A new conjecture on the divisor summatory function offering a much higher prediction accuracy than Dirichlet's divisor problem approach

*

Wiki-like transdisciplinary article (Open development interval: 2018 - ?)
- working paper preprint[1] -
Version 1.0 (17.06.2018)

*

Andrei-Lucian Drăgoi[2] (**version 1.0 published on: 17.06.2018; last update on: 17.06.2018**)
Pediatrician specialist with both interdisciplinary and transdisciplinary independent research in physical sciences
(Bucharest, Romania), E-mail: dr.dragoi@yahoo.com

*

Important note: The latest (free) version of this article can be downloaded from this URL.

*

Abstract

This paper presents a new conjecture on the divisor summatory function (also in relation with prime numbers), offering a much higher prediction accuracy than Dirichlet's divisor problem approach.

Keywords: conjecture; divisor function; divisor summatory function; prime numbers; Dirichlet's divisor problem

Important note (1). This atypical URL-rich paper (which maximally exploits the layer of hyperlinks in this document), chooses to use Wikipedia links for all the important terms used. The main motivation for this approach was that each Wikipedia web-article contains all the main reference (included as endnotes) on the most important terms used in this paper: it's simply the most practical way to cite entire collections of important articles/books without using an overwhelming list of footnote/endnote references. The secondary motivation (for using Wikipedia hyperlinks directly included in keywords) was to assure a "click-away" distance to short encyclopedic monographs on all the (important) terms used in this paper, so that the flow of reading to be minimally interrupted.
Important note (2). This paper also exploits the advantages of the hierarchic tree-like model of presenting informational content which is very easy to be kept updated and well organized.

---

[1] Online paper preprint DOI: 10.13140/RG.2.2.19770.49603
[2] Andrei-Lucian Dragoi research pages on: ResearchGate, Academia.edu, ViXra, GSJournal;

I. Introduction

1) **Introduction to the divisor function**. From the number theory literature and given a matrix $M_n = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 2 \\ 4 & 3 \\ \vdots & \vdots \\ n & d_n \end{pmatrix}$ which counts (on its 2nd column) **the number of (trivial plus non-trivial) positive divisors** (d_n) for each (non-zero) natural number $n \geq 1$ (A000005 OIES sequence), d_n (in simplified notation and usually noted $d(n)$ or $\tau(n)$; aka "the divisor function" $\boxed{d(n) = (\nu_1+1)(\nu_2+1)...(\nu_k+1)}$ with $\nu_1, \nu_2 ... \nu_k$ being the exponents of the prime factorization[URL2] of $\boxed{n = p_1^{\nu_1} p_2^{\nu_2} ... p_k^{\nu_k} = \prod_{j=1}^{k} p_j^{\nu_j}}$; also known in sigma notation as the special case $\boxed{\sigma_0(n) = \sum_{d|n} d^0 \left[= d(n) \right]}$ of $\boxed{\sigma_x(n) = \sum_{d|n} d^x}$ [URL] with x being a real or complex number) appears in a number of remarkable identities (including relationships on the Riemann zeta function and the Eisenstein series of modular forms) and has some well-known properties like [URL1, URL2, URL3, URL4]:

a) $\boxed{d_n < 2\sqrt{n}}$, for any (non-zero) natural number $n \geq 1$;

b) In 1838, P. G. L Dirichlet showed that the (arithmetic) average number of divisors $\boxed{d_{n(av)} = \sum_{k=1}^{n} d_k / n}$ has the property $\boxed{d_{n(av)} \cong d_{n(av)(D)} \left[= \ln(n) + (2\gamma - 1) \right]}$ [URL-MathWorld] and that divisor summatory function (**DSF**) $\boxed{S_n = \sum_{k=1}^{n} d_k \left(= n \cdot d_{n(av)} \right)}$ (with the simplified notation S_n replacing the standard notation of DSF $\boxed{D(n) = \sum_{k=1}^{n} d(k)}$) has the property that $\boxed{S_{n(D)} = n\left[\ln(n) + (2\gamma - 1) \right] + O\left(n^\theta\right)}$ for any natural number $n \geq 1$ (this "S_n" predicted by Dirichlet was abbreviated as $S_{n(D)}$ or "**DSn**" so that to be distinguished from S_n and to be compared with the other predicted $S_{n(pr)}$ proposed by the conjecture presented in this paper) [URL1, URL2, URL3], with the following explanations, definitions and notations:

i) Euler–Mascheroni (gamma) constant γ (the limiting difference between the harmonic series and the natural logarithm) is predefined as $\boxed{\gamma = \lim_{n \to \infty} \left(\sum_{k=1}^{n} k^{-1} - \ln(n) \right) = \int_1^\infty \left(\frac{1}{floor(x)} - 1/x \right) \cong 0.5772}$ (with floor function $floor(x)$ predefined as the greatest integer $i \leq x$ and x being a real number)

ii) the big O (Bachmann–Landau or asymptotic) notation is predefined such as: $f(x) \in O(g(x))$ or $f(x) = O(g(x))$ **if and only if** there exists both a real constant $c > 0$ and a finite real number x_0, such that $f(x) \le c \cdot g(x)$ for any real number $x \ge x_0$

iii) the theta exponent (θ) of non-leading term $O\left(n^{\theta}\right)$ of DSn is the "target" of Dirichlet's divisor problem (**DDP**) which is: **to find the smallest value of** θ **(noted** $\theta_{\min}$**) for which** $\boxed{S_n - n\left[\ln(n) + (2\gamma - 1)\right] = O\left(n^{\theta_{\min}+t}\right)}$ **for any** $t > 0$**.** Until present, it is widely conjectured that $\theta_{\min} \overset{conj.}{\underset{?}{=}} 1/4$ and $\theta_{\min}$ was already formally demonstrated to be in double closed interval $[1/4, 131/416(\cong 0.3149)]$ by M.N. Huxley in 2003. DDP is one of the major arithmetical problems still unsolved up to present, but the new conjecture presented in this paper offers a practical alternative method to approximate S_n (also in relation with prime numbers), independently to DSn and its predicted $S_{n(D)}$.

iv) One consequence of $S_{n(D)}$ definition is that a randomly chosen (non-zero) natural number $n \ge 1$ has an expected number of divisors $\boxed{d_n \cong \left[d_{n(av)} \cong \ln(n) + (2\gamma - 1)\right] \cong \ln(n)}\left(< 2\sqrt{n}\right)$, which implies that $\boxed{S_{n(D)} \cong \sum_{k=1}^{n} \ln(k) \cong \ln(n!)}$. The **graph** of the ratio $\boxed{d_n / \ln(n) \cong 1}$ (with red linear trend line added) and the **graph** of its absolute error (in base-10 logarithmic scale) $\boxed{\log_{10}\left|1 - d_n / \ln(n)\right|}$ (which indicates an interesting oscillating accuracy of the approximation $\boxed{d_n / \ln(n) \cong 1}$) are presented next.

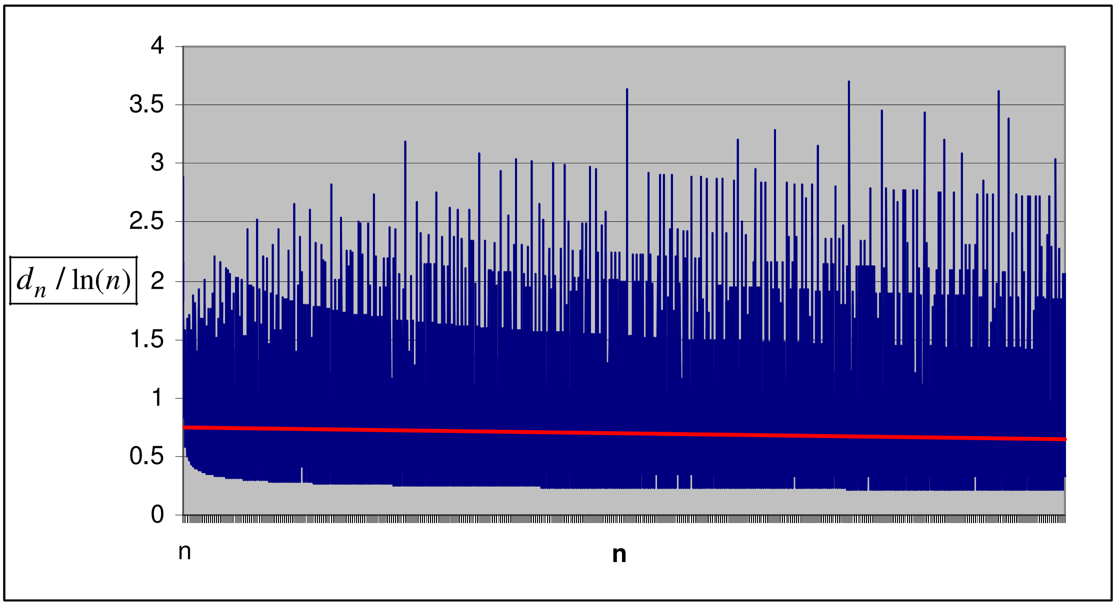

Figure Intro-1a. The values of the ratio $d_n / \ln(n)$ for the natural number $n \in \left[1, 10^4\right]$.

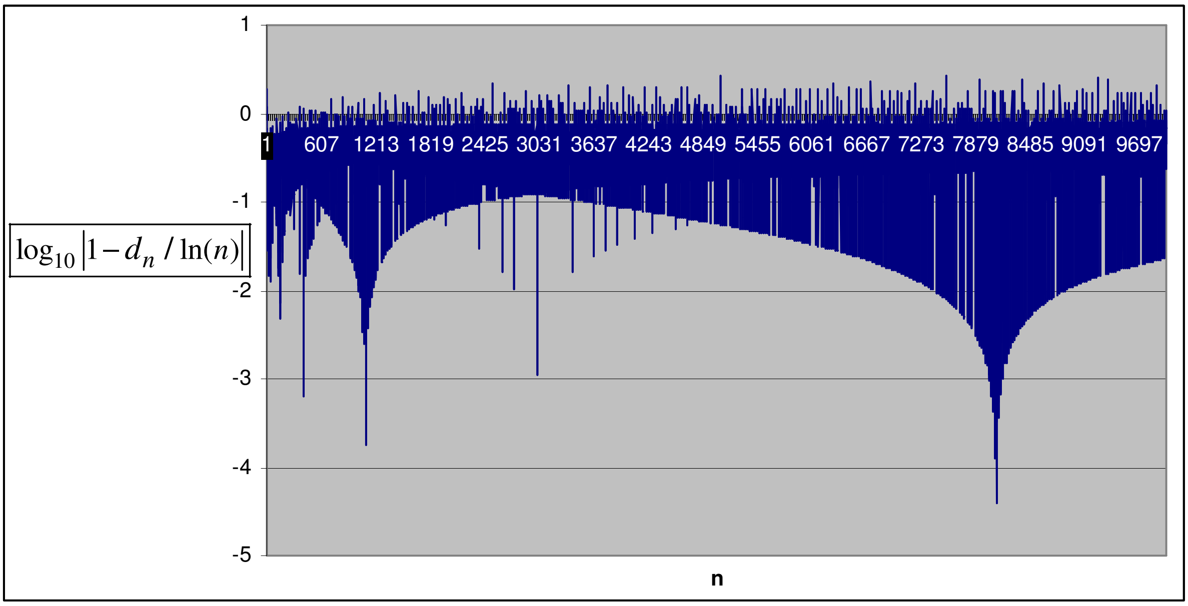

Figure Intro-1b. The values of $\log_{10}|1-d_n/\ln(n)|$ for the natural number $n\in[1,10^4]$.

***

II. Some new conjectures on the divisor function

1) **New conjectures on the divisor function**. The author of this paper discovered some new conjectures on $\boxed{S_n = \sum_{k=1}^{n} d_k}$ with relatively high accuracy in predicting S_n and which can be used as alternatives to Dirichlet's DSn: **see next**.

2) **Conjecture no. 1 (C1)**. $\boxed{E_n = \left(\frac{e^{S_n/n}}{3\sqrt{n}}\right)^2 \cong 1}$, an approximate equality which becomes progressively more exact with the growth of the natural number $n \geq 1$ to infinity. C1 additionally states that $\boxed{E_n < 1}$ for any natural number $n \geq 1082$. **See the next graph**.

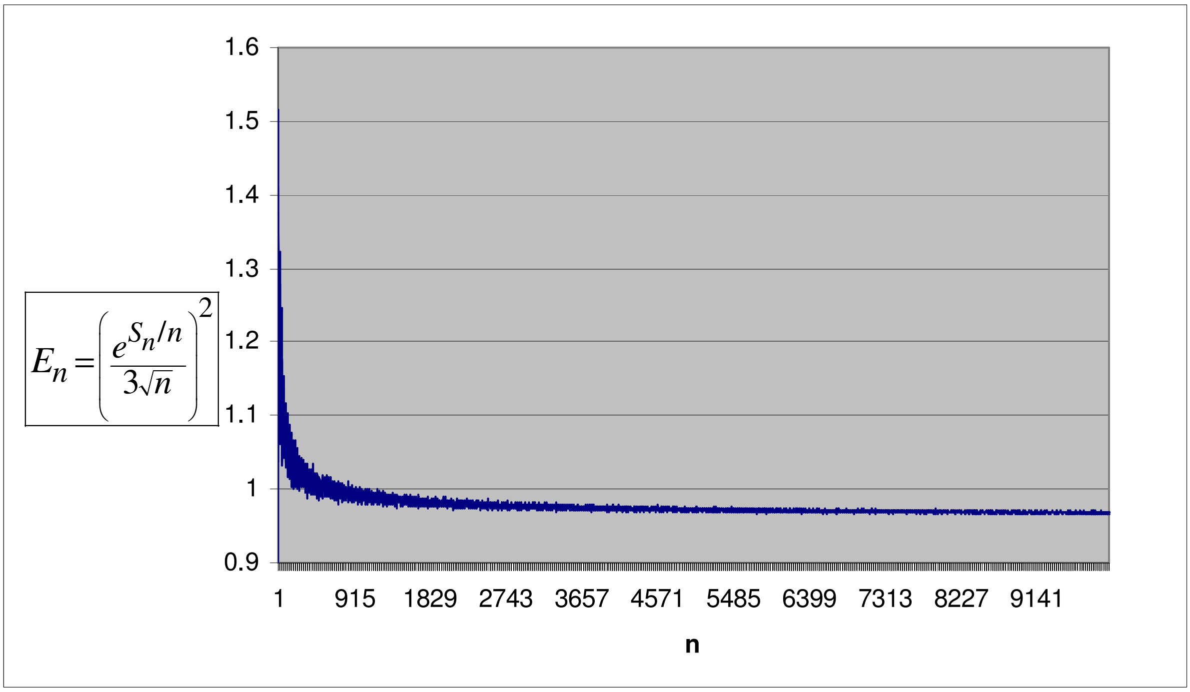

Figure C1-1a. The values of E_n for the natural numbers $n \in \left[1, 10^4\right]$

a) Interestingly, the values of the function $\boxed{\log_{10}\left|1 - E_n\right|}$ (which measures the closeness of $\boxed{E_n\,(\cong 1)}$ to value 1: the absolute error measured logarithmically), tends to stabilize its values very close to -1.5 so that $\boxed{E_n \cong 1 - 10^{-3/2} \cong 0.968}$ for $\boxed{n \geq 1082}$: **see the next graph**.

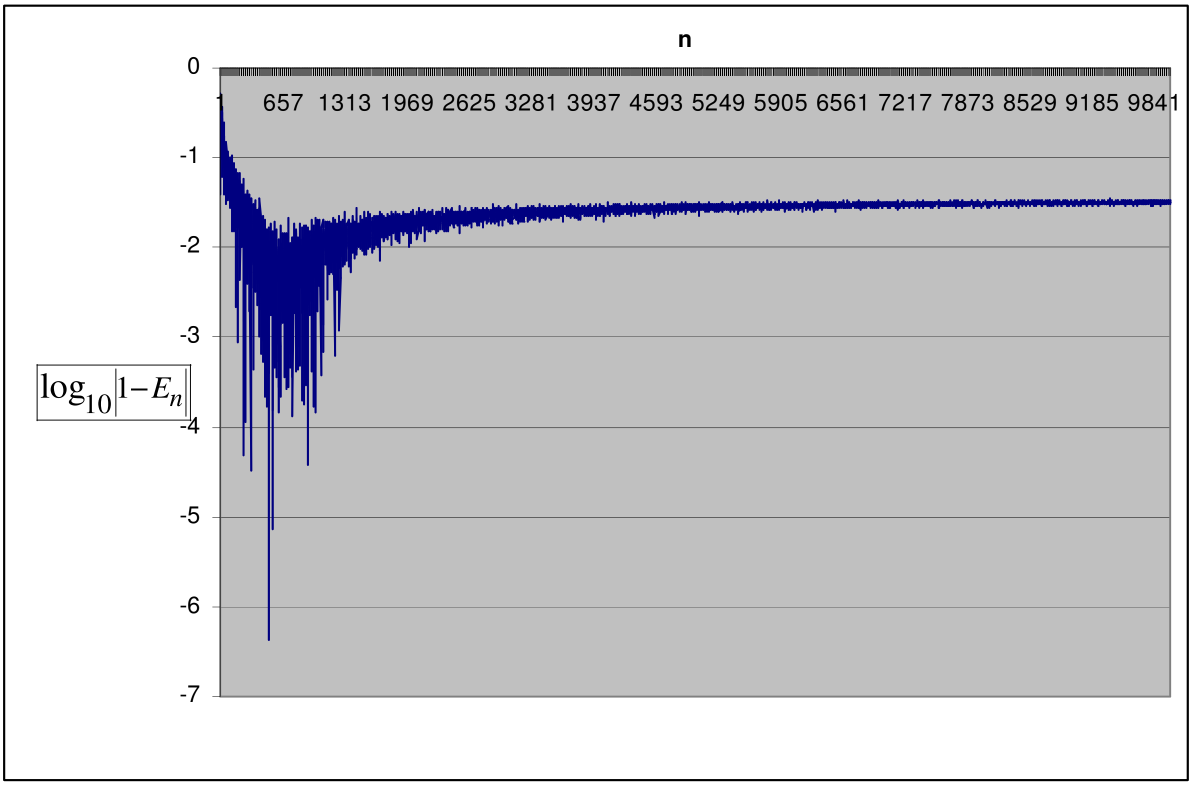

Figure C1-1b. The values of $\boxed{\log_{10}|1-E_n|}$ for the natural numbers $n \in \left[1, 10^4\right]$

3) Actually, the value $\boxed{x = 1 - 10^{-3/2} \cong 0.968}$ appears as the real "target" around which E_n tends to stabilize: **see the next graph of the function** $\boxed{\log_{10}|E_n - x|}$.

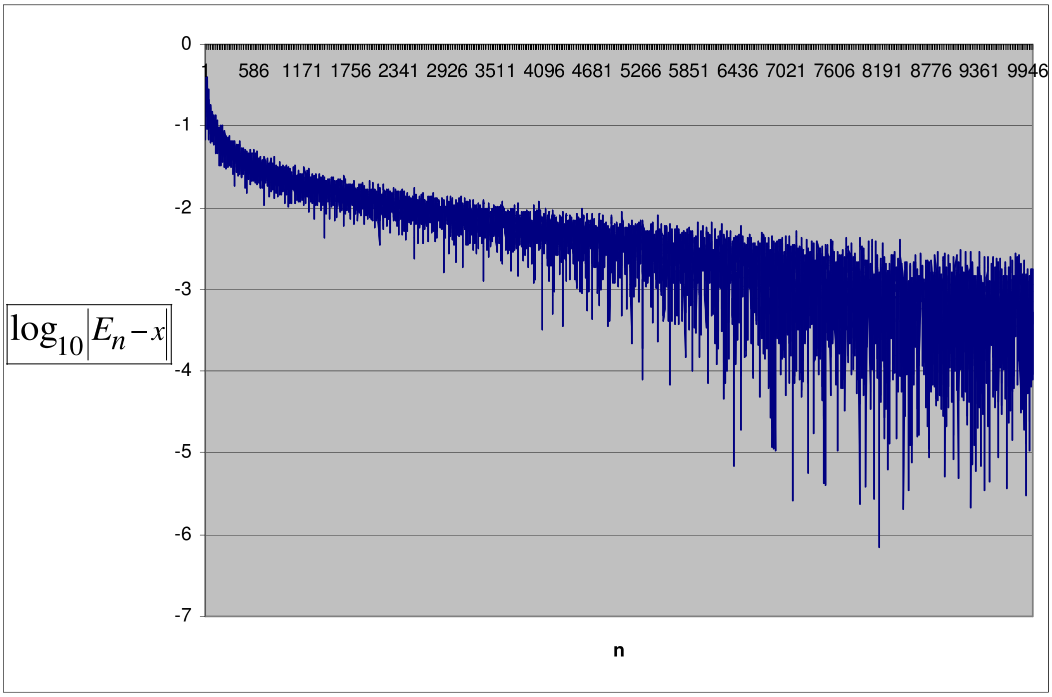

Figure C1-1c. The values of $\boxed{\log_{10}|E_n - x|}$ for the natural numbers $n \in \left[1, 10^4\right]$

a) **Redefinition (1)**. Based on the progressive decrease of $\boxed{\log_{10}|E_n - x|}$, C1 can be refined and rewritten as $\boxed{E_n = \left(\frac{e^{S_n/n}}{3\sqrt{n}}\right)^2 \cong x\left(=1-10^{-3/2} \cong 0.968\right)}$. **Note**. C1 allows to rapidly predict (with relative high accuracy) the value of S_n for any $n \geq 1$, such as $\boxed{S_n \cong n\ln\left(3\sqrt{xn}\right)}$. Defining the predicted (pr) $\boxed{S_{n(pr)} = n\ln\left(3\sqrt{xn}\right)}$, the ratio $\boxed{S_{n(pr)} / S_n \cong 1}$ is **graphed below**. The absolute error measured logarithmically as $\boxed{\log_{10}\left|1 - S_{n(pr)} / S_n\right|}$ is also **graphed below (with a red linear trend line added)**: from this (second) graph, one can observe that $\boxed{\log_{10}\left|1 - S_{n(pr)} / S_n\right| \cong -\log_{10}(n)}$, which is equivalent to $\boxed{\left|1 - S_{n(pr)} / S_n\right| \cong 1/n}$ and $\boxed{S_{n(pr)} / S_n \cong 1 - 1/n}$ (as also seen from the graph of $\boxed{S_{n(pr)} / S_n \cong 1}$)

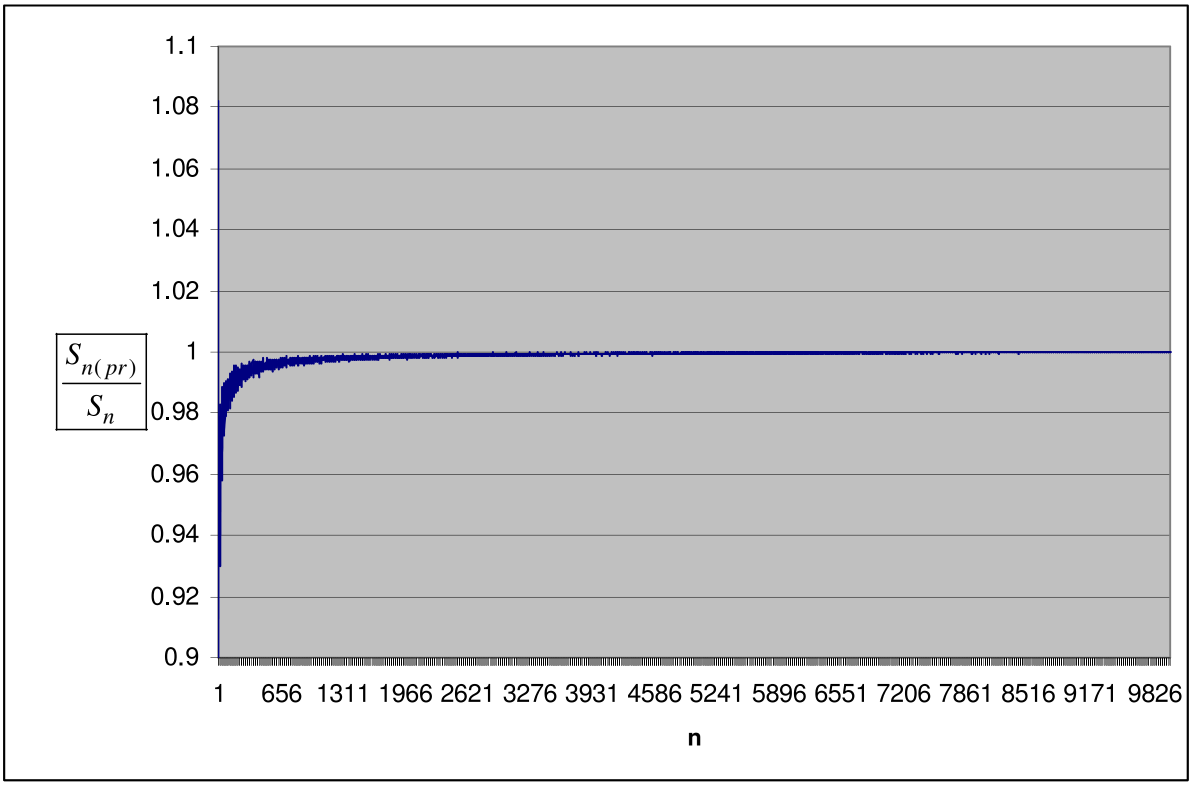

Figure C1-1d. The values of the ratio $\boxed{S_{n(pr)} / S_n \cong 1}$ for the natural numbers $n \in \left[1, 10^4\right]$

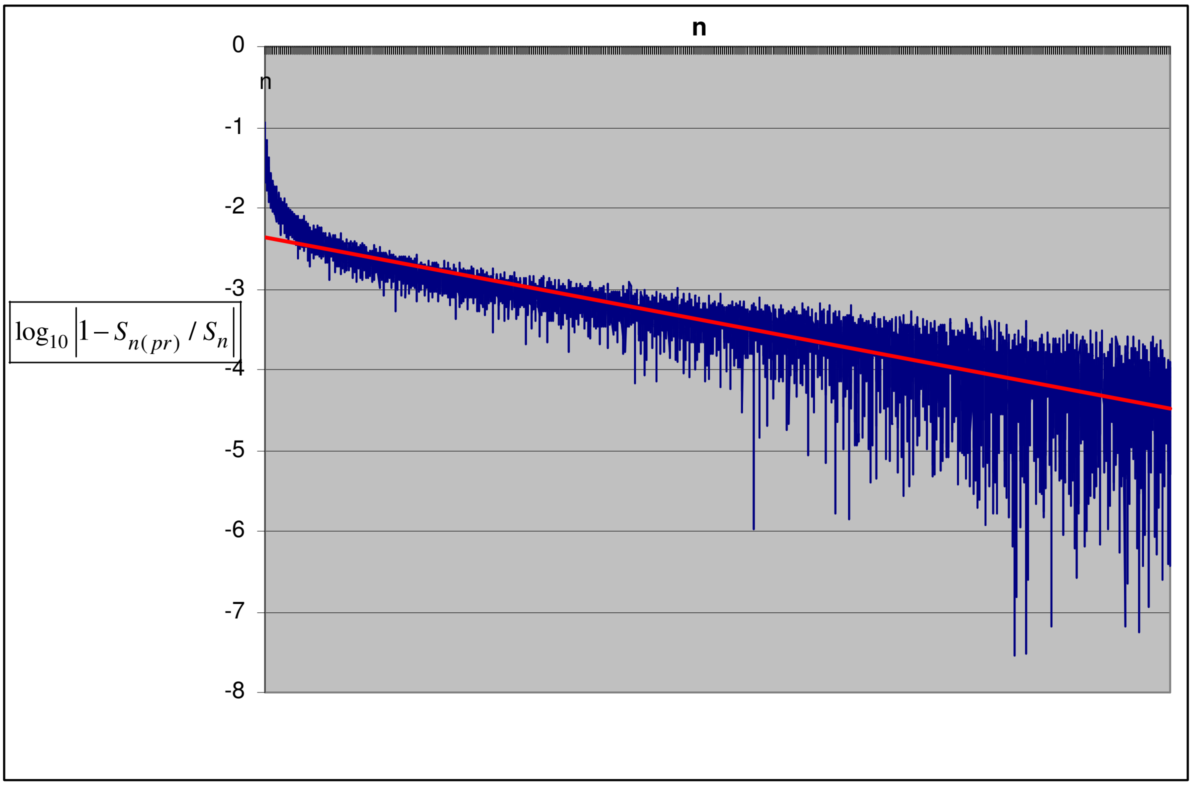

Figure C1-1e. The values of $\boxed{\log_{10}\left|1-S_{n(pr)}/S_n\right|}$ for the natural numbers $n\in\left[1,10^4\right]$

b) **Redefinition (2)**. Based on the additional property $\boxed{S_{n(pr)}/S_n\cong 1-1/n}$ (which is equivalent to $\boxed{S_n\cong S_{n(pr)}/\left(1-1/n\right)}$), C1 can be refined as $\boxed{S_n\cong n\ln\left(3\sqrt{xn}\right)/\left(1-1/n\right)}$, with $\boxed{x=1-10^{-3/2}\cong 0.968}$ and natural number $n\geq 1$. (Re)defining $\boxed{S_{n(pr2)}=n\ln\left(3\sqrt{xn}\right)/\left(1-1/n\right)}$, the ratio $\boxed{S_{n(pr2)}/S_n\cong 1}$ and its associated $\boxed{\log_{10}\left|1-S_{n(pr2)}/S_n\right|}$ is **graphed next.**

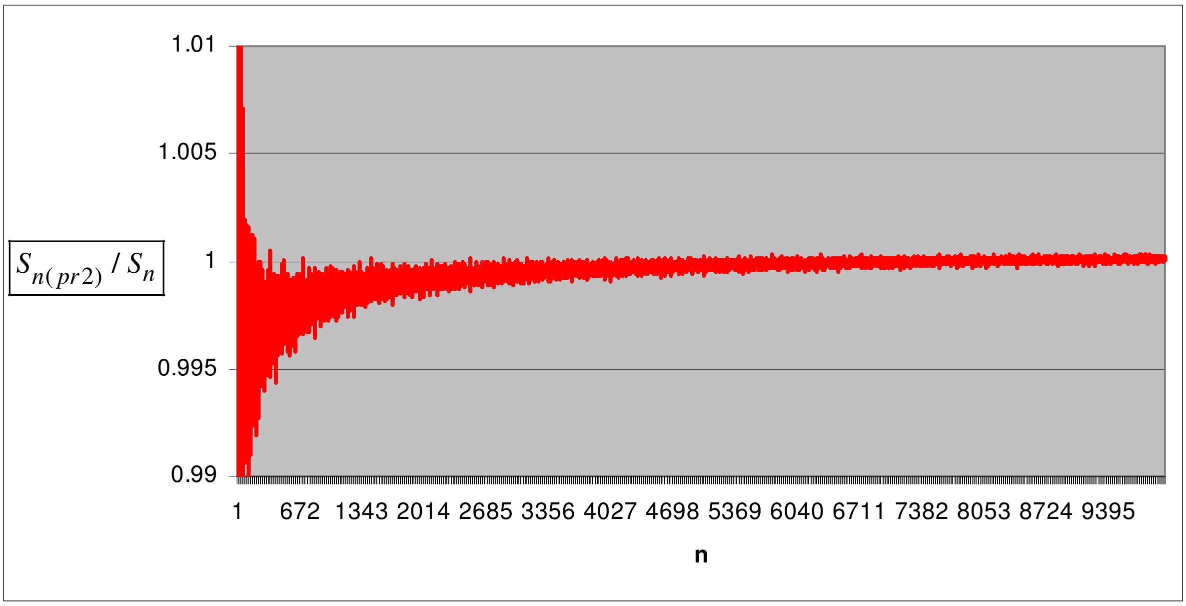

Figure C1-2a. The values of the ratio $\boxed{S_{n(pr2)}/S_n \cong 1}$ for the natural numbers $n \in \left[1, 10^4\right]$.

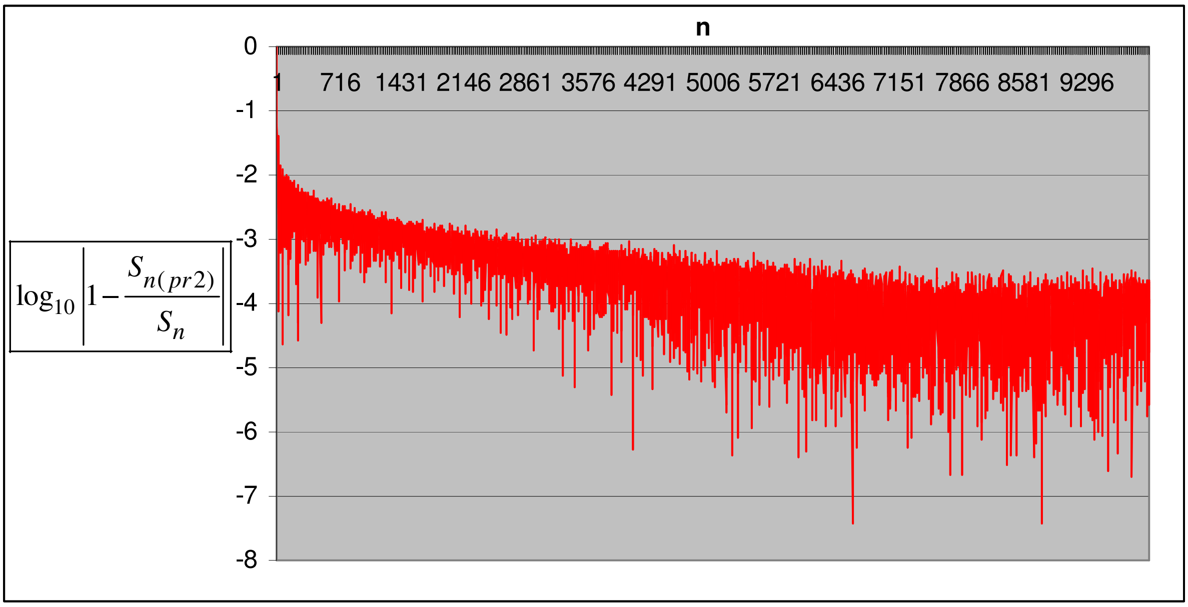

Figure C1-2b. The values of $\boxed{\log_{10}\left|1-S_{n(pr2)}/S_n\right|}$ for the natural numbers $n \in \left[1, 10^4\right]$.

c) **Redefinition (3)**. The graph of $\boxed{\log_{10}\left|1-S_{n(pr2)}/S_n\right|}$ one can also observe that $\boxed{\log_{10}\left|1-S_{n(pr2)}/S_n\right| \cong -\log_{10}(n)}$, which is equivalent to $\boxed{\left|1-S_{n(pr2)}/S_n\right| \cong 1/n}$ and $\boxed{S_{n(pr2)}/S_n \cong 1-1/n}$: this implies that $\boxed{n\ln\left(3\sqrt{xn}\right)/\left(1-1/n\right) \cong S_n\left(1-1/n\right)}$, which is equivalent to $\boxed{S_n \cong n\ln\left(3\sqrt{xn}\right)/\left(1-1/n\right)^2}$ so that **a predicted** $S_{n(pr3)}$ **can be further refined as**

$\boxed{S_{n(pr3)} = n\ln\left(3\sqrt{xn}\right)/\left(1-1/n\right)^2}$. **C1 additionally states that the function** $\boxed{S_{n(\sup)} = n\ln\left(3\sqrt{n}\right)/\left(1-1/n\right)^2}$ **is a superior limit for** S_n **for any natural number** $\boxed{n \geq 1082}$**, so that** $\boxed{S_n = O\left(S_{n(\sup)}\right)}$ **: see the next graph of the ratio** $\boxed{S_{n(\sup)} / S_n \underset{(n\geq 1082)}{\overset{>}{\cong}} 1}$**.**

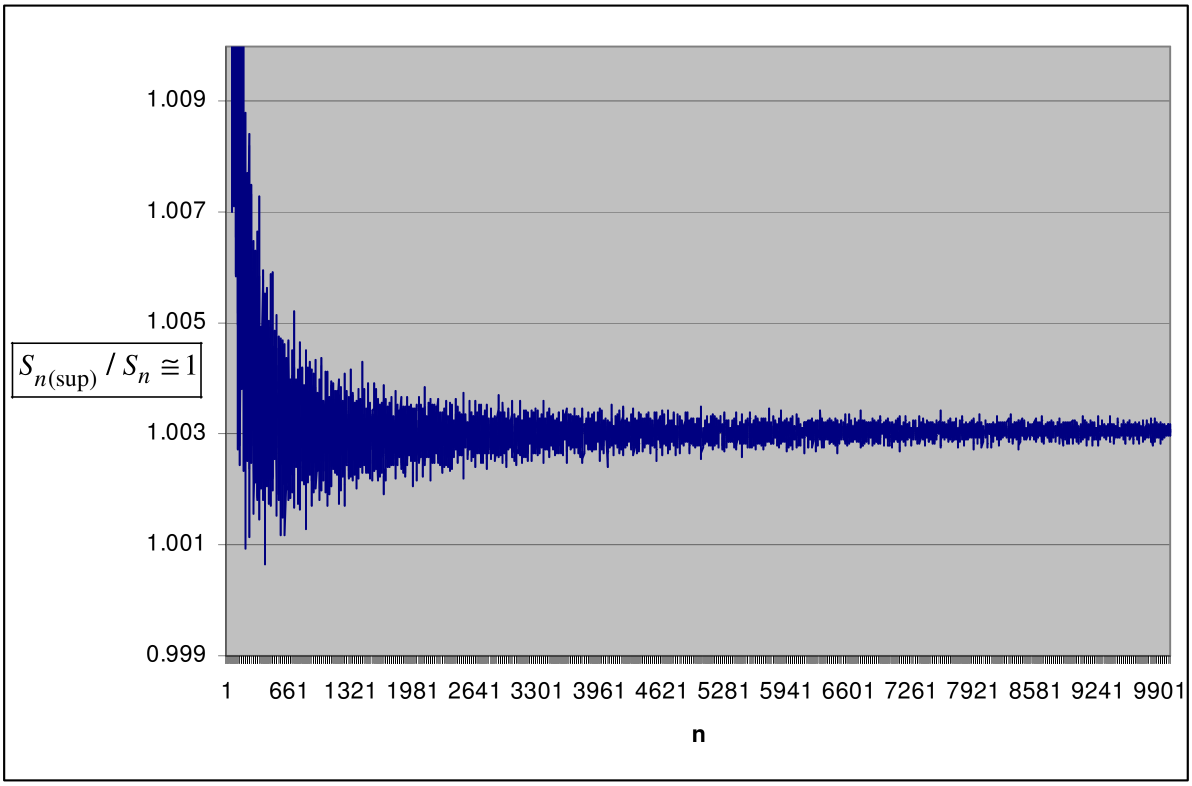

Figure C1-3a. The values of the ratio $\boxed{S_{n(\sup)} / S_n \cong 1}$ for the natural numbers $n \in \left[1, 10^4\right]$.

d) **Redefinition (4)**. $S_{n(pr3)}$ **supports further refining with even higher accuracies, by using an "accessory" function** $\boxed{f(n) = n^{-n/\ln(n)}}$**, so that** $\boxed{S_{n(pr4)} = n\ln\left(3\sqrt{xn}\right)/\left(1-1/n\right)^{2-f(n)}}$**.**

e) The predicted (arithmetic) average number of divisors $\boxed{d_{n(av)(pr3)} = S_{n(pr3)} / n = \ln\left(3\sqrt{xn}\right)/\left(1-1/n\right)^2}$ and $\boxed{d_{n(av)(pr4)} = S_{n(pr4)} / n = \ln\left(3\sqrt{xn}\right)/\left(1-1/n\right)^{2-f(n)}}$ (with $\boxed{f(n) = n^{-n/\ln(n)}}$) can be compared with DSn prediction $\boxed{d_{n(av)(D)} = \ln(n) + (2\gamma - 1)}$. For example, $d_{n(av)(pr3)}$ generates much more accurate predictions for $\boxed{d_{n(av)} = \sum_{k=1}^{n} d_k / n = S_n / n}$ than $d_{n(av)(D)}$ does: for comparison, the ratios $d_{n(av)(pr3)} / d_{n(av)}$ and $d_{n(av)(D)} / d_{n(av)}$ are **graphed next** in red and blue respectively.

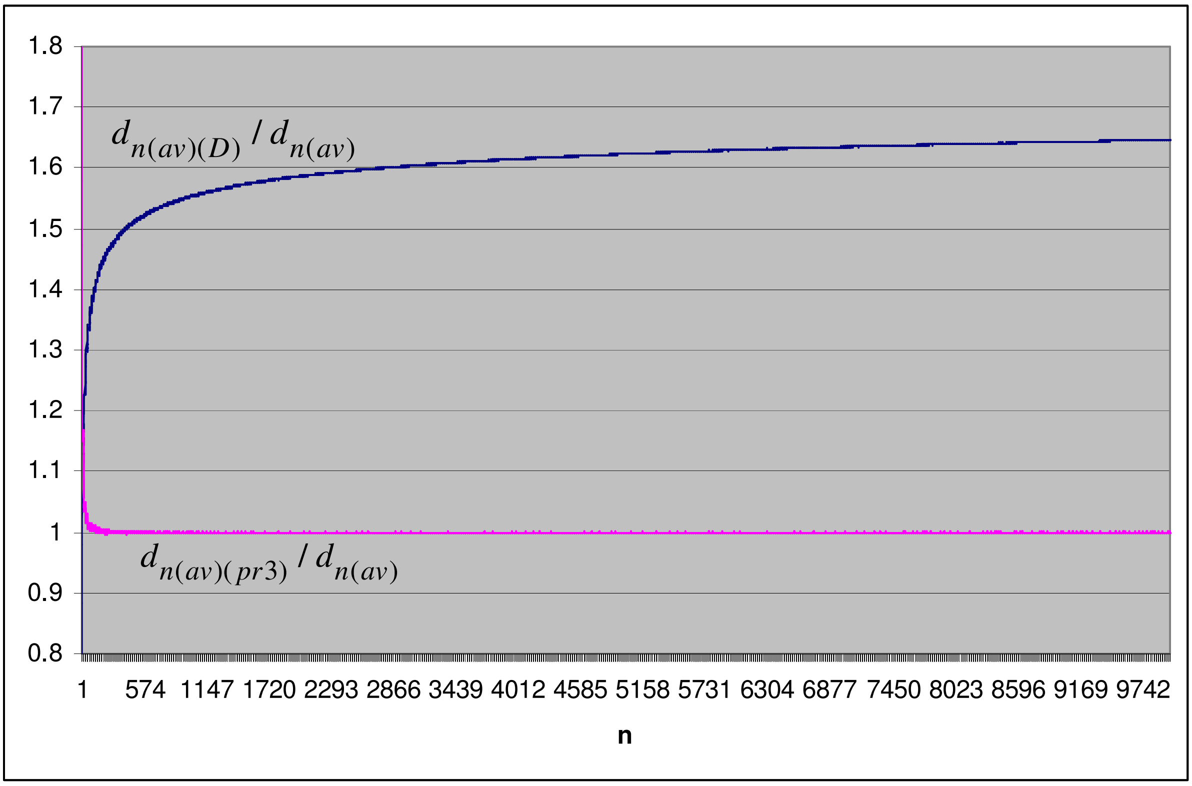

Figure C1-4. A comparison between the ratios $d_{n(av)(pr3)} / d_{n(av)}$ (in red) and $d_{n(av)(D)} / d_{n(av)}$ (in blue), to emphasize the much higher accuracy of C1 when compared to DSn in predicting $d_{n(av)}$ for the natural numbers $n \in \left[1,10^4\right]$.

f) **Remark**. C1 is also indirectly related to the prime-number theorem because an important element which "slows down" the progressive growth of S_n (and the growth of the exponential E_n implicitly, which is conjectured to remain subunitary for any $n \ge 1082$) is the frequency of prime numbers (a frequency mainly defined by the prime number theorem as $\boxed{n / P_n \cong 1/\ln(P_n)}$, also based on the prime-counting function P_n, usually noted $\pi(n)$) which primes (**p**) all have $d_p = 2$, a d_p value which acts like a "brake" and slowing the growth of S_n and E_n implicitly.

4) **Final conclusion**. Conjecture 1 (**C1**) has a major advantage of Dirichlet's estimation of $d_{n(av)}$ (**DSn**) (and $\boxed{S_n = n \cdot d_{n(av)}}$ implicitly), as C1 predicts $d_{n(av)}$ (and S_n implicitly) with much higher accuracy: $\boxed{d_{n(av)(pr3)} = S_{n(pr3)} / n = \ln\left(3\sqrt{xn}\right) / \left(1 - 1/n\right)^2}$ and $\boxed{d_{n(av)(pr4)} = S_{n(pr4)} / n = \ln\left(3\sqrt{xn}\right) / \left(1 - 1/n\right)^{2-f(n)}}$, with $\boxed{f(n) = n^{-n/\ln(n)}}$.