On the new method for finding sum of an infinite series in which $\frac{1}{n}$ $(n \in N)$ is common from every term such that $n \to \infty$

Tejas Chandrakant Thakare

K.T.H.M. College, Nashik, India
Mobile No. +918796784098/ E-mail: thakarecd9@gmail.com

Abstract: Using method of integration as the limit of sum we can easily evaluate sum of an infinite series in which $\frac{1}{n}$ is common from every term such that $n \to \infty$ $(n \in N)$. However in this method we do some rigorous calculations before integration. In this paper, in order to minimize the labor involved in this process I propose an alternative new method for finding the sum of an infinite series in which $\frac{1}{n}$ is common from every term such that $n \to \infty$.

Keywords: Integration; Infinite series; Limit; Sum.

1) Introduction:

We can evaluate sum of an infinite series in which $\frac{1}{n}$ is common from every term such that $n \to \infty$ by using method of integration as the limit of sum [1-2]. If $f(x)$ is function define on [a, b] then we have

$$\int_a^b f(x)dx = \lim_{n\to\infty} h[f(a+dx)+f(a+2dx)+\cdots+f(a+(n-1)dx)+f(a+ndx)] \tag{1}$$

Where n is a total number of strips in which area under $f(x)$ is divided, dx is a width of an individual strip and $ndx = b - a$. We can evaluate sum of an infinite series in which $\frac{1}{n}$ is common from every term such that $n \to \infty$ by converting eq. (1) in following definite integral

$$\int_0^1 f(x)dx = \lim_{n\to\infty}\left[\sum_{r=1}^{n}\frac{1}{n} f\left(\frac{r}{n}\right)\right] \tag{2}$$

Where $a = o$, $b = 1$ and $ndx = 1 - 0$ or $n = \frac{1}{dx}$. Generally we takes help of method of integration as the limit of sum to evaluate sum S of an infinite series in which $\frac{1}{n}$ is common from every term such that $n \to \infty$. Method of integration as the limit of sum to evaluate sum S of an infinite series in which $\frac{1}{n}$ is common from every term such that $n \to \infty$ is demonstrated in the following example. So consider operation on one such infinite series

$$\text{Therefore, } S = \lim_{n\to\infty}\left[\frac{n^2}{(n^2+1)^{\frac{3}{2}}}+\frac{n^2}{(n^2+2^2)^{\frac{3}{2}}}+\frac{n^2}{(n^2+3^2)^{\frac{3}{2}}}+\cdots+\frac{n^2}{[n^2+(n-1)^2]^{\frac{3}{2}}}\right] \tag{3}$$

$$\text{Therefore, } S = \lim_{n\to\infty}\left[\sum_{r=1}^{n-1}\frac{n^2}{(n^2+r^2)^{\frac{3}{2}}}\right] \tag{4}$$

Therefore, $S = \lim_{n\to\infty}\left[\sum_{r=1}^{n-1}\frac{1}{(1+\frac{r^2}{n^2})^{\frac{3}{2}}}\cdot\frac{1}{n}\right]$ (5)

Therefore, $S = \int_0^1 \frac{1}{(1+x^2)^{\frac{3}{2}}}dx$ (6)

Integral (6) can be solved by substituting $x = tan\phi$. Therefore after solving definite integral (6) we get the sum of series (3), which is

Therefore, $S = \frac{1}{\sqrt{2}}$ (7)

So by using above method one can find the sum S of any infinite series in which $\frac{1}{n}$ is common from every term such that $n \to \infty$ however this method is pretty laborious. To minimize this labor involved in this process I propose an alternative simple method to find sum S of an infinite series in which $\frac{1}{n}$ is common from every term such that $n \to \infty$. For our convenience I shall call this new method as S-method. In section 2 there is systematic description of the S-method and section 3 gives conclusions of this study.

2) The S-Method:

Using S-method we can easily evaluate sum S of an infinite series in which $\frac{1}{n}$ is common from every term such that $n \to \infty$ so for that we have to follow one algorithm. In this algorithm firstly without considering limit $n \to \infty$ we have to equate $r = 1$ term of the series (2) with $r = 1$ term of given infinite series in which $\frac{1}{n}$ is common from every term such that $n \to \infty$. After equating $r = 1$ term of both series we have to replace n by $\frac{1}{x}$ afterwards we will get a function of x ($f(x)$). By integrating obtained function $f(x)$ within limit 0 to 1 we will get sum of the given infinite series in which $\frac{1}{n}$ is common from every term such that $n \to \infty$. I will follow this algorithm of S-method to evaluate sum of series (3). Now by equating $r = 1$ term of series (2) with $r = 1$ term of series (3), we get

$$\frac{1}{n}f\left(\frac{1}{n}\right) = \frac{n^2}{(n^2+1)^{\frac{3}{2}}} \quad (8)$$

Now replace n of eq. (8) by $\frac{1}{x}$, we get

$$f(x) = \frac{1}{(1+x^2)^{\frac{3}{2}}} \quad (9)$$

Now integrating eq. (9) w.r.t x within limit 0 to 1, we get

$$S = \int_0^1 f(x)dx = \int_0^1 \frac{1}{(1+x^2)^{\frac{3}{2}}}dx \quad (10)$$

Here definite integral (10) is also appeared in the process of calculation of sum of series (3) by using method of integration as the limit of sum. Solving definite integral (10) we will get the sum of series (3), which is

$S = \frac{1}{\sqrt{2}}$

It follows that calculation of the sum of series (3) is simple if we use S-method. In fact we can find sum S of any infinite series in which $\frac{1}{n}$ is common from every term such that $n \to \infty$. For instance consider another such series

$$S = \lim_{n\to\infty}\left[\frac{n}{(n+1)\sqrt{2n+1}} + \frac{n}{(n+2)\sqrt{2(2n+2)}} + \cdots + \frac{n}{(2n)\sqrt{n\,.\,3n}}\right] \tag{11}$$

The value of sum of series (11) by using S-method and method of integration as the limit of sum is $\frac{\pi}{3}$. Though we get same result by using both methods, S-method reduces labor involved in the process of calculation of sum of an infinite series in which $\frac{1}{n}$ is common from every term such that $n \to \infty$.

3) Conclusions:

Using S-method we can find sum of an infinite series in which $\frac{1}{n}$ is common from every term such that $n \to \infty$. S-method not only gives accurate result but also reduces labor involved in the process of calculation of sum of an infinite series in which $\frac{1}{n}$ is common from every term such that $n \to \infty$.

References:

[1] R. D. Sharma, Mathematics for IIT-JEE, Dhanpat Rai Publications (2015) Vol. 1, p. 11.31.

[2] H. C. Verma, Concepts of physics, Bharati Bhawan Publishers & Distributors (1999) Vol. 1, p. 19-20.