Dark matter and the energy-momentum of the gravitational field

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Abstract

The Λ Cold Dark Matter cosmological model assumes general relativity is correct. However, the Einstein equation does not contain a symmetric tensor which describes the energy-momentum of the gravitational field itself. Recently, a modified equation of general relativity was developed which contains the missing tensor and completes the Einstein equation. An exact static solution was obtained from the modified Einstein equation in a spheroidal metric describing the gravitational field outside of its source, which does not contain dark matter. The flat rotation curves for a class of galaxies were calculated and the baryonic Tully-Fisher relation followed directly from the gravitational energy-momentum tensor. The Newtonian rotation curves for galaxies with no flat orbital curves, and those with rising rotation curves for large radii were described as examples of the flexibility of the orbital rotation curve equation.

Keywords
general relativity, line element, dark matter, galaxy, rotation curves

1. Introduction

The ΛCDM model describes the formation of galaxies after the Big Bang from cooled baryonic matter gravitationally attracted into a dark matter skeleton. Dark matter in the ΛCDM model also provides the additional mass required to describe the flat rotation curves observed in many galaxies. However, no dark matter particles have been detected and there have been several attempts to explain the flat rotational curves without dark matter.

The leading candidate is a *phenomenological* model of Modified Newtonian dynamics (MOND) introduced by Milgrom [1]. The Newtonian force F is modified according to

$$F = m\mu(\frac{a}{a_0})a \tag{1}$$

where a_0 is a fundamental acceleration $\approx 1.2 \times 10^{-10} m/s^2$. μ is a function of the ratio of the acceleration relative to a_0 which tends to one for $a \gg a_0$ and tends to $\frac{a}{a_0}$ for $a \ll a_0$. MOND successfully explains many, but not all, mass discrepancies observed in galactic data. However, it has no covariant roots in Einstein's equation or cosmological theory. MOND and ΛCDM were thoroughly discussed by McGaugh in [2].

Other alternatives to dark matter were reviewed by Mannheim in [3] with references therein. In particular, Moffat [4] used a nonsymmetric gravitational theory without dark matter to obtain the flat rotation curves of some galaxies. The bimetric theory of Milgrom [5] involved two metrics as independent degrees of freedom to obtain a relativistic formulation of MOND.

Different approaches to the missing matter problem include dipolar dark matter, which was introduced by Bernard, Blanchet and Heisenberg in [6] to solve the problems of cold dark matter at galactic scales and reproduce the phenomenology of MOND. The theory involves two different species of dark matter particles which are separately coupled to the two metrics of bigravity and are linked together

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by an internal vector field. In [7], a theory of emergent gravity (EG) which claims a possible breakdown in general relativity, was introduced by Verlinde that provided an explanation for Milgrom's phenomenological fitting formula in reproducing the flattening of rotation curves. Campigotto, Diaferio and Fatibenec [8] showed conformal gravity cannot describe galactic rotation curves without the aid of dark matter. On the other hand, a logical analysis based on observational data was presented by Kroupa in [9] to support the conjecture that dark matter does not exist.

The existence of dark matter is based on the validity of general relativity. However, Einstein's equation does not contain a symmetric tensor that describes the energy-momentum of the gravitational field itself. It was recently shown for the first time by Nash in [10], that the symmetric tensor $\Phi_{\alpha\beta}$ describes the energy-momentum of the gravitational field, and completes Einstein's equation of general relativity. Φ, the trace of $\Phi_{\alpha\beta}$ with respect to the metric, dynamically replaces the cosmological constant. The positive values of Φ describe dark energy; $\Phi < 0$ represents the energy of the gravitational field interacting with itself. It was shown how the dark energy density could explain the expansion and acceleration of the early universe and establish the cosmological vacuum energy density. This led to an explanation of why the vacuum energy density is so small and yet important in the present epoch. It is therefore natural to investigate if the complete Einstein equation can explain dark matter. As a first step in that direction, this article explores if $\Phi_{\alpha\beta}$ can explain the rotation curves of different types of galaxies without the need for dark matter or a new theory of gravity.

2. The modified Einstein equation in a spheroidal spacetime

In curved spacetime on a 4-dimensional time oriented Lorentzian manifold with metric, $(M, g_{\alpha\beta})$, the modified Einstein equation

$$-\frac{8\pi G}{c^4}\tilde{T}_{\alpha\beta} + G_{\alpha\beta} + \Phi_{\alpha\beta} = 0 \tag{2}$$

was developed in [10]. $\tilde{T}_{\alpha\beta}$ is the total matter energy-momentum tensor which represents all types of matter, including dark matter. In this article, it is assumed dark matter does not exist and that baryonic matter and other possible sources of matter such as neutrinos, produce the gravitational field. The symmetric tensor $\Phi_{\alpha\beta}$ defined by

$$\Phi_{\alpha\beta} = \frac{1}{2}(\nabla_\alpha X_\beta + \nabla_\beta X_\alpha) + u^\lambda(u_\alpha \nabla_\beta X_\lambda + u_\beta \nabla_\alpha X_\lambda) \tag{3}$$

represents the energy-momentum of the gravitational field itself. u is a unit vector collinear with the regular vector X, which is one of $(X, -X)$ in the non-vanishing line element field. Its trace with respect to the metric, Φ, satisfies the global constraint

$$\int g_{\alpha\beta}\Phi^{\alpha\beta}\sqrt{-g}d^4x = \int \Phi\sqrt{-g}d^4x = 0 \tag{4}$$

where $\Phi = \nabla_\alpha X_\beta(g^{\alpha\beta} + 2u^\alpha u^\beta)$. The positive values of Φ describe dark energy and negative values of Φ represent the energy of the gravitational field interacting with itself; $\Phi = 0$ is the condition for free fall.

In a region of spacetime where there is no matter, $\tilde{T}_{\alpha\beta} = 0$ and the field equations must satisfy

$$G_{\alpha\beta} + \Phi_{\alpha\beta} = 0. \tag{5}$$

Spheroidal solutions to these nonlinear equations are now investigated. The spheroidal behaviour of the metric is to be determined from a particular solution to (5) in a spacetime described by a metric of the form

$$ds^2 = -e^\nu c^2dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2) \tag{6}$$

where ν and λ are functions of t, r and θ. The non-zero connection coefficients (Christoffel symbols) are:

$$\begin{aligned}
\Gamma^0_{00} = \frac{1}{2}\partial_0\nu,\ \Gamma^0_{01} = \frac{1}{2}\partial_1\nu,\ \Gamma^0_{02} = \frac{1}{2}\partial_2\nu,\ \Gamma^0_{11} = \frac{1}{2}\partial_0\lambda e^{\lambda-\nu},\\
\Gamma^1_{00} = \frac{1}{2}\partial_1\nu e^{\nu-\lambda},\ \Gamma^1_{01} = \frac{1}{2}\partial_0\lambda,\ \Gamma^1_{11} = \frac{1}{2}\partial_1\lambda,\ \Gamma^1_{12} = \frac{1}{2}\partial_2\lambda,\ \Gamma^1_{22} = -re^{-\lambda},\ \Gamma^1_{33} = -r\sin^2\theta e^{-\lambda},\\
\Gamma^2_{00} = \frac{1}{2r^2}\partial_2\nu e^{\nu},\ \Gamma^2_{11} = \frac{1}{2r^2}\partial_2\lambda e^{\lambda},\ \Gamma^2_{12} = \frac{1}{r},\ \Gamma^2_{33} = -\sin\theta\cos\theta,\\
\Gamma^3_{13} = \frac{1}{r},\ \Gamma^3_{23} = \cot\theta.
\end{aligned} \tag{7}$$

The unit vectors u_β are orthogonal and satisfy

$$u^\alpha u_\alpha = -1 \tag{8}$$

with $u^\alpha u_\beta = 0$ for $\alpha \neq \beta$. As a first step to understand this highly nonlinear set of equations given by (5) with the constraint (8) in this metric, u_3 is chosen to vanish. This requires

$$X_3 = 0 \tag{9}$$

because u_α is collinear with X_α. All other components of X_α are non-zero.

Static solutions to (5) are sought which require the components of the line element field to satisfy

$$\partial_0 X_\alpha = 0, \tag{10}$$

and from the metric,

$$\partial_0\lambda = 0,\ \ \partial_0\nu = 0. \tag{11}$$

The components of $\mathit{\Phi}_{\alpha\beta}$ are then:

$$\mathit{\Phi}_{00} = (1+2u_0u^0)(-\frac{1}{2}e^{\nu-\lambda}\nu' X_1 - \frac{1}{2r^2}e^{\nu}\partial_2\nu X_2), \tag{12}$$

$$\mathit{\Phi}_{11} = (1+2u_1u^1)({X_1}' - \frac{1}{2}\lambda' X_1 - \frac{1}{2r^2}e^{\nu}\partial_2\lambda X_2), \tag{13}$$

$$\mathit{\Phi}_{22} = (1+2u_2u^2)(\partial_2 X_2 + re^{-\lambda}X_1), \tag{14}$$

$$\mathit{\Phi}_{33} = r\sin^2\theta e^{-\lambda}X_1 + \sin\theta\cos\theta X_2, \tag{15}$$

the Ricci scalar, which from (5) equals Φ, is

$$R = e^{-\lambda}(-\nu'' - \frac{1}{2}\nu'^2 + \frac{1}{2}\lambda'\nu' - \frac{2}{r}\nu' + \frac{2}{r}\lambda' - \frac{2}{r^2}) + \frac{1}{r^2}(-\frac{1}{2}\partial_2\nu^2 - \partial_2\partial_2\nu - \partial_2\nu\cot\theta + 2), \tag{16}$$

and the components of the Einstein tensor are:

$$G_{00} = \frac{1}{r^2}e^{\nu-\lambda}(r\lambda' - 1 + e^{\lambda}) + \frac{e^{\nu}}{4r^2}\partial_2\nu\partial_2\lambda, \tag{17}$$

$$G_{11} = \frac{1}{r^2}(1 + r\nu' - e^{\lambda}) + \frac{e^{\lambda}}{2r^2}[\partial_2\partial_2\lambda + \partial_2\partial_2\nu + \frac{1}{2}\partial_2\lambda^2 + \frac{1}{2}\partial_2\nu^2 + \frac{1}{2}\partial_2\lambda\partial_2\nu + \cot\theta(\partial_2\lambda) + \partial_2\nu], \tag{18}$$

$$G_{22} = \frac{r^2e^{-\lambda}}{2}[\nu'' + (\frac{1}{2}\nu' + \frac{1}{r})(\nu' - \lambda')] - \frac{1}{2}\partial_2\partial_2\lambda - \frac{1}{2}\partial_2\lambda^2 + \frac{1}{2}\cot\theta\partial_2\nu, \tag{19}$$

$$G_{33} = \sin\theta^2[\frac{r^2e^{-\lambda}}{2}(-\frac{\lambda'}{r} + \frac{\nu'}{r} + \nu'' + \frac{1}{2}\nu'^2 - \frac{1}{2}\lambda'\nu') + \frac{1}{4}\partial_2\nu^2 + \frac{1}{2}\partial_2\partial_2\nu - \frac{1}{2}\cot\theta\partial_2\lambda] \tag{20}$$

where the prime denotes ∂_1.

These equations are greatly simplified by setting

$$\nu = -\lambda. \tag{21}$$

Thus, a class of static spheroidal solutions to (5) are sought with the restrictions (9),(10),(11) and (21).

Since $e^{\lambda-\nu}(\mathit{\Phi}_{00}+G_{00})+\mathit{\Phi}_{11}+G_{11}=0$ from (5),

$$(\frac{\lambda'}{2}X_1+\frac{1}{2r^2}e^{\lambda}\partial_2\lambda X_2)(1+2u_0u^0)+(X_1'-\frac{\lambda'}{2}X_1-\frac{1}{2r^2}e^{\lambda}\partial_2\lambda X_2)(1+2u_1u^1)=0. \tag{22}$$

If we choose $\frac{\lambda'}{2}X_1=X_1'-\frac{\lambda'}{2}X_1$,

$$X_1=a_1e^{\lambda} \tag{23}$$

where a_1 is an arbitrary but non-zero constant with the dimensions of L^{-1}. Then (22) becomes

$$(\frac{X_1'}{2}+\frac{1}{2r^2}e^{\lambda}\partial_2\lambda X_2)(1+2u_0u^0)+(\frac{X_1'}{2}-\frac{1}{2r^2}e^{\lambda}\partial_2\lambda X_2)(1+2u_1u^1)=0 \tag{24}$$

from which

$$\partial_2\lambda=a_1\lambda' r^2\frac{u_2u^2}{(u_0u^0-u_1u^1)X_2} \tag{25}$$

using $1+u_0u^0+u_1u^1=-u_2u^2$.

From (5) and (23) in the interval $0<\theta<\pi$, $G_{22}+\mathit{\Phi}_{22}=0$ gives

$$(-\lambda''+\lambda'^2-\frac{2}{r}\lambda')+\frac{e^{\lambda}}{r^2}(-\frac{1}{2}\partial_2\lambda^2-\partial_2\partial_2\lambda-\partial_2\lambda\cot\theta)+\frac{2e^{\lambda}}{r^2}(\partial_2X_2+a_1r)(1+2u_2u^2)=0 \tag{26}$$

and $G_{33}+\mathit{\Phi}_{33}=0$ yields

$$-\lambda''+\lambda'^2-\frac{2}{r}\lambda'+\frac{e^{\lambda}}{r^2}(\frac{1}{2}\partial_2\lambda^2-\partial_2\partial_2\lambda-\partial_2\lambda\cot\theta)+\frac{2e^{\lambda}}{r^2}(a_1r+X_2\cot\theta)=0. \tag{27}$$

Subtracting (26) from (27) requires

$$\cot\theta X_2-\partial_2X_2+\frac{1}{2}\partial_2\lambda^2-2u_2u^2(\partial_2X_2+a_1r)=0. \tag{28}$$

Choosing $\cot\theta X_2=\partial_2X_2$ gives

$$X_2=a_2\sin\theta \tag{29}$$

where $a_2\neq 0$ is an otherwise arbitrary dimensionless constant, and demands

$$\partial_2\lambda^2=4u_2u^2(a_2\cos\theta+a_1r). \tag{30}$$

Equation (27) can now be expressed as

$$-\lambda''+\lambda'^2-\frac{2}{r}\lambda'+\frac{e^{\lambda}}{r^2}(\frac{1}{2}\partial_2\lambda^2-\partial_2\partial_2\lambda-\partial_2\lambda\cot\theta)+\frac{2e^{\lambda}}{r^2}(a_1r+a_2\cos\theta)=0. \tag{31}$$

From (25) and (30), the derivative terms in $\partial_2\lambda$ can be neglected if u_2u^2 is restricted to be very small but non-zero. Assuming $\partial_2\partial_2\lambda$ can also be neglected, equation (31) is then approximated by

$$-\lambda''+\lambda'^2-\frac{2}{r}\lambda'+\frac{2e^{\lambda}}{r^2}(a_1r+a_2\cos\theta)=0,\ \ 0<\theta<\pi \tag{32}$$

which, for a fixed value of $\cos\theta$, has the exact solution

$$\lambda=-\ln(\frac{c_1}{r}+c_2-a_1r-2a_2\cos\theta\ln r),\ \ 0<\theta<\pi,\ \ 0<r<\infty \tag{33}$$

where c_1 and c_2 are arbitrary constants.

3. The radial force and galactic rotation curves

The radial force on an object of mass m can now be calculated from (33). Using the conventional relationship of the Newtonian potential ϕ to g_{00},

$$\phi = \frac{c^2}{2}(e^\nu - 1), \tag{34}$$

the radial force F_r is

$$\begin{aligned} F_r &= -m\partial_1\phi \\ &= \frac{mc^2}{2}\left(\frac{c_1}{r^2} + \frac{2a_2\cos\theta}{r} + a_1\right). \end{aligned} \tag{35}$$

Choosing

$$c_1 = -\frac{2GM}{c^2}, \tag{36}$$

where M represents the total mass of the galaxy composed of mainly baryonic matter and no dark matter, we arrive at the modified Newtonian force

$$F_r = -\frac{GMm}{r^2} + \frac{mc^2a_2\cos\theta}{r} + \frac{mc^2a_1}{2}. \tag{37}$$

The correction terms to the Newtonian force come from the non-zero components of the line element field in the energy-momentum tensor $\Phi_{\alpha\beta}$. The components of the line element field can change their sign, which means a_j can change to $-a_j$ with j=1,2 in this restricted metric. Thus the middle term is gravitationally attractive and represents the "dark matter" correction if $a_2\cos\theta < 0$ in the interval $0 < \theta < \pi$. It is the term that gives rise to the flat rotation curves. The third constant term is positive and repulsive if $a_1 > 0$. This describes the repulsive dark energy force in the present epoch. However, during a part of the previous decelerating epoch observed by Riess et al. [11], $a_1 < 0$. They used the Hubble telescope to provide the first conclusive evidence for cosmic deceleration that preceded the current epoch of cosmic acceleration.

Assuming a circular orbit about a point mass, it follows that the orbital velocity of a star rotating in the galaxy satisfies

$$v^2 = v_N^2 - a_2c^2\cos\theta - \frac{a_1c^2}{2}r \tag{38}$$

where v_N^2 is the Newtonian term

$$v_N^2 = \frac{GM}{r}. \tag{39}$$

Equation (38) demands an upper limit to r describing a large but finite galaxy.

Because $a_1 \neq 0$, it is possible for the Newtonian force to balance the dark energy force,

$$v_N^2 - \frac{a_1c^2}{2}r = 0 \tag{40}$$

in (38). Then,

$$v^2 = -a_2c^2\cos\theta, \;\; a_2\cos\theta < 0 \tag{41}$$

describes a specific class of galaxies with a flat orbital rotation curve. From (39) and (40), we obtain the Tully-Fisher relation

$$v_N^4 = \frac{GMc^2a_1}{2}, \;\; a_1 > 0. \tag{42}$$

This result holds for any finite r in contrast to EG which holds only for large r as determined by Lelli, McGaugh and Schombert [12]. With $\frac{c^2a_1}{2} := a_0$, the Tully-Fisher relation in MOND is evident.

The importance of the radial acceleration relative to the rotation curves of galaxies was discussed by Lelli, McGaugh, Schombert, and Pawlowski in [13] where it was determined that late time galaxies (spirals and irregulars), early time galaxies (ellipticals and lenticulars), and the most luminous dwarf spheroidals follow the same baryonic Tully-Fisher relation. The observed acceleration correlates well with that expected from the distribution of baryons.

Equation (37), which does not include dark matter in this analysis, is general enough to describe the rotation curves of many types of galaxies. For example, galaxy NGC4261 has a relatively flat rotation curve but starts to rise at larger radii, reaching velocities of 700 km s^{-1} at 100 kpc [11]. That requires a_1 in (38) to be negative which was interpreted above. As another example, both $c^2 a_1 r$ and $c^2 a_2$ could be small enough relative to $\frac{GM}{r}$ so that the Newtonian term is dominant. Galaxies with no flat rotation curves have recently been observed by van Dokkum et.al [14]. It should also be remembered that equation (38) came from an approximation to equation (31) which could be used to model galaxies in greater detail. Furthermore, equation (31) is a restricted version of the general equation (5) which provides additional variables that may explain even more aspects of cosmology now attributed to dark matter.

However, it is still possible that dark matter particles may exist. As a part of (2) in the total matter energy-momentum tensor $\tilde{T}_{\alpha\beta}$, they would contribute to the gravitational field outside of its source along with baryonic matter in equation (5) and therefore in (37). But any dark matter contribution to the gravitational field would play a much lesser role because of the existence of $\Phi_{\alpha\beta}$.

4. Conclusion

The modified Einstein equation contains a symmetric tensor that was missing in the theory of general relativiy. $\Phi_{\alpha\beta}$ completes the Einstein equation which requires $G_{\alpha\beta} + \Phi_{\alpha\beta} = 0$ in a region of spacetime outside of the source of the gravitational field. This is in contrast to general relativity where $G_{\alpha\beta} = 0$ because of the missing tensor for the energy-momentum of the gravitational field. $\Phi_{\alpha\beta}$ introduces correction terms to the Newtonian force and the equation for the orbital rotation curves of galaxies.

An exact static solution was obtained from the modified Einstein equation in a restricted spheroidal metric describing the gravitational field outside of its source, which does not contain dark matter. The flat rotation curves for a class of galaxies were calculated and the baryonic Tully-Fisher relation followed directly from the energy-momentum tensor of the gravitational field. The Newtonian rotation curves for galaxies with no flat orbital curves, and those with rising rotation curves for large radii were described as examples of the flexibility of the orbital rotation curve equation.

The results obtained from the *complete* Einstein equation thus far are able to substantially describe the missing mass problem attributed to dark matter. Further mathematical and detailed numerical analyses to explore the ability of the energy-momentum tensor of the gravitational field to replace dark matter in cosmology, are fully warranted. This rigourous analysis with comparison to astronomical data may still point to the existence of dark matter to some extent. But even if that is the case, the gravitational role of dark matter is substantially reduced by the impact of the energy-momentum tensor of the gravitational field.

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