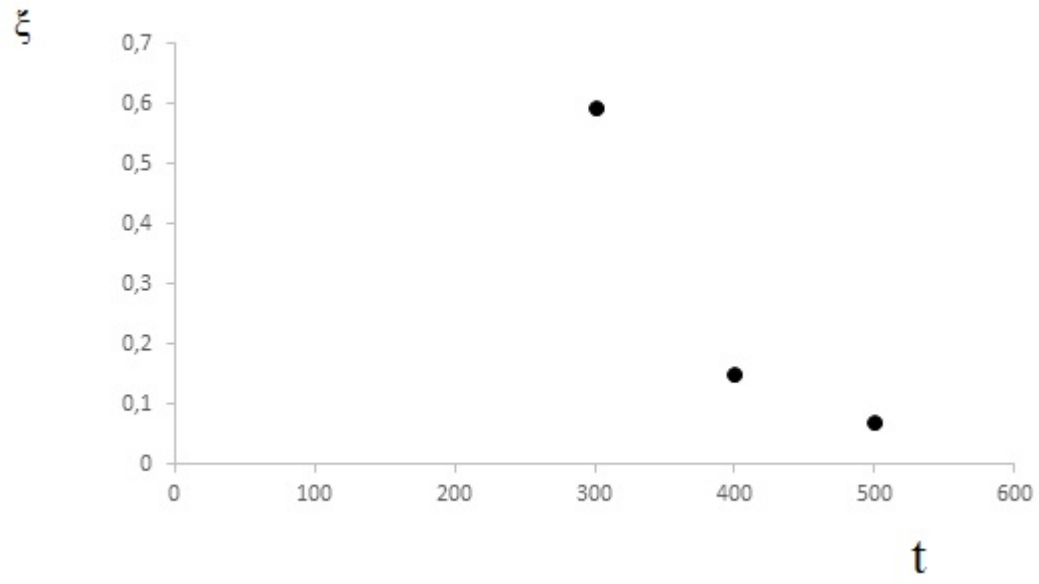

Fig. 13.

As a result of numerical simulation, it was found that the rate of particle evaporation from the ring decreases with time. In particular, the dependence of the average fraction of evaporated particles on time was obtained (Fig. 13). This is a power law of the following form:

$$\xi \sim \frac{1}{t^{4.6}} \tag{4.4}$$

Thus, the system relaxes ("cools") in a power law. We did not consider small times (0-200 years), because at such times the particles still do not have time to overcome a distance sufficient to consider them evaporated. We note that the power law was also observed in other systems (such as Saturn rings).

However, a direct comparison with these results cannot be made, because in the case of Saturn, Saturn itself and its satellites play an important role in the dynamics of the rings. In our case, there is no central body. As is known, power laws are characteristic, for example, for self-organized criticality (see, for example, [25]). Indeed, the existence of conditions close to critical, in this case is possible, because the first evaporating particles have the largest energy, and other particles influence them the most. The remaining particles distort each other's movements weaker and weaker (see, also, [18]).

A study of the average fraction of evaporated particles for the random initial direction of the particles and the ordered motion in one direction was also made, and the influence of the initial eccentricity was also studied. As a result of the simulation, it turned out that these factors have little effect on the average fraction of evaporated particles.

4.2. The investigation of the fraction of pairs of evaporated particles

In the study of the fraction of evaporating particles, an interesting question is, what the fraction of pairs in the total number of evaporating particles is. Calculations were made in the framework of previous calculations with virial ratios $K = -U$, $K = -0.4U$ and configurations of the directions of particle motion with a random direction of motion of the particles along the circular orbit and counterclockwise.

Let us consider the dependence of the average fraction of pairs of evaporated particles on their total number for the virial ratio $K = -U$ and the configuration of the random direction of particle motion (Figure 14).

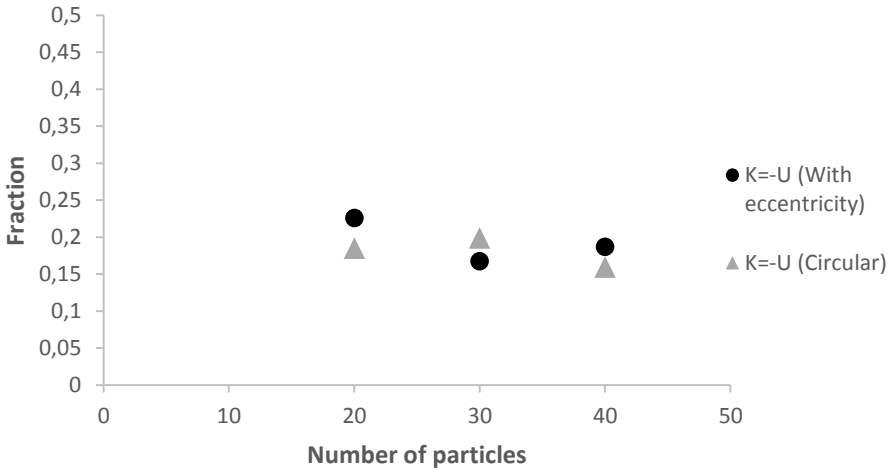

Fig 14.

From the calculations it can be concluded that the fraction of evaporated particle pairs for the virial ratio $K = -U$ is large enough, but the configuration of the directions of movements strictly along circular orbits or in orbits with eccentricity has little effect on this value.

4.3. Semi-phenological model of evaporation of ring
Полуфеноменологическая модель испарения кольца

The numerical calculations of the evolution of the ring of material points with gravity for various initial configurations, as well as in the presence of a central body and without it, allow us to construct an analytical model of this process.

As the basic assumptions of the model, we accept the following:

- we will consider the balance of particles in the system as a whole,

- random processes, which are the consequence of chaotization of trajectories, will be modeled using Gaussian noise.

Since the energy barrier that prevents evaporation of particles is created by all other particles of the ring, as well as by the central body, we first calculate the gravitational potential at the middle of the width of the ring. On the basis of the law of interaction between particles, we obtain the potential of an infinitely thin ring of mass m at an arbitrary point:

$$d\varphi = -\frac{d\alpha}{2\pi} G \frac{m}{R} \frac{1}{\left(1 - 2\cos\alpha \frac{r_0}{R} + \left(\frac{r_0}{R}\right)^2\right)^{1/2}} .$$

Here r_0 is the coordinate of an arbitrary point, R is the radius of the ring (see Figure 15).

We introduce n - the average number of particles per unit radius of the ring. Then one can find the potential of a ring of finite width at its center:

$$\varphi(l) = -Gm \int_{1-l/2}^{1+l/2} dy \int_0^{2\pi} \frac{d\alpha}{2\pi} \frac{n}{\left(1 - 2y\cos\alpha + (y)^2\right)^{1/2}} . \quad (4.5)$$

Here $y = r_0/R$, $l = L / R$ is the dimensionless width of the ring.

The potentials difference multiplied by the particle mass, $(0 - \varphi(l))m$, will be the average potential barrier for particles in the ring E. In the presence of noise, a particle can overcome this

barrier with a certain probability. Then we write the equation of balance of particles in the ring in the following form:

$$\frac{dn}{dt} = -An^2 exp\left(-\frac{E}{\Theta_{eff}}\right). \qquad (4.6)$$

Here Θeff – is the effective temperature of the noise.

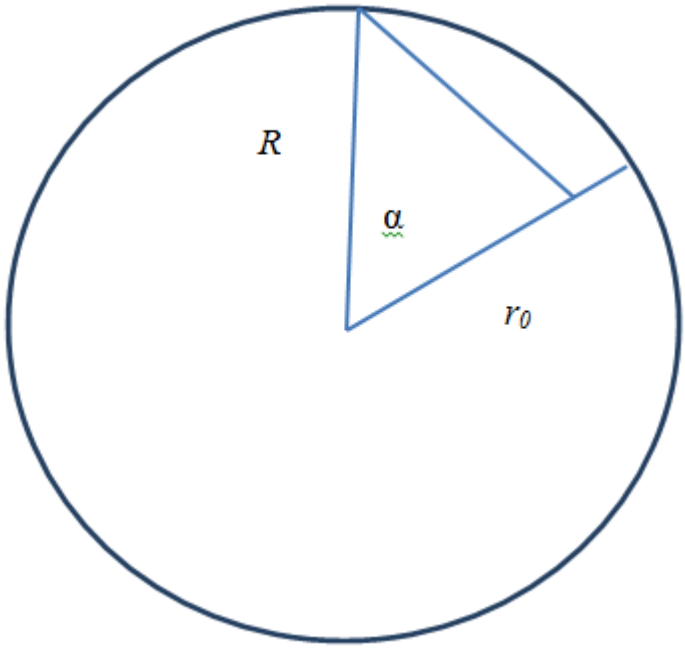

Fig.15.

The escape of particles from the ring is due to the close approach of two particles, each of which receives a large energy. Let us further assume that the noise temperature is proportional to the density of the particle orbits. Then we finally get:

$$\frac{dn}{dt} = -An^2 exp\left(-\frac{B}{n}\right).$$

Here, A and B are phenomenological coefficients that cannot be found within the framework of this model.

Let us consider two cases: in the presence and in the absence of a massive central body.

In the case of a massive central body, a potential barrier for particles is created mainly by this body, i.e. the value of B can be regarded as constant. Then we obtain a solution of (4.6) in the form:

$$n = \frac{B}{ln\left(ABt + exp\left(\frac{B}{n_0} \right) \right)}.$$

Here n_0 – is the initial density of particles. For the evaporation rate we obtain:

$$\frac{dn}{dt} = -\frac{B}{\left[ln\left(ABt + exp\left(\frac{B}{n_0} \right) \right) \right]^2} \frac{AB}{ABt + exp\left(\frac{B}{n_0} \right)}.$$

In the absence of a central body, a potential barrier is created only by the ring itself. In this case, the barrier value, according to (4.5), is proportional to the average particle concentration. In this case we obtain the equation:

$$\frac{dn}{dt} = -An^2 exp(-\alpha).$$

Here, α will not depend on the particle concentration. The solution of this equation will have the form:

$$n = \frac{n_0}{n_0 A exp(-\alpha) t + 1}.$$

Evaporation rate will take the form:

$$\frac{dn}{dt} = -\frac{n_0}{\left(n_0 A exp(-\alpha) t + 1 \right)^2} n_0 A exp(-\alpha).$$

Thus, in the presence of a massive central body, evaporation will be relatively slow, since the particle barrier remains approximately constant during evaporation. In the absence of a massive central body, relaxation is faster, since during the evaporation process the barrier for particles itself decreases, which in turn leads to acceleration of evaporation.

The result obtained for the evaporation rate (proportionality $1/t^2$ for sufficiently large t) qualitatively coincides with the one found numerically. As a result of numerical calculations, the evaporation rate was found to be proportional to $1/t^{2.8}$ [18] and $1/t^{4.6}$ in this paper. The discrepancy between the theory and the numerical experiment in the degrees of the dependence of the rate of evaporation on time can be related both with the approximation of the model and with a relatively small number of numerical experiments. In addition, particles at the outer edge of the ring, for which the potential barrier is minimal, can contribute to the high rate of evaporation.

The construction of a more rigorous (distributed) model encounters difficulties associated with the analytic determination of the noise spectrum and the dependence of the intensity of the random process on the coordinates. This issue requires separate consideration.

4.4. Power spectrum and autocorrelation functions

The autocorrelation function is an important quantitative characteristic of random and chaotic processes. Its physical meaning is that it characterizes the relationship between events for the same signal, taken with a time shift. The autocorrelation function and the power spectrum are determined by the following formulas

$$R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)x(t+\tau)dt$$

$$S(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau$$

In this case, the signal means the dependence of the coordinates of one body on time.

We introduce for the ring of particles with gravity the flow function *f(t)* of particles leaving the system through the boundary:

$$f(t) = \sum_i \delta(t - t_i), \tag{4.7}$$

where t_i is the time when the i-th particle crosses the boundary line and evaporates.

Then the power spectrum $s(\omega)$ in this case can also be defined as follows [26]:

$$s(w) = lim_{T\to\infty} \frac{1}{2T} \left| \int_{-T}^{T} f(t) exp(iwt) dt \right|^2 \tag{4.8}$$

The particle flow is understood as the fraction of the departing particles at a fixed time interval. The data on the fraction were averaged over a set of experiments for a given initial configuration for specific values of N (the number of particles in the system). The model considered is in many respects similar to the traffic model for cars with two lanes [26].

We construct graphs for the signal, which will be the flow function $f(t)$ written above, as well as its autocorrelation function and power spectrum for the initial configuration indicated above in experiments with the number of particles in the system $N = 40$.

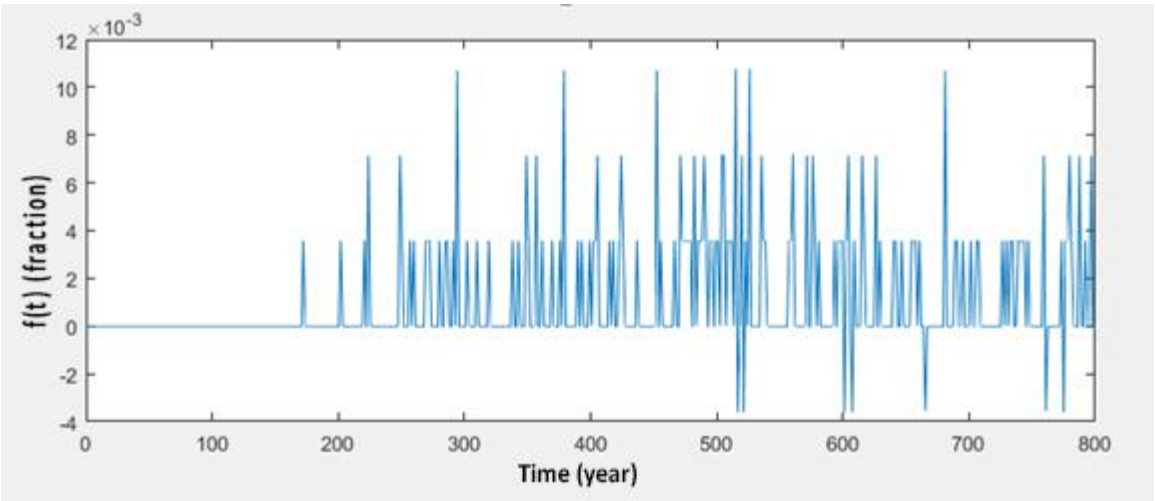

Figure 16.

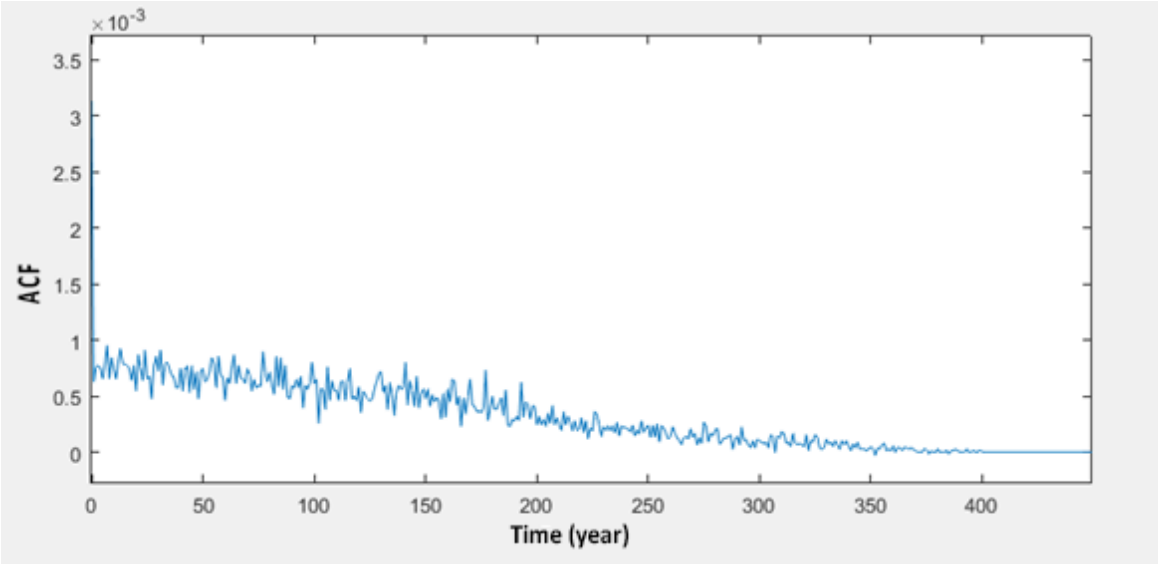

Figure 17.

According to Figure 19, it can be concluded that the signal is randomized since the autocorrelation function decays with time (see, for example, [27]).

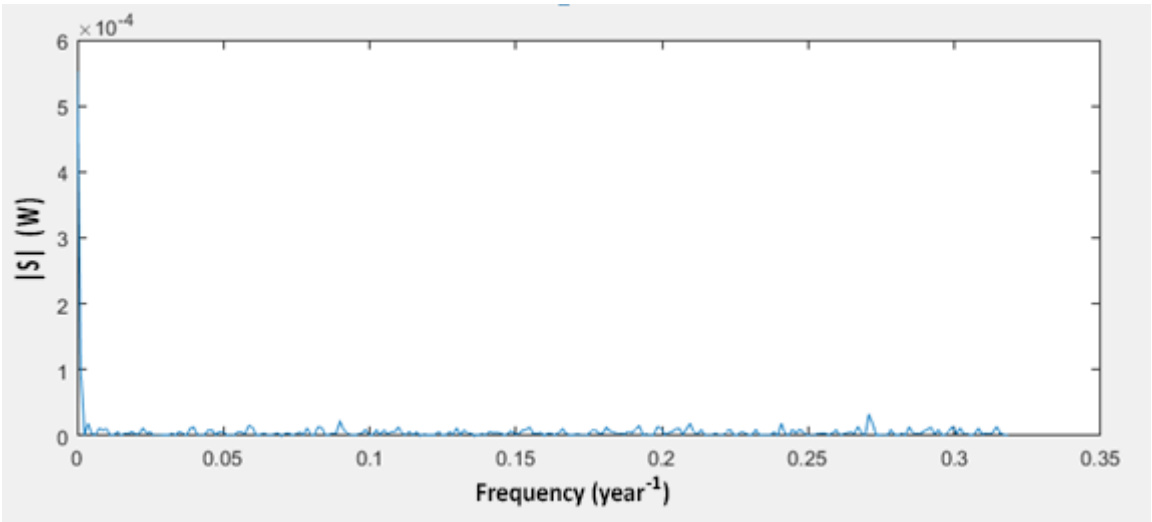

Figure 18.

From Fig. 18 it can be seen that the power spectrum has many peaks. Calculations for the case with the number of particles $N = 20$ are shown in the figures.

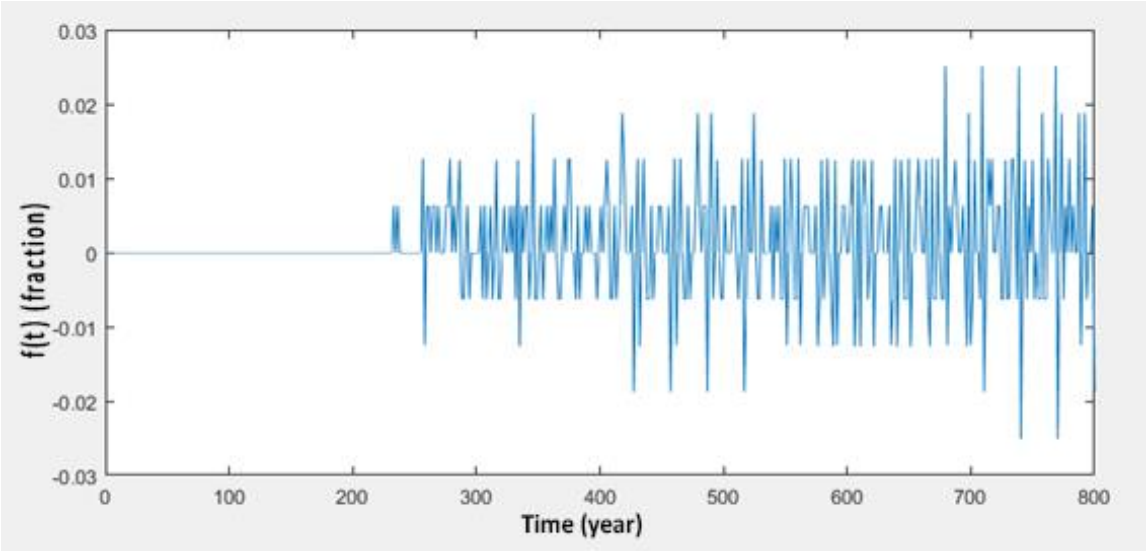

Figure 19.

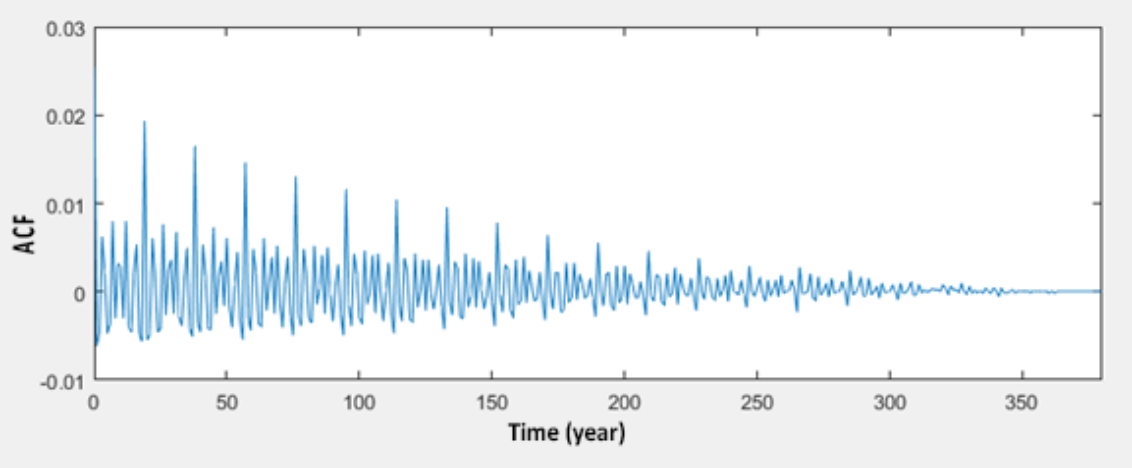

Figure 20.

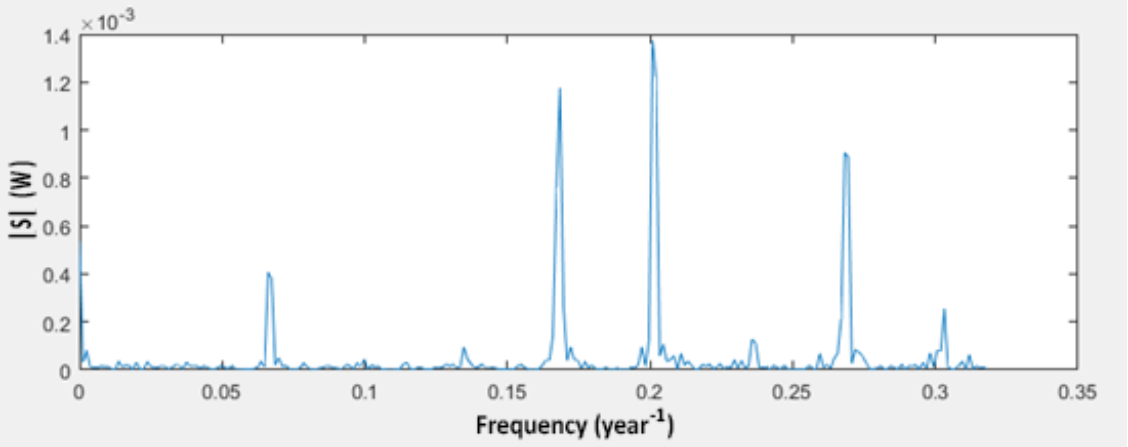

Figure 21.

According to the data from the graphs for the experiments with N = 20, it can be seen that the signal, its autocorrelation function and the power spectrum differ significantly from the data for the experiments with N = 40. This can indicate how much the initial conditions affect the process. In this case, the initial conditions for the coordinates especially strongly influence, since they are assigned each time in a random manner with respect to the angle φ for each particle in its orbit located in a ring with internal radius 1 AU and external one 30 AU. The number of particles in the system is also important.

Let us compare graphs of signals, ACF and power spectrum for the same virial ratio $K = -0.4U$, but for experiments with initial particle velocity vectors directed in a random direction tangentially to the circular orbit at N = 20, 30, 40.

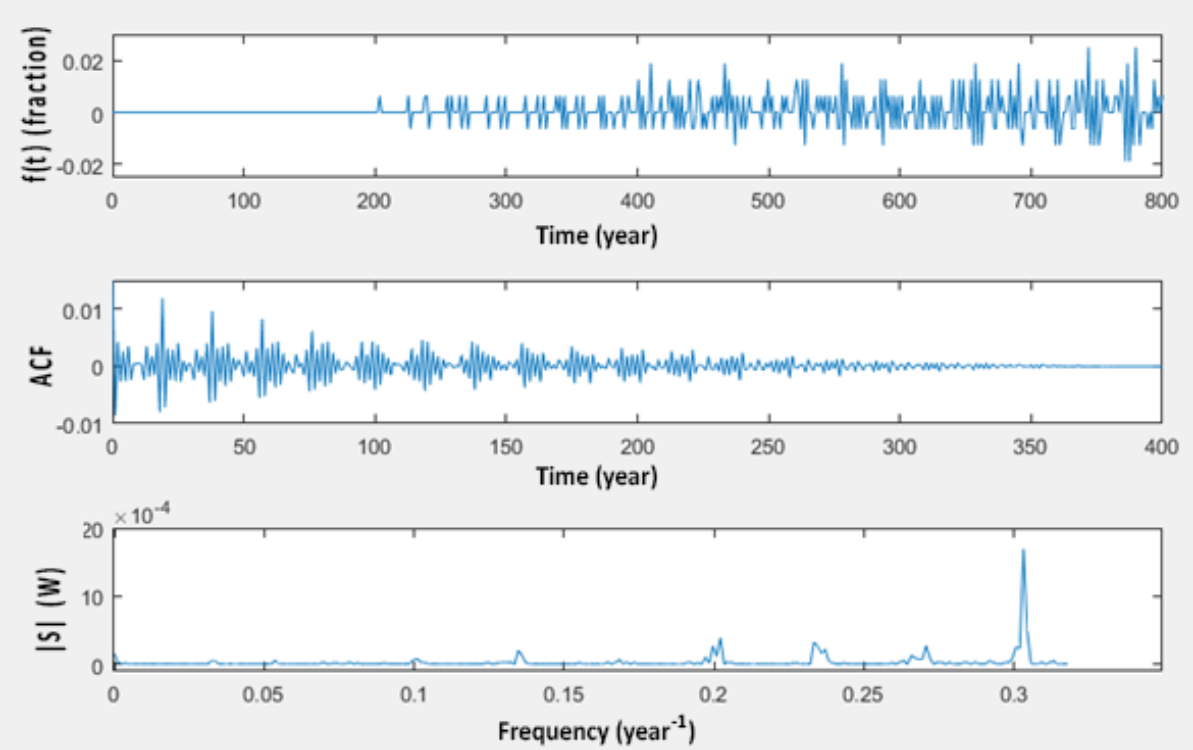

Figure 22.

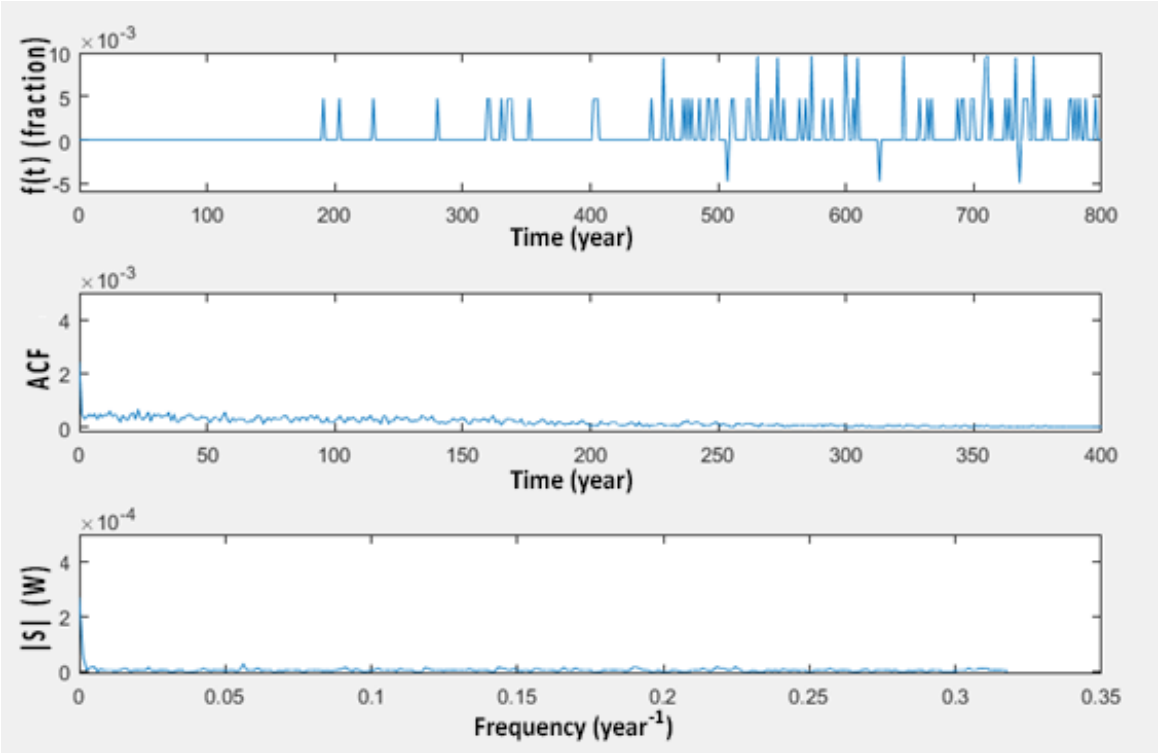

Figure 23.

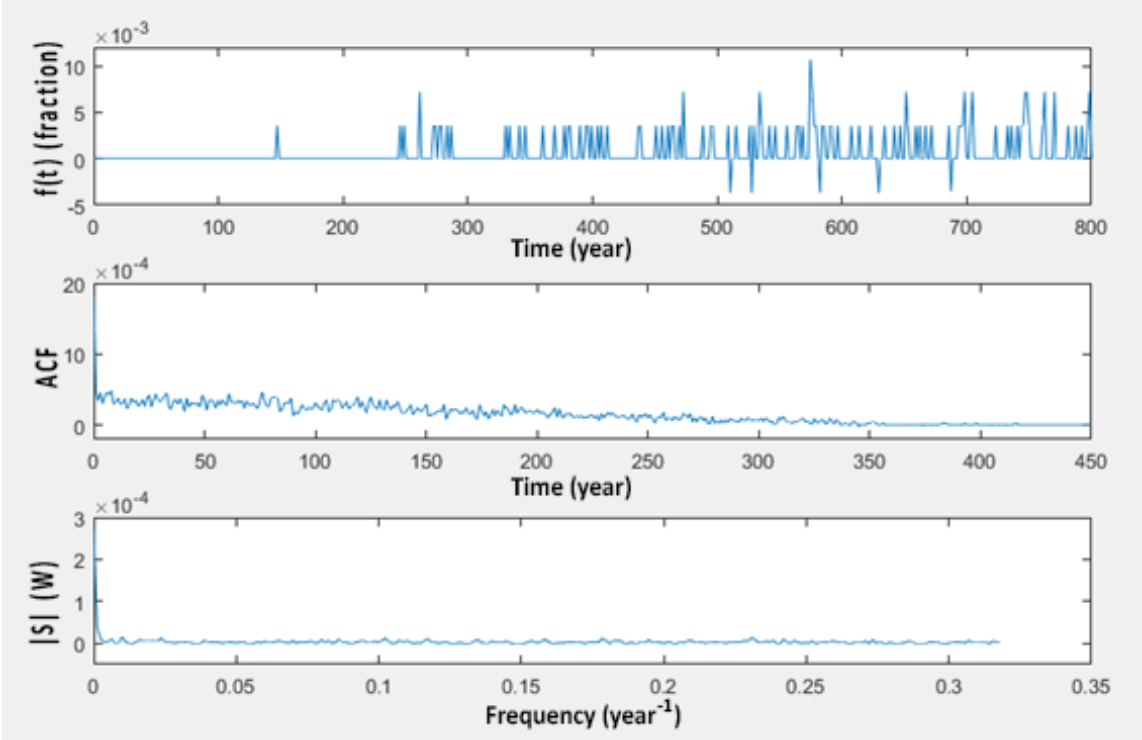

Figure 24.

In the case of N = 20 (Fig. 22), the chaotic behavior of the signal is similar to that of the configuration for the directions of the velocity vectors strictly counterclockwise (Figure 19). The behavior of their autocorrelation functions is also similar, but there is a striking difference between N = 20 and N = 30, 40.

At N = 40 for both configurations the behavior of the signal is somewhat similar, and there are practically no negative values. And there is a noticeable significant difference in the nature of the signal between N = 20 and N = 30, 40 and some similarity between N = 30 and N = 40. It can be said that an increase in the number of particles in the system does not lead to chaotization of this signal.

Let us approximate the power spectrum for the last three signals (Figures 22-24) by the power function $S_N(f) \sim \frac{1}{f^{\beta}}$.

Table 1

Configuration \ Number of particles, N	20	30	40
All particles are directed tangentially to the circular orbit counterclockwise	-0.4199	-0.4314	-0.5776
Random direction tangentially to the circular orbit	-0.3209	-0.5162	-0.5478

The emergence of a $1/f^{\beta}$ - noise is associated with the system's tendency to regulate its dynamics - particles bringing together too closely have to fly away from the system, as they receive too much momentum (a similar behavior is observed in the traffic model of cars).

Also in both systems clusters of particles are observed. It can be assumed that this is promoted by the choice of a particular configuration of the system with a particle ring size of 1-30 AU. Wherein the interaction of the particles is not so strong that initially being close to each other, small groups of particles can form a cluster, and move in it for a sufficiently long period of time until it disintegrates or exchange of its elements with particles from other clusters or single particles occurs.

The results of the numerical simulation can be used both for understanding the origin of planets from a protoplanetary cloud, and for understanding the evolution of stellar clusters and galaxies. In particular, to model the evolution of galaxies, it is necessary to take into account

dark matter, which is a significant part of the mass of the galaxy. If the only interaction in which dark matter participates is gravity, then the results obtained by us can be important for understanding the evolution of dark matter, and its evaporation in galaxies. In particular, the problem of casp-halo for dark matter remains largely unresolved. One of the promising ones is also the problem of the formation of galaxies from primodial perturbations in the early stages of the evolution of the universe.

In the future, we plan to construct a distributed analytical model for the evolution of the ring of material points, with the help of which it would be possible to calculate the spectrum of fluctuations of the departing particles.

Conclusion

As a result of the numerical simulation of the evolution of the gravitating disk with and without the central body, time dependence of the distribution function is obtained. Wherein, the effective diffusion coefficient decreases. This corresponds to the reduction of system chaotization degree. The dependence of the fraction of the evaporated material points on the width of the disk, their mass and initial distance from the central body is obtained. The maximum fraction of the evaporated particles tends to 0.4. The dependence of the maximum Lyapunov exponent on the width of the ring is obtained.

For the case of the absence of a central body, the dependence of fraction of particles that left the system on the virial number is obtained. The dependence of the fraction of pairs of particles leaving the ring on the initial configuration of the system is investigated. The average fraction of such particles turned out to be 0.2.

Semi-phenomenological model for the evaporation of a ring with a central body and without it was constructed.

Power spectra and autocorrelation functions for some system configurations are analyzed. The power spectra of the obtained signals were approximated by a power function of the form $1/f^{\beta}$.

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Figure captions

Fig.1 Initial configuration of bodies.

Fig.2 The velocity distribution function of the material points (20 material points with masses m = 318, width of the ring 29 a.u. after 25000 and 50000 step, respectively).

Fig. 3 The velocity distribution function of the material points (20 material points with masses m = 150, width of the ring 8 a.u. after 25000 and 50000 step, respectively).

Fig. 4 The distribution function of the material points for the radius-vector (20 material points with masses m = 318, ring's width 29 a.u. after 25000 and 50000 steps, respectively).

Fig. 5 The distribution function of the material points for the radius-vector (20 material points with masses m = 150, ring's width 8 a.u. after 25000 and 50000 steps, respectively).

Fig. 6 Dependence of dispersion of the position of particle $\mathbf{D_r}$ on time, where 1 step = 0,01 year.

Fig. 7 Dependence of dispersion of the velocity of particle $\mathbf{D_v}$ on time, where 1 step = 0.01 year.

Fig. 8 The dependence of the fraction of the evaporated particles ***p*** on the mass of the particles ***m*** and on the width of the ring ***L*** during 500 years.

Fig. 9 The dependence of the mass of a particle m on the width of the ring L corresponding to the boundary of the condensed (ordered) phase.

Fig. 10 - The distribution function on the fraction of evaporated particles for the case $N = 20$ and the virial ratio $K = -0.8U$ for configurations of random directions and domains

Fig. 11 - Dependence of the fraction of evaporated particles on the number of particles and virial ratios

Fig. 12 - Dependence of the mean-square fluctuation on the number for various virial ratios in logarithmic coordinates, where $\ln N$ is plotted horizontally, and $\ln f$ is $\ln\left(\frac{\left(\left\langle\left(\Delta N_E\right)^2\right\rangle\right)^{1/2}}{\left\langle N_E\right\rangle}\right)$ is plotted vertically