The Similarity between Rules for Essentially Adequate Quaternionic and Complex Differentiation

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Abstract This paper is the third paper of the cycle devoted to the theory of essentially adequate quaternionic differentiability. It is established that the quaternionic holomorphic ($\mathbb{H}$ -holomorphic) functions, satisfying the essentially adequate generalization of Cauchy-Riemann's equations, make up a very remarkable class: generally non-commutative quaternionic multiplication behaves as commutative in the case of multiplication of $\mathbb{H}$ -holomorphic functions. Everyone can construct such $\mathbb{H}$-holomorphic functions by replacing a complex variable as a single whole by a quaternionic one in expressions for complex holomorphic functions, and thereafter verify their commutativity. This property, which is confirmed by a lot of $\mathbb{H}$-holomorphic functions, gives conclusive evidence that the developed theory is true. The rules for quaternionic differentiation of combinations of $\mathbb{H}$-holomorphic functions find themselves similar to those from complex analysis: the formulae for differentiation of sums, products, ratios, and compositions of $\mathbb{H}$-holomorphic functions as well as quaternionic power series, are fully identical to their complex analogs. The example of using the deduced rules is considered and it is shown that they reduce essentially the volume of calculations. The base notions of complex Maclaurin series expansions are adapted to the quaternion case.

***Keywords*:** quaternionic holomorphic functions, quaternionic analysis, quaternionic generalization of Cauchy-Riemann's equation, rules for differentiating, sums, products, ratios, and compositions of quaternionic functions

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1 Introduction

This paper is a continuation of papers [1] and [2], in which we were building the mathematically coherent theory of quaternionic differentiation based upon the concept of essentially adequate differentiability [1]. In this paper we will now show that such a theory is fully similar to complex one.

We denote an independent quaternionic variable, as in [1] and [2], by $p = x + yi + zj + uk = a + bj \in \mathbb{H}$**,** where i, j, k are quaternionic basis vectors**;** x, y, z, u are real values; $a = x + yi$, $b = z + ui$ are complex constituents of the representation of quaternions in the Cayley–Dickson doubling form $a + bj$, and $\mathbb{H}$ denotes the 4-dimensional quaternion space. Respectively, quaternion-valued (briefly, quaternionic) functions $\psi(p) = \psi_1(x,y,z,u) + \psi_2(x,y,z,u)i + \psi_3(x,y,z,u)j + \psi_4(x,y,z,u)k$, where $\psi_1, \psi_2, \psi_3, \psi_4$ are real-valued functions of real variables x, y, z, u, are denoted in the Cayley–Dickson doubling form by $\psi(p) = \phi_1 + \phi_2 j$, where $\phi_1 = \phi_1(a,b) = \psi_1 + \psi_2 i$ and $\phi_2 = \phi_2(a,b) = \psi_3 + \psi_4 i$. The Cayley–Dickson doubling form (procedure) is a basic representation form in the theory in question. In accordance with the concept of essentially adequate differentiability a quaternionic derivative is defined as a limiting value of the difference quotient $\frac{\Delta\psi}{\Delta p}$ as Δp tends to 0. At that the limit is required to be independent not only of the limiting path (as in complex analysis), but also of the way of quaternion division: on the left or on the right. Such an independence is called [1] the "independence of a quaternionic derivative of a way of its computation".

A quaternionic function is said [1] to be quaternion-holomorphic (briefly, $\mathbb{H}$-holomorphic) at a point p, if it has a quaternionic derivative independent of a way of its computation in some open connected neighborhood $G_4 \in \mathbb{H}$ of a point $p \in G_4$. We use further the following equivalent definition of a $\mathbb{H}$- holomorphic function [1].

Definition 1.1. *It is assumed that the constituents $\phi_1(a,b)$ and $\phi_2(a,b)$ of a quaternionic function $\psi(p) = \psi(a,b) = \phi_1 + \phi_2 j$ possess continuous first-order partial derivatives with respect to $a, \bar{a}, b$, and $\bar{b}$ in some open connected neighborhood $G_4 \in \mathbb{H}$ of a point $p \in G_4$. Then a function $\psi(p)$ is said to be $\mathbb{H}$-holomorphic, and denoted by $\psi_H(p)$ at a point p if and only if the functions $\phi_1(a,b)$ and $\phi_2(a,b)$ satisfy in G_4 the following quaternionic generalization* [1] *of Cauchy-Riemann's equations:*

$$\begin{cases} 1)\ (\partial_a \phi_1| = (\partial_{\overline{b}} \overline{\phi}_2|, & 2)\ (\partial_a \phi_2| = -(\partial_{\overline{b}} \overline{\phi}_1|, \\ 3)\ (\partial_a \phi_1| = (\partial_b \phi_2|, & 4)\ (\partial_{\overline{a}} \phi_2| = -(\partial_{\overline{b}} \phi_1|. \end{cases} \tag{1.1}$$

Here and in the sequel, the complex conjugation is denoted as usual (for example, $\overline{a}$ or $\overline{\phi}_2$); the brackets $(..|$ with the closing vertical bar indicate that the transition $a = \overline{a} = x$ has been already performed in expressions enclosed in brackets. We will use the shorthand ∂_t to represent a partial derivative with respect to a variable t. For examples, $\partial_a \phi_1$designates a partial derivative of ϕ_1 with respect to a.

We see that, the ℍ-holomorphicity conditions (1.1) are defined so that during the verification of quaternionic holomorphicity of any quaternionic function we have to do the transition $a = \overline{a} = x$ in already computed expressions for partial derivatives of functions $\phi_1$and ϕ_2 and their conjugations in order to use them further in the equations (1.1). However, this does not mean that we deal with triplets in general; this transition cannot be initially done for quaternionic variables and functions [1]. Any quaternionic function of a quaternionic variable remains the same 4-dimensional quaternionic function regardless of whether we check its quaternionic holomorphicity or not.

The essentially adequate quaternionic ℍ-holomorphicity condition (1.1) differs from the famous Cauchy-Riemann-Fueter equations, which we characterize as inessentially adequate conditions [1,2]. At this point, we also need to recall some theorems and corollaries obtained in [1]. (The last two numbers of numbering remains the same as in [1]).

Theorem 1.4.4 *(an extension of complex holomorphicity to quaternionic one). Let a complex function $\psi_C(\xi): G_2 \to \mathbb{C}$ be ℂ-holomorphic everywhere in a connected open set $G_2 \subseteq \mathbb{C}$, except, possibly, at certain singularities. Then a ℍ-holomorphic function $\psi_H(p)$ of the same kind as $\psi_C(\xi)$ can be constructed (without change of a functional dependence form) from $\psi_C(\xi)$ by replacing a complex variable $\xi \in G_2$ in an expression for $\psi_C(\xi)$ by a quaternionic variable $p \in G_4 \subseteq \mathbb{H}$, where G_4 is defined (except, possibly, at certain singularities) by the relation $G_4 \supset G_2$ in the sense that G_2 exactly follows from G_4 upon transition from p to ξ.*

Theorem 1.4.5 *Let a continuous quaternion function $\psi_H(p) = \phi_1(a,b) + \phi_2(a,b)j$, where $\phi_1(a,b)$ and $\phi_2(a,b)$ are differentiable with respect to a, $\overline{a}$, b and $\overline{b}$, be ℍ-holomorphic everywhere in its domain of definition $G_4 \subseteq \mathbb{H}$. Then its quaternion derivative, defined by*

$$\psi_H(p)' = \phi_1^{(\prime)} + \phi_2^{(\prime)} j\ ,$$

where

$$\phi_1^{(\prime)} = \partial_a \phi_1(a,b) + \partial_{\overline{a}} \phi_1(a,b) = \partial_{a,\overline{a}} \phi_1, \qquad \phi_2^{(\prime)} = \partial_a \phi_2(a,b) + \partial_{\overline{a}} \phi_2(a,b) = \partial_{a,\overline{a}} \phi_2,$$

,

is also ℍ-holomorphic in G_4, except, possibly, at certain singularities. If a quaternion function $\psi(p)$ is once ℍ-differentiable in G_4, then it possesses derivatives of all orders in G_4, each one ℍ-holomorphic.

Corollary 1.4.6 *All expressions for derivatives of a ℍ-holomorphic function $\psi_H(p)$ of the same kind as a ℂ-holomorphic function $\psi_C(\xi)$ have the same forms as the expressions for corresponding derivatives of a function $\psi_C(\xi)$.*

We use basically the generalized quaternionic formula [1], which follows from Theorem 1.4.5, for computing a ℍ-holomorphic quaternionic derivative of all orders. According to this formula, a k'th derivative of a ℍ-holomorphic function $\psi_H(p)$ is defined by

$$\psi_H^{(k)}(p) = \phi_1^{(k)} + \phi_2^{(k)} \cdot j, \tag{1.2}$$

where complex constituents $\phi_1^{(k)}$ and $\phi_2^{(k)}$ are expressed by

$$\phi_1^{(k)} = \partial_a \phi_1^{(k-1)} + \partial_{\overline{a}} \phi_1^{(k-1)} = \partial_{a,\overline{a}} \phi_1^{(k-1)}, \tag{1.2a}$$

$$\phi_2^{(k)} = \partial_a \phi_2^{(k-1)} + \partial_{\overline{a}} \phi_2^{(k-1)} = \partial_{a,\overline{a}} \phi_2^{(k-1)}; \tag{1.2b}$$

$\phi_1^{(k-1)}$ and $\phi_2^{(k-1)}$ are the constituents of the $(k-1)'$th derivative of $\psi_H(p)$, represented in the Cayley–Dickson doubling form as $\psi(p)^{(k-1)} = \phi_1^{(k-1)} + \phi_2^{(k-1)} \cdot j$, $k \geq 1$; $\phi_1^{(0)} = \phi_1(a,b)$ and $\phi_2^{(0)} = \phi_2(a,b)$. For simplicity, we denote the first and the second derivatives: $\psi_H^{(1)}(p)$ and $\psi_H^{(2)}(p)$ by primes, i. e. by $\psi_H'(p)$ and $\psi_H''(p)$, respectively.

The purposes of this article are to develop some general differentiation rules for finding derivatives of combinations of ℍ-holomorphic functions, which retain the ℍ-holomorphicity property, as well as establish new properties of the class of ℍ-holomorphic functions, defined by equations (1.1).

We will show that the rules of quaternionic differentiation of combinations of ℍ-holomorphic functions within the framework of the essentially adequate theory are the same as the rules of complex differentiation.

In the sequel, we use the quaternionic multiplication rule [1]

$$q_1 \cdot q_2 = (a_1 + a_2 \cdot j) \cdot (b_1 + b_2 \cdot j) = \left(a_1 b_1 - a_2 \overline{b}_2\right) + \left(a_1 b_2 + a_2 \overline{b}_1\right) \cdot j, \tag{1.3}$$

represented in the Cayley–Dickson doubling form for arbitrary quaternions $q_1 = a_1 + a_2 \cdot j$ and $q_2 = b_1 + b_2 \cdot j$, where by "$\cdot$" is denoted the quaternionic multiplication. The quaternionic multiplication is generally non-commutative, i. e. $q_1 \cdot q_2 \neq q_2 \cdot q_1$, where q_1 and q_2 are arbitrary quaternionic functions. However further, we will show that in the case of ℍ-holomorphic functions, say, $\psi_H(p)$ and $\varphi_H(p)$, defined as above, the quaternionic multiplication behave as commutative: $\psi_H(p) \cdot \varphi_H(p) = \varphi_H(p) \cdot \psi_H(p)$.

In the sequel, we also use the identity

$$jz = \overline{z} j, \text{ for any } z \in \mathbb{C}. \tag{1.4}$$

When it is obvious that the quaternion multiplication is used, we can omit its notation, i. e. the dot operator "$\cdot$".

We also recall that the constituents of a ℍ-holomorphic function satisfy in $G_4 \subseteq \mathbb{H}$ the following system of equations [1,2]:

$$1)\ \partial_b\phi_2 = \partial_{\overline{b}}\overline{\phi}_2, \quad 2)\ \partial_a\phi_2 = -\partial_{\overline{b}}\phi_1\,,$$
$$3)\ \partial_{\overline{a}}\phi_1 = \partial_a\overline{\phi}_1\,, \quad 4)\ \partial_{\overline{a}}\phi_2 = -\partial_{\overline{b}}\overline{\phi}_1, \tag{1.5}$$

if the second mixed partial derivatives of ϕ_1 and ϕ_2 don't vanish in G_4 (This is not needed for equation (1.5-1)). These equations follow with necessity [2] from the ℍ-holomorphicity equations (1.1). Note that we do not need to perform the transition $a = \overline{a} = x$ in (1.5). These equations are valid for any function $\psi_H(p) = \phi_1 + \phi_2 j$, which is constructed by means of replacing a complex variable as a single whole by a quaternionic variable in an expression for an arbitrary ℂ-holomorphic function, depending only on complex variable as a single whole. This fact exist in reality independently from whether any theory exists or not. Everyone can verify this fact when considering any ℍ-holomorphic function constructed by mentioned replacement of variables.

Using these equations and base expression (1.2) we can construct equivalent formulae for quaternionic derivatives [1], which generalize the corresponding counterparts from the complex analysis based on Cauchy-Riemann's equations. For example, the first derivative can be represented as

$$\psi_H(p)' = \partial_a\phi_1(a,b) + \partial_{\overline{a}}\phi_1(a,b) - \left(\partial_{\overline{b}}\phi_1(a,b) + \partial_{\overline{b}}\overline{\phi}_1(a,b)\right)\cdot j,$$
$$\psi_H(p)' = \left(\partial_a\phi_1(a,b) + \partial_a\overline{\phi}_1(a,b)\right) + \left(-\partial_{\overline{b}}\phi_1(a,b) + \partial_{\overline{a}}\phi_2(a,b)\right) j.$$

2 Rules for differentiating combinations of ℍ-holomorphic functions

When constructing ℍ-holomorphic functions from ℂ-holomorphic we prefer to deal with the basic initial complex functions $\psi(\xi)$, depending only on a complex variable $\xi = x + zj$ as a single whole. Then by replacing a complex variable ξ as a single whole by a quaternionic p in an expression for an initial ℂ-holomorphic function (*without change of its functional dependence form*) we obtain a ℍ- holomorphic function $\psi(p)$ in accordance with the theorem about an extension of complex holomorphicity to quaternionic one (Theorem 1.4.4). We can also consider any compositions of such functions as basic initial functions. For example, the composite functions e^{p^2}, $ln(sinp)$ and $cos\left(e^{p^2}\right)$ are ℍ-holomorphic functions constructed from ℂ-holomorphic initial functions e^{ξ^2}, $ln(sin\xi)$ and $cos\left(e^{\xi^2}\right)$. All these functions are Liouvillian elementary functions [1].

When considering ℍ-holomorphic functions in [1] and [2], we meant that the ℍ-holomorphicity of quaternionic functions (in particular, functions, involving the imaginary unit j as a multiplicative factor [2]) and their combinations must be verified by using equations (1.1). However, it is tedious to compute partial derivatives of function combinations every time in order to substitute them into equations (1.1), when we only need to know whether a function combination is ℍ-holomorphic. Further, we establish the rules retaining the ℍ-holomorphicity property for combinations of ℍ-holomorphic functions and their derivatives. By using such rules the verification of ℍ-holomorphicity will not be necessary every time. They allow us also to compute the derivatives of almost all combinations of functions we are likely to encounter, if we know the derivatives of base functions, from which combinations are constructed.

2.1 The rule for multiplying by a constant

A ℍ-holomorphic function $f_H(p)$ multiplied by a constant r is also ℍ-holomorphic only if a constant is a real number:

$$\psi_H(p) = r f_H(p) \tag{2.1}$$

The full ℍ-holomorphic derivative of a ℍ-holomorphic function $f_H(p)$ multiplied by a real constant r is the following:

$$\left(r f_H(p)\right)' = r f_H'(p) \tag{2.2}$$

Proof. Let a quaternionic function $f_H(p) = f_1(a,b) + f_2(a,b)j$ be ℍ-holomorphic. Then it satisfies the conditions of ℍ-holomorphicity (1.1) as follows:

$$\begin{cases} 1)\ (\partial_a f_1| = (\partial_{\overline{b}}\overline{f}_2|, \quad 2)\ (\partial_a f_2| = -(\partial_{\overline{b}}\overline{f}_1|, \\ 3)\ (\partial_a f_1| = (\partial_b f_2|, \quad 4)\ (\partial_{\overline{a}} f_2| = -(\partial_{\overline{b}} f_1|. \end{cases} \tag{2.3}$$

Consider the quaternionic product $\psi(p) = C\cdot f_H(p)$, where we initially suppose that the constant $C = C_1 + C_2 j$ is a quaternionic constant, i. e. C_1 and C_2 are complex constants. We need to define such values of constants C_1 and C_2, for which the function $C\cdot f_H(p)$ is ℍ-holomorphic.

Using the multiplication rule (1.3) for quaternions, we obtain

$$\psi(p) = C\cdot f_H(p) = (C_1 + C_2 j)\cdot(f_1 + f_2 j) = \left(C_1 f_1 - C_2\overline{f}_2\right) + \left(C_1 f_2 + C_2\overline{f}_1\right)\cdot j = \phi_1 + \phi_2\cdot j,$$

whence

$$\phi_1 = C_1 f_1 - C_2\overline{f}_2,\ \phi_2 = C_1 f_2 + C_2\overline{f}_1, \tag{2.4}$$

and, respectively,

$$\overline{\phi}_1 = \bar{C}_1\overline{f}_1 - \bar{C}_2 f_2,\ \overline{\phi}_2 = \bar{C}_1\overline{f}_2 + \bar{C}_2 f_1. \tag{2.5}$$

To prove that a constant can be only a real constant it suffices to use the equation

$$\partial_b\phi_2 = \partial_{\overline{b}}\overline{\phi}_2, \tag{2.6}$$

following with necessity [1], [2] for ℍ-holomorphic functions from the ℍ-holomorphicity conditions (1.1-1) and (1.1-3). Differentiating ϕ_2 with respect to b and $\overline{\phi}_2$ with respect to $\overline{b}$, and substituting the obtained derivatives into (2.6), we get the equation

$$\partial_b\phi_2 = C_1\partial_b f_2 + C_2\partial_b\overline{f}_1 = \partial_{\overline{b}}\overline{\phi}_2 = \bar{C}_1\partial_{\overline{b}}\overline{f}_2 + \bar{C}_2\partial_{\overline{b}}f_1,$$

that must be satisfied, if the function $\psi(p) = C \cdot f_H(p)$ will be ℍ-holomorphic. Since $f_H(p)$ is ℍ-holomorphic, the equality $\partial_b f_2 = \partial_{\overline{b}}\overline{f}_2$ holds true in this condition , hence $C_1\partial_b f_2 = \bar{C}_1\partial_{\overline{b}}\overline{f}_2$ can only be satisfied if $C_1 = \bar{C}_1$. The derivative $\partial_b\overline{f}_1$ does not belong to the system (2.3), hence the condition $\partial_b\overline{f}_1 = \partial_{\overline{b}}f_1$ cannot be in principle satisfied. In this case, the equality $C_2\partial_b\overline{f}_1 = \bar{C}_2\partial_{\overline{b}}f_1$ can only be satisfied if $C_2 = \bar{C}_2 = 0$. It is not difficult to show that the function $\psi(p) = C \cdot f_H(p)$ satisfies all equations of the system (1.1) if $C_1 = \bar{C}_1$ and $C_2 = 0$. For example, consider equation (1.1-2). If equation (1.1-2) must be satisfied, then differentiating the functions ϕ_2 and $\overline{\phi}_1$ defined, respectively, by (2.4) and (2.5) we get the following equation:

$$(\partial_a\phi_2| = C_1(\partial_a f_2| + C_2(\partial_a\overline{f}_1| = -(\partial_{\overline{b}}\overline{\phi}_1| = -\bar{C}_1(\partial_{\overline{b}}\overline{f}_1| + \bar{C}_2(\partial_{\overline{b}}f_2|, \tag{2.7}$$

which must be satisfied as well.

Since, in accordance with the equation (2.3-2), we have $(\partial_a f_2| = -(\partial_{\overline{b}}\overline{f}_1|$, the partial equality $C_1(\partial_a f_2| = -\bar{C}_1(\partial_{\overline{b}}\overline{f}_1|$ will be satisfied only if $C_1 = \bar{C}_1$. On the other hand, since the function $(\partial_a\overline{f}_1|$ doesn't belong to the system (2.3), i. e. the equality $(\partial_a\overline{f}_1| = (\partial_{\overline{b}}f_2|$ cannot be satisfied when using the system (2.3), we conclude that the equality $C_2(\partial_a\overline{f}_1| = \bar{C}_2(\partial_{\overline{b}}f_2|$ can be in general satisfied (in a neighborhood of ℍ-holomorphicity $G_4 \in \mathbb{H}$ of a point $p \in G_4$) only if C_2 =0.

Thus, the function $\psi(p) = C \cdot f_H(p)$ satisfies the equation (1.1-2) only if $C_1 = \bar{C}_1 = r \in \mathbb{R}$ and C_2 =0, i. e. $C = r \in \mathbb{R}$. Quite analogously we can verify the validity of the rest ℍ-holomorphicity equations (1.1) if $C_1 = \bar{C}_1 = r$ and C_2 =0. The same results we obtain for the functions $\psi(p) = f_H(p) \cdot C$.

We see that it is impossible to multiply a ℍ-holomorphic function $f_H(p)$ by a complex or a quaternionic constant, if we want to obtain the ℍ-holomorphicity of a function multiplied by a constant. Thus, the ℍ-holomorphicity of a ℍ-holomorphic function multiplied by a constant, where a constant can only be a real constant, is proved.

However, there are exceptions to this rule for functions jp, pj, and jpj, considered in [2], when $C_1 = 0$, $C_2 = 1$, and $f_H(p) = p = a + bj$ is a power function of degree 1. In this case, we no longer need to restrict C_2 to be zero, since it turns out that the coefficients C_2 on the left-hand and on the right-hand sides of each equation of system (1.1) are multiplied by corresponding partial derivatives of $f_1 = a$ or $f_2 = b$, which both have the same value (0 or 1). For example, consider the function $\psi(p) = jp$ and equation (1.1-2). Equation (2.7) for $C_1 = 0$ and $C_2 = 1$ becomes

$$(\partial_a\phi_2| = 1 \cdot (\partial_a\overline{f}_1| = -(\partial_{\overline{b}}\overline{\phi}_1| = 1 \cdot (\partial_{\overline{b}}f_2|.$$

Since $\partial_a\overline{f}_1 = 0$ and $\partial_{\overline{b}}f_2 = 0$, we get the identity $(\partial_a\phi_2| = 1 \cdot 0 = -(\partial_{\overline{b}}\overline{\phi}_1| = 1 \cdot 0$. Equation (1.1-2) is satisfied, since the derivatives $\partial_a\overline{f}_1$ and $\partial_{\overline{b}}f_2$ equal the same zero value. We skip further details (see [2]), only would like to make paragraph 1) of section 4 in [2] a little clearer: the above result of multiplying by j cannot be extended to any ℍ-holomorphic function $f(p)$ constructed from ℂ-holomorphic one, if partial derivatives of its constituents do not become the identical constants in the above sense, in particular, to the function $f(p) = p^n$, where $n > 1$.

Since the derivatives of the constituents of a quaternionic constant vanish, it is evident that the ℍ-holomorphicity property is also retained when adding or subtracting a quaternionic constant. We can rewrite (2.1) as follows:

$$\psi_H(p) = rf_H(p) \pm q = rf_1 + rf_2 \cdot j \pm q,$$

where $r \in \mathbb{R}$, and $q \in \mathbb{H}$. Applying (1.2) for $k = 1$ to this expression we obtain the following formula for computing the ℍ-holomorphic derivative of a ℍ-holomorphic function $f_H(p)$ multiplied by a constant r:

$$\psi'_H(p) = \big(rf_H(p)\big)' = \partial_{a,\overline{a}}(rf_1) + \partial_{a,\overline{a}}(rf_2) \cdot j = r\big(\partial_{a,\overline{a}}f_1 + \partial_{a,\overline{a}}f_2 \cdot j\big) = rf'_H(p),$$

where $r \in \mathbb{R}$, $\partial_{a,\overline{a}} = \partial_a + \partial_{\overline{a}}$. Thus, formula (2.2) is proved. Q.E.D.

It is not superfluous to note that this general result coincides in its special case of $f(p) = p$ with the result of the "classic" theory of quaternionic differentiability [3, 4] based on the well-known Cauchy-Riemann-Fueter equations: the only function, possessing both left- and right-hand derivatives, is $\psi(p) = rp \pm q$, where $r \in \mathbb{R}$ and $q \in \mathbb{H}$ [3]. We regard the Cauchy-Riemann-Fueter equations to the "inessentially adequate" ℍ-holomorphicity conditions [1].

2.2 The sum rule

A sum $\psi_H(p) = \sum_{l=1}^{n} f_l(p)$ of a finite number n of ℍ-holomorphic functions $f_l(p)$ is ℍ-holomorphic as well. ***The full*** *ℍ-holomorphic quaternionic derivative of the sum $\psi_H(p) = \sum_{l=1}^{n} f_l(p)$ is the following:*

$$\psi'_H(p) = \left(\sum_{l=1}^{n} f_l(p)\right)' = \sum_{l=1}^{n} f'_l(p). \tag{2.8}$$

Proof. Let the functions $f_l(p) = f_{1,l} + f_{2,l} \cdot j$, $l = 1,2 \dots n$, be ℍ-holomorphic. Then each of them satisfies the ℍ-holomorphicity equations (1.1) as follows:

$$\begin{cases} 1)\ (\partial_a f_{1,l}| = (\partial_{\overline{b}}\overline{f_{2,l}}|, & 2)\ (\partial_a f_{2,l}| = -(\partial_{\overline{b}}\overline{f_{1,l}}|, \\ 3)\ (\partial_a f_{1,l}| = (\partial_b f_{2,l}|, & 4)\ (\partial_{\overline{a}} f_{2,l}| = -(\partial_{\overline{b}} f_{1,l}|, \end{cases} \tag{2.9}$$

If we summarize the functions $f_1(p)$, $f_2(p)$,… $f_n(p)$ by applying component-wise addition, then we get:

$$\sum_{l=1}^{n} f_l(p) = \big(f_{1,1} + f_{1,2} + \ldots + f_{1,n}\big) + \big(f_{2,1} + f_{2,2} + \ldots + f_{2,n}\big) \cdot j = \phi_1 + \phi_2 \cdot j,$$

whence

$$\phi_1 = f_{1,1} + f_{1,2} + \ldots + f_{1,n}, \qquad \phi_2 = f_{2,1} + f_{2,2} + \ldots + f_{2,n}.$$

The derivatives that we need to substitute into equation (1.1-1) are the following: $\partial_a \phi_1 = \partial_a f_{1,1} + \partial_a f_{1,2} + \ldots + \partial_a f_{1,n}$ and $\partial_{\overline{b}} \overline{\phi}_2 = \partial_{\overline{b}} \overline{f_{2,1}} + \partial_{\overline{b}} \overline{f_{2,2}} + \ldots + \partial_{\overline{b}} \overline{f_{2,n}}$. After performing the transition $a = \overline{a} = x$ in them, we can rewrite equation (1.1-1) as follows:

$$(\partial_a \phi_1| = (\partial_a f_{1,1}| + (\partial_a f_{1,2}| + \ldots + (\partial_a f_{1,n}| = (\partial_{\overline{b}} \overline{\phi}_2| = (\partial_{\overline{b}} \overline{f_{2,1}}| + (\partial_{\overline{b}} \overline{f_{2,2}}| + \ldots + (\partial_{\overline{b}} \overline{f_{2,n}}|.$$

Since by virtue of equation (2.9-1) the first term on the left side of this expression is equal to the first term on the right side, the second term on the left side is equal to the second term on the right side, and so on, we can state that equation (1.1-1) is satisfied. Analogously, we can show that the rest equations of the system (1.1) are satisfied as well. The sum $\sum_{l=1}^{n} f_l(p)$ is $\mathbb{H}$-holomorphic. Applying base formulae (1.2), (1.2a), and (1.2b) with $k = 1$ to functions ϕ_1 and ϕ_2 we obtain the first derivative of the sum $\sum_{l=1}^{n} f_l(p)$ as follows:

$$\begin{aligned} \psi_H'(p) = (\textstyle\sum_{l=1}^{n} f_l(p))' &= \partial_{a,\overline{a}} \phi_1 + \partial_{a,\overline{a}} \phi_2 \cdot j \\ &= \partial_{a,\overline{a}} (f_{1,1} + f_{1,2} + \ldots + f_{1,n}) + \partial_{a,\overline{a}} (f_{2,1} + f_{2,2} + \ldots + f_{2,n}) \cdot j \\ &= (\partial_{a,\overline{a}} f_{1,1} + \partial_{a,\overline{a}} f_{2,1} \cdot j) + (\partial_{a,\overline{a}} f_{1,2} + \partial_{a,\overline{a}} f_{2,2} \cdot j) + \ldots + (\partial_{a,\overline{a}} f_{1,n} + \partial_{a,\overline{a}} f_{2,n} \cdot j) \\ &= (f_1(p))' + (f_2(p))' + \ldots + (f_n(p))' = \textstyle\sum_{l=1}^{n} f_l'(p). \end{aligned}$$

The formula (2.8) is proved. Q.E.D.

It is evident that this rule applies to a difference of a finite number of $\mathbb{H}$-holomorphic functions as well.

2.3 The linearity rule

A sum *of a finite number n of $\mathbb{H}$-holomorphic functions $f_l(p)$ each multiplied by a real constant r_l*:

$$\psi_H(p) = \sum_{l=0}^{n} r_l f_l(p) \tag{2.10}$$

is $\mathbb{H}$-holomorphic as well. *The full $\mathbb{H}$-holomorphic derivative of $\psi_H(p) = \sum_{l=1}^{n} r_l f_l(p)$ is defined by*

$$\psi_H'(p) = (\sum_{l=0}^{n} r_l f_l(p))' = \sum_{l=0}^{n} r_l f_l'(p). \tag{2.11}$$

Proof. Using rules (2.1) and (2.2), we prove expression (2.11) immediately. Q.E.D.

Recall that in accordance with [1] the derivative designation $\frac{d}{dp}$ is reserved to solve tasks in physical 3-dimensional space, i. e. after transition $a = \overline{a} = x$: $\frac{d\psi_H(p)}{dp} = ((\psi_H(p))'|$.

2.4 The product rule

A product $\psi_H(p) = f_H(p) \cdot g_H(p)$ of two $\mathbb{H}$-holomorphic functions $f_H(p)$ and $g_H(p)$ is $\mathbb{H}$-holomorphic as well. Its full quaternionic derivative can be calculated by the following formula:

$$\psi_H'(p) = (f_H(p))' \cdot g_H(p) + f_H(p) \cdot (g_H(p))' \tag{2.12}$$

Proof. If a $\mathbb{C}$-holomorphic function $\psi_C(\xi)$ of a complex variable $\xi = x + zj$ is a product $\psi_C(\xi) = f_C(\xi) g_C(\xi)$ of two $\mathbb{C}$-holomorphic functions $f_C(\xi)$ and $g_C(\xi)$, then by replacing ξ by p in $\psi_C(\xi)$ (without change of a functional dependence form) we get a $\mathbb{H}$-holomorphic function $\psi_H(p) = f_H(p) \cdot g_H(p)$ (see Theorem 1.4.4). To prove the validity of the formula (2.12) it suffices to use the main consequence of the theorem about the $\mathbb{H}$-holomorphicity of derivatives of all orders (see Corollary 1.4.6), which are calculated by using (1.2). This corollary says that all expressions for derivatives of a $\mathbb{H}$-holomorphic function $\psi_H(p)$ of the same kind as a $\mathbb{C}$-holomorphic function $\psi_C(\xi)$ have the same forms as the expressions for corresponding derivatives of a function $\psi_C(\xi)$. Therefore, if the complex derivative of $\psi_C(\xi)$ has the form $\psi_C'(\xi) = (f_C(\xi))' \cdot g_C(\xi) + f_C(\xi) \cdot (g_C(\xi))'$, then the quaternionic derivative of $\psi_H(p)$ obtained from $\psi_C(\xi)$ by replacing ξ as a single whole by p must have the analogous form $\psi_H'(p) = (f_H(p))' \cdot g_H(p) + f_H(p) \cdot (g_H(p))'$, i. e. expression (2.12) is valid. Q.E.D.

Theorem 2.4.1

If quaternionic functions are $\mathbb{H}$-holomorphic, i. e. satisfy equations (1.1), then their quaternionic multiplication behaves as commutative.

Proof. It was shown in [2] that the general expressions for constituents $\phi_1(a,b)$ and $\phi_2(a,b)$ of a $\mathbb{H}$-holomorphic function $\psi(p) = \phi_1 + \phi_2 j$, satisfying the $\mathbb{H}$-holomorphicity equations (1.1), are the following:

$$\phi_1(a,b) = A[a, \overline{a}, (b\overline{b})] = A, \tag{2.13}$$

$$\phi_2(a,b) = B[(a\overline{a}), (a\overline{a})_m, (b\overline{b})]b = Bb, \tag{2.14}$$

where $B = \overline{B}$ and $(a\overline{a})_m$ is $(a\overline{a})_m = a^m \overline{a}^0 + a^{(m-1)} \overline{a}^1 + a^{(m-2)} \overline{a}^2 + \ \ldots + a^2 \overline{a}^{(m-2)} + a^1 \overline{a}^{(m-1)} + a^0 \overline{a}^m$ [2] or another symmetric form invariant under complex conjugation. Let the functions $f_H = f_1 + f_2 \cdot j$ and $g_H = g_1 + g_2 \cdot j$ be $\mathbb{H}$-holomorphic. This means that they satisfy the $\mathbb{H}$-holomorphicity equations (1.1), where instead of functions ϕ_1 and ϕ_2 we put respectively f_1 and f_2 in the case of the function f_H or instead of ϕ_1 and ϕ_2 respectively g_1 and g_2 in the case of the function g_H.

In accordance with general forms (2.13) and (2.14) we have the following representations for constituents of f_H and g_H:

$$f_1 = A_f[a, \overline{a}, (b\overline{b})] = A_f, \tag{2.15}$$

$$f_2 = B_f[(a\overline{a}), (a\overline{a})_m, (b\overline{b})]b = B_f b; B_f = \overline{B_f}, \tag{2.16}$$

$$g_1 = A_g[a, \overline{a}, (b\overline{b})] = A_g, \tag{2.17}$$

$$g_2 = B_g\left[(a\overline{a}),(a\overline{a})_m,\left(b\overline{b}\right)\right]b = B_g b;\ B_g = \overline{B_g}. \tag{2.18}$$

In accordance with the rule of quaternion multiplication (1.3) we obtain the following expressions:

$$f_H \cdot g_H = (f_1 + f_2 \cdot j)\cdot(g_1 + g_2 \cdot j) = (f_1 g_1 - f_2\overline{g_2}) + (f_1 g_2 + f_2\overline{g_1})\cdot j = \phi_{1(f\cdot g)} + \phi_{2(f\cdot g)}\cdot j, \tag{2.19}$$

$$g_H \cdot f_H = (g_1 + g_2 \cdot j)\cdot(f_1 + f_2 \cdot j) = \left(f_1 g_1 - \overline{f}_2 g_2\right) + \left(\overline{f}_1 g_2 + f_2 g_1\right)\cdot j = \phi_{1(g\cdot f)} + \phi_{2(g\cdot f)}\cdot j. \tag{2.20}$$

In general, quaternion multiplication is non-commutative, i. e., $f \cdot g \neq g \cdot f$ for arbitrary quaternionic functions f and g. However, we now show that the introduced ℍ-holomorphic functions possess properties such that the quaternionic multiplication of these functions behaves as commutative: $f_H \cdot g_H = g_H \cdot f_H$. In other words, we will now prove that in the case of ℍ-holomorphic functions f_H and g_H the following equalities are valid:

$$f_1 g_1 - f_2\overline{g_2} = f_1 g_1 - \overline{f}_2 g_2, \tag{2.21}$$

$$f_1 g_2 + f_2\overline{g_1} = \overline{f}_1 g_2 + f_2 g_1. \tag{2.22}$$

Using (2.16) and (2.18), we obtain the following equalities:

$$f_2\overline{g_2} = B_f b\overline{B_g b} = B_f B_g b\overline{b},$$

$$\overline{f}_2 g_2 = \overline{B_f b} B_g b = B_f B_g b\overline{b},$$

whence

$$f_2\overline{g_2} = \overline{f}_2 g_2.$$

Thus, we have proved that equality (2.21) is satisfied for ℍ-holomorphic functions f_H and g_H. The only thing left to do is to prove that the equality (2.22) must be satisfied for ℍ-holomorphic functions f_H and g_H.

In accordance with (2.19) we have for the constituent $\phi_{2(f\cdot g)}$ of the quaternionic product $f_H \cdot g_H$ the following expression:

$$\phi_{2(f\cdot g)} = f_1 g_2 + f_2\overline{g_1} \tag{2.23}$$

as well as its complex conjugation

$$\overline{\phi_{2(f\cdot g)}} = \overline{f}_1\overline{g}_2 + \overline{f}_2 g_1. \tag{2.24}$$

Given the ℍ-holomorphicity of the quaternionic product $f_H \cdot g_H$ (see the product rule), we can write $\phi_{2(f\cdot g)}$ and its conjugation in the general form (2.14) as follows:

$$\phi_{2(f\cdot g)} = Bb, \quad \overline{\phi_{2(f\cdot g)}} = \overline{B}\overline{b} = B\overline{b},$$

whence

$$B = \frac{\phi_{2(f\cdot g)}}{b} = \frac{\overline{\phi_{2(f\cdot g)}}}{\overline{b}}. \tag{2.25}$$

Substituting (2.23) and (2.24) into (2.25), we obtain the following expression:

$$B = \frac{f_1 g_2 + f_2\overline{g_1}}{b} = \frac{\overline{f_1}\overline{g_2} + \overline{f_2} g_1}{\overline{b}}. \tag{2.26}$$

By substituting expressions (2.16) and (2.18) and their conjugates into (2.26), we have

$$B = \frac{f_1 B_g b + B_f b\overline{g_1}}{b} = \frac{\overline{f_1} B_g\overline{b} + B_f\overline{b} g_1}{\overline{b}},$$

whence

$$f_1 B_g + B_f\overline{g_1} = \overline{f}_1 B_g + B_f g_1. \tag{2.27}$$

Further, multiplying both sides of (2.27) by b, gives

$$f_1 B_g b + B_f b\overline{g_1} = \overline{f}_1 B_g b + B_f b g_1$$

Finally, using (2.18), (2.16) in the last expression, we obtain the following equality

$$f_1 g_2 + f_2\overline{g_1} = \overline{f}_1 g_2 + f_2 g_1,$$

which coincides with (2.22). This completes the proof of this theorem in whole. Q.E.D.

Thus, the class of ℍ-holomorphic functions in question is in fact a special class of quaternionic functions, for which in principle non-commutative quaternionic multiplication becomes commutative. Surely, one can argue that such a kind of commutativity is not a remarkable fact, since its foundation was initially artificially laid in ℍ-holomorphicity equations (1.1) when defining the quaternionic derivative as a limit of a difference quotient $\frac{\Delta\psi}{\Delta p}$ as $\Delta p \to 0$, which must be independent of the way of quaternion division: on the left or on the right [1]. However that is not quite true. Firstly, such an independence is not artificial, since we needed to impose the requirement of unambiguity on the mathematical fact that there exist two quaternionic derivatives (left and right). This unambiguity follows from the physical reality: an unambiguity of a derivative as an expression of unambiguity of any field strength in physical space [1]. Secondly, even if some doubt in principles of such a theory of quaternionic differentiability would arise, then there nevertheless exist the remarkable fact (can be always directly verified) that by replacing a complex variable as a single whole by a quaternionic one in expressions for complex differentiable functions we obtain quaternionic functions, whose quaternionic multiplication behave as commutative. As noted in [1], "each point of any real line is at the same time a point of some plane and space as a whole, and therefore any characterization of differentiability at a point must be the same regardless of whether we think of that point as a point on the real axis or a point in the complex plane, or a point in space". The observed commutativity of quaternionic multiplication in the case of ℍ-holomorphic functions as well as the similarity between quaternionic and complex rules for finding derivatives of combinations of ℍ-holomorphic functions are consequence of such a point of view realized in the presented theory of essentially adequate quaternionic differentiability [1, 2].

Since the expression $f_H \cdot g_H = g_H \cdot f_H$ is true, we can obtain the product rule in the following form:

$$\psi'_H(p) = (f_H \cdot g_H)' = \left(g_H(p)\right)' \cdot f_H(p) + g_H(p)\cdot\left(f_H(p)\right)',$$

which is equivalent to (2.12).

Example 2.4.2.

To demonstrate the commutativity of quaternionic multiplication in the case of ℍ-holomorphic functions consider two ℍ-holomorphic functions: the power function $f_H = p^2$ and the quaternionic natural logarithmic function (its principal branch) $g_H = \ln p$. As shown in [1], we have in the Cayley–Dickson doubling form the following expressions:

$$p^2 = (a+bj)^2 = \phi_{1(p^2)} + \phi_{2(p^2)} \cdot j,$$

where

$$\phi_{1(p^2)} = \left(a^2 - b\bar{b}\right), \quad \phi_{2(p^2)} = b(a+\bar{a})$$

and

$$\ln p = \phi_{1(\ln p)} + \phi_{2(\ln p)} \cdot j,$$

where

$$\phi_{1(\ln p)} = \ln|p| + \frac{(a-\bar{a})\mathrm{Arccos}\frac{a+\bar{a}}{2|p|}}{2V}, \quad \phi_{2(\ln p)} = \frac{b\mathrm{Arccos}\frac{a+\bar{a}}{2|p|}}{V},$$

$$V = \bar{V} = \sqrt{y^2+z^2+u^2} = \frac{\sqrt{4\left(a\bar{a}+b\bar{b}\right)-(a+\bar{a})^2}}{2}, \quad p, V \neq 0.$$

Using the rule of quaternionic multiplication (1.3) we obtain

$$p^2 \cdot \ln p = \left(\phi_{1(p^2)} + \phi_{2(p^2)} \cdot j\right) \cdot \left(\phi_{1(\ln p)} + \phi_{2(\ln p)} \cdot j\right) = \phi_{1(p^2 \cdot \ln p)} + \phi_{2(p^2 \cdot \ln p)} \cdot j,$$

where

$$\phi_{1(p^2 \cdot \ln p)} = \phi_{1(p^2)}\phi_{1(\ln p)} - \phi_{2(p^2)}\overline{\phi_{2(\ln p)}}$$

$$= \left(a^2 - b\bar{b}\right)\left[\ln|p| + \frac{(a-\bar{a})\mathrm{Arccos}\frac{a+\bar{a}}{2|p|}}{2V}\right] - b(a+\bar{a})\frac{\bar{b}\mathrm{Arccos}\frac{a+\bar{a}}{2|p|}}{V},$$

$$\phi_{2(p^2 \cdot \ln p)} = \phi_{1(p^2)}\phi_{2(\ln p)} + \phi_{2(p^2)}\overline{\phi_{1(\ln p)}}$$

$$= \left(a^2 - b\bar{b}\right)\frac{b\mathrm{Arccos}\frac{a+\bar{a}}{2|p|}}{V} + b(a+\bar{a})\left[\ln|p| - \frac{(a-\bar{a})\mathrm{Arccos}\frac{a+\bar{a}}{2|p|}}{2V}\right]$$

$$= \frac{b\left(a^2+\bar{a}^2\right)\mathrm{Arccos}\frac{a+\bar{a}}{2|p|}}{2V} - \frac{b^2\bar{b}\mathrm{Arccos}\frac{a+\bar{a}}{2|p|}}{V} + b(a+\bar{a})\ln|p|.$$

Interchanging the order of multiplication, we have in accordance with (1.3) the following expression:

$$\ln p \cdot p^2 = \left(\phi_{1(\ln p)} + \phi_{2(\ln p)} \cdot j\right) \cdot \left(\phi_{1(p^2)} + \phi_{2(p^2)} \cdot j\right) = \phi_{1(\ln p \cdot p^2)} + \phi_{2(\ln p \cdot p^2)} \cdot j,$$

where

$$\phi_{1(\ln p \cdot p^2)} = \phi_{1(\ln p)}\phi_{1(p^2)} - \phi_{2(\ln p)}\overline{\phi_{2(p^2)}}$$

$$= \left[\ln|p| + \frac{(a-\bar{a})\mathrm{Arccos}\frac{a+\bar{a}}{2|p|}}{2V}\right]\left(a^2 - b\bar{b}\right) - \frac{b\mathrm{Arccos}\frac{a+\bar{a}}{2|p|}}{V}\bar{b}(a+\bar{a}),$$

$$\phi_{2(\ln p \cdot p^2)} = \phi_{1(\ln p)}\phi_{2(p^2)} + \phi_{2(\ln p)}\overline{\phi_{1(p^2)}}$$

$$= \left[\ln|p| + \frac{(a-\bar{a})\mathrm{Arccos}\frac{a+\bar{a}}{2|p|}}{2V}\right]b(a+\bar{a}) + \frac{b\mathrm{Arccos}\frac{a+\bar{a}}{2|p|}}{V}\left(\bar{a}^2 - b\bar{b}\right)$$

$$= \frac{b\left(a^2+\bar{a}^2\right)\mathrm{Arccos}\frac{a+\bar{a}}{2|p|}}{2V} - \frac{b^2\bar{b}\mathrm{Arccos}\frac{a+\bar{a}}{2|p|}}{V} + b(a+\bar{a})\ln|p|.$$

Comparing the obtained expressions for $\phi_{1(p^2 \cdot \ln p)}$ and $\phi_{1(\ln p \cdot p^2)}$ we easily see that $\phi_{1(p^2 \cdot \ln p)}$ and $\phi_{1(\ln p \cdot p^2)}$ are equal. Analogously, by comparing the expressions for $\phi_{2(p^2 \cdot \ln p)}$ and $\phi_{2(\ln p \cdot p^2)}$ we see that they are also identical. Thus, the equality $p^2 \cdot \ln p = \ln p \cdot p^2$ holds, and hence the quaternionic multiplication of ℍ-holomorphic functions $f_H = p^2$ and $g_H = \ln p$ behaves as commutative. As an illustration of Theorem 2.4.1 we could present various as well as more complicated combinations of functions such as, for example, $p \cdot e^{p^2}$, $e^p \cdot \ln p$, $\sin p \cdot \cos p$, $p \cdot e^p \cdot \ln p$, and so on. All of them and a bunch of other researched products of ℍ-holomorphic functions behave as commutative. However a more detailed illustration of these matters is beyond the scope of the present paper.

2.5 The chain rule

If a ℍ-holomorphic function $\psi_H(p)$ is $\psi_H(p) = f\left(g_H(p)\right)$, where $g_H(p)$ is ℍ-holomorphic, then

$$\psi'_H(p) = f'_g\left(g_H(p)\right) \cdot g'_H(p), \tag{2.28}$$

where $f'_g\left(g_H(p)\right)$ denotes the derivative of $f\left(g_H(p)\right)$ with respect to $\left(g_H(p)\right)$.

Proof. The proof is quite analogues to the proof of the product rule 2.4. Q.E.D.

2.6 The quotient rule.

If there exists a $\mathbb{H}$-holomorphic multiplicative inverse $\frac{1}{g_H(p)}$ of a $\mathbb{H}$-holomorphic function $g_H(p)$, and a function $f_H(p)$ is $\mathbb{H}$-holomorphic, then the following quotient rule is valid:

$$\left(\frac{f_H(p)}{g_H(p)}\right)' = \frac{f_H'(p)\cdot g_H(p) - f_H(p)\cdot g_H'(p)}{\left(g_H(p)\right)^2}.$$

Proof. If the functions $g_H(p)$ and $\frac{1}{g_H(p)}$ are $\mathbb{H}$-holomorphic, then in accordance with the chain rule 2.5 we have as follows:

$$\left(\frac{1}{g_H(p)}\right)' = -\frac{g_H'(p)}{\left(g_H(p)\right)^2}. \tag{2.29}$$

Further, using (2.29) and the product rule 2.4 we obtain

$$\left(\frac{f_H(p)}{g_H(p)}\right)' = \left(f_H(p)\cdot\frac{1}{g_H(p)}\right)' = f_H'(p)\cdot\frac{1}{g_H(p)} - f_H(p)\cdot\frac{g_H'(p)}{\left(g_H(p)\right)^2} = \frac{f_H'(p)\cdot g_H(p) - f_H(p)\cdot g_H'(p)}{\left(g_H(p)\right)^2}.$$

This calculation is possible, since the quaternionic multiplication of $\mathbb{H}$-holomorphic functions behaves as commutative.

Q.E.D.

2.7 The example of efficiency of the sum, product and chain rules

We consider the composite $\mathbb{H}$-holomorphic function $\psi_H(p) = e^{p^2}$. First of all, we will represent this function in the Cayley–Dickson doubling form $e^{p^2} = \phi_1 + \phi_2 \cdot j$. Then we will compute its first derivative $\left(e^{p^2}\right)' = \phi_1' + \phi_2' \cdot j$ by using the base formula (1.2). Analogously, we will compute its second derivative $\left(e^{p^2}\right)'' = \phi_1'' + \phi_2'' \cdot j$. Further, we will compute the first and second derivatives of this function by using the formulae (2.8), (2.12), and (2.28), and then compare these results with the results obtained when using the formula (1.2). In accordance with Theorems 1.4.4 and 1.4.5 this function and its derivatives of all orders are $\mathbb{H}$-holomorphic. In Appendix A, by direct verifying equations (1.1), we will illustrate as example that the second derivative of the function e^{p^2} is $\mathbb{H}$-holomorphic.

First we represent the quaternionic function p^2 in the "pure complex" form, i. e. as a sum of real and imaginary parts. By direct multiplication and using (1.4), we get $p^2 = (a+bj)\cdot(a+bj) = \left(a^2 - b\overline{b}\right) + (a+\overline{a})b\cdot j$. Taking into account that $a = x + yi$, $a^2 = (x^2 - y^2) + 2xyi$, $b\overline{b} = (z+ui)(z-ui) = z^2 + u^2$, and $a + \overline{a} = 2x$, we get $p^2 = (x^2 - y^2 - z^2 - u^2) + \frac{yi+zj+uk}{\sqrt{y^2+z^2+u^2}}\left(2x\sqrt{y^2+z^2+u^2}\right)$. If we denote the value of $\frac{yi+zj+uk}{\sqrt{y^2+z^2+u^2}}$ by r, for which the identity $r^2 = -1$ can be verified directly, we can regard the expression for p^2 as a purely "complex" form:

$$p^2 = (x^2 - y^2 - z^2 - u^2) + r\left(2x\sqrt{y^2+z^2+u^2}\right), \tag{2.30}$$

since $x^2 - y^2 - z^2 - u^2$ and $2x\sqrt{y^2+z^2+u^2}$ have real values and the imaginary unit r plays a role of the complex imaginary unit. Taking into account that the quaternionic formula (see Example 5.3 in [1]) $e^p = e^{x+rv} = e^x(\cos v + r\sin v)$ is valid when representing a quaternion by a "purely complex" expression $p = x + rv$, where x and v are real values and r is a purely imaginary unit quaternion, we obtain the expression for e^{p^2} as follows:

$$e^{p^2} = e^{(x^2-y^2-z^2-u^2)}\left[\cos\left(2x\sqrt{y^2+z^2+u^2}\right) + r\sin\left(2x\sqrt{y^2+z^2+u^2}\right)\right].$$

Substituting $r = \frac{yi+zj+uk}{\sqrt{y^2+z^2+u^2}}$ into this expression, we obtain

$$e^{p^2} = \phi_1 + \phi_2\cdot j = e^{(x^2-y^2-z^2-u^2)}\left[\cos\left(2x\sqrt{y^2+z^2+u^2}\right) + \frac{yi+zj+uk}{\sqrt{y^2+z^2+u^2}}\sin\left(2x\sqrt{y^2+z^2+u^2}\right)\right], \tag{2.31}$$

whence

$$\phi_1 = e^{(x^2-y^2-z^2-u^2)}\left[\cos\left(2x\sqrt{y^2+z^2+u^2}\right) + \frac{y\sin\left(2x\sqrt{y^2+z^2+u^2}\right)}{\sqrt{y^2+z^2+u^2}}i\right], \tag{2.32}$$

$$\phi_2 = e^{(x^2-y^2-z^2-u^2)}\frac{\sin\left(2x\sqrt{y^2+z^2+u^2}\right)}{\sqrt{y^2+z^2+u^2}}b. \tag{2.33}$$

Substituting the relations $x = \frac{a+\overline{a}}{2}$, $y = \frac{a-\overline{a}}{2i}$, $z = \frac{b+\overline{b}}{2}$, $u = \frac{b-\overline{b}}{2i}$ into (2.32) and (2.33), and introducing the designations:

$$V = \sqrt{y^2+z^2+u^2} = \frac{\sqrt{4\left(a\overline{a}+b\overline{b}\right)-(a+\overline{a})^2}}{2}, \tag{2.34}$$

$$\beta = e^{(x^2-y^2-z^2-u^2)} = e^{\frac{a^2+\overline{a}^2-2b\overline{b}}{2}}, \tag{2.35}$$

$$\theta = (a+\overline{a})V, \tag{2.36}$$

we obtain finally the expressions for constituents ϕ_1 and ϕ_2 of the quaternionic function $e^{p^2} = \phi_1 + \phi_2\cdot j$ as follows:

$$\phi_1 = \beta\left[\cos(\theta) + \frac{(a-\overline{a})\sin(\theta)}{2V}\right], \tag{2.37}$$

$$\phi_2 = \frac{\beta\sin(\theta)}{V}b. \tag{2.38}$$

Consider the first derivative of the function e^{p^2}. Now, we will calculate the partial derivatives of ϕ_1 and ϕ_2 with respect to each of $a, \overline{a}$. For $\partial_a\phi_1$ we obtain

$$\partial_a\phi_1 = \left[\cos(\theta) + \frac{(a-\overline{a})\sin(\theta)}{2V}\right]\partial_a\beta + \beta\partial_a\left[\cos(\theta) + \frac{(a-\overline{a})\sin(\theta)}{2V}\right]. \tag{2.39}$$

To proceed, we first compute the partial derivatives of intermediate functions with respect to a and $\overline{a}$:

$$\partial_a V = \partial_a \frac{\sqrt{4(a\overline{a}+b\overline{b})-(a+\overline{a})^2}}{2} = -\frac{a-\overline{a}}{4V}, \tag{2.40}$$

$$\partial_a \theta = V - \frac{a^2-\overline{a}^2}{4V}, \tag{2.41}$$

$$\partial_a \beta = \beta a, \tag{2.42}$$

$$\partial_{\overline{a}} V = \frac{a-\overline{a}}{4V}, \tag{2.43}$$

$$\partial_{\overline{a}} \theta = V + \frac{a^2-\overline{a}^2}{4V}, \tag{2.44}$$

$$\partial_{\overline{a}} \beta = \beta \overline{a}. \tag{2.45}$$

$$\partial_a \left[\cos(\theta) + \frac{(a-\overline{a})\sin(\theta)}{2V}\right] = -\sin(\theta)\partial_a\theta + \frac{[\sin(\theta)+(a-\overline{a})\cos(\theta)\partial_a\theta]V-(a-\overline{a})\sin(\theta)\partial_a V}{2V^2}, \tag{2.46}$$

$$\partial_{\overline{a}} \left[\cos(\theta) + \frac{(a-\overline{a})\sin(\theta)}{2V}\right] = -\sin(\theta)\partial_{\overline{a}}\theta + \frac{[-\sin(\theta)+(a-\overline{a})\cos(\theta)\partial_{\overline{a}}\theta]V-(a-\overline{a})\sin(\theta)\partial_{\overline{a}} V}{2V^2}, \tag{2.47}$$

Substituting (2.42) and (2.46) into (2.39), we obtain after some algebra the following expression for $\partial_a\phi_1$:

$$\partial_a\phi_1 = \beta a\left[\cos(\theta) + \frac{(a-\overline{a})\sin(\theta)}{2V}\right] + \beta\left\{-\sin(\theta)\partial_a\theta + \frac{[\sin(\theta)+(a-\overline{a})\cos(\theta)\partial_a\theta]V-(a-\overline{a})\sin(\theta)\partial_a V}{2V^2}\right\}. \tag{2.48}$$

Differentiating (2.37) with respect to $\overline{a}$, and using (2.45) and (2.47), we quite analogously get the following expression for the derivative $\partial_{\overline{a}}\phi_1$:

$$\partial_{\overline{a}}\phi_1 = \beta \overline{a}\left[\cos(\theta) + \frac{(a-\overline{a})\sin(\theta)}{2V}\right] + \beta\left\{-\sin(\theta)\partial_{\overline{a}}\theta + \frac{[-\sin(\theta)+(a-\overline{a})\cos(\theta)\partial_{\overline{a}}\theta]V-(a-\overline{a})\sin(\theta)\partial_{\overline{a}} V}{2V^2}\right\}. \tag{2.49}$$

Subsistuting (2.40), (2.41) into (2.48), and (2.43), (2.44) into (2.49), we finally have

$$\partial_a\phi_1 = a\beta\gamma + \beta\left\{-\sin(\theta)\left(V - \frac{a^2-\overline{a}^2}{4V}\right) + \frac{\sin(\theta)}{2V} + \frac{(a-\overline{a})}{2V}\left(V - \frac{a^2-\overline{a}^2}{4V}\right)\cos(\theta) + \frac{(a-\overline{a})^2\sin(\theta)}{8V^3}\right\}, \tag{2.50}$$

$$\partial_{\overline{a}}\phi_1 = \overline{a}\beta\gamma + \beta\left\{-\sin(\theta)\left(V + \frac{a^2-\overline{a}^2}{4V}\right) - \frac{\sin(\theta)}{2V} + \frac{(a-\overline{a})}{2V}\left(V + \frac{a^2-\overline{a}^2}{4V}\right)\cos(\theta) - \frac{(a-\overline{a})^2\sin(\theta)}{8V^3}\right\}, \tag{2.51}$$

where

$$\gamma = \cos(\theta) + \frac{(a-\overline{a})\sin(\theta)}{2V}. \tag{2.52}$$

Given (2.50) and (2.51) and according to (1.2a), we now obtain the expression for the constituent ϕ_1' of the first derivative $\left(e^{p^2}\right)' = \phi_1' + \phi_2' \cdot j$:

$$\phi_1' = \partial_{a,\overline{a}}\phi_1 = \partial_a\phi_1 + \partial_{\overline{a}}\phi_1 = (a+\overline{a})\beta\gamma + \beta[-2V\sin(\theta) + (a-\overline{a})\cos(\theta)]. \tag{2.53}$$

To obtain the expression for the constituent $\phi_2' = \partial_a\phi_2 + \partial_{\overline{a}}\phi_2$ of the first derivative of e^{p^2} we need to compute the partial derivatives $\partial_a\phi_2$ and $\partial_{\overline{a}}\phi_2$. The computation gives the following results:

$$\partial_a\phi_2 = b\beta\left[\frac{4aV^2+(a-\overline{a})}{4V^3}\right]\sin(\theta) + b\beta\left[1 - \frac{a^2-\overline{a}^2}{4V^2}\right]\cos(\theta), \tag{2.54}$$

$$\partial_{\overline{a}}\phi_2 = b\beta\left[\frac{4\overline{a}V^2-(a-\overline{a})}{4V^3}\right]\sin(\theta) + b\beta\left[1 + \frac{a^2-\overline{a}^2}{4V^2}\right]\cos(\theta). \tag{2.55}$$

Then summarizing the expressions (2.54) and (2.55) in accordance with (1.2b), we get the constituent ϕ_2' as follows:

$$\phi_2' = \partial_{a,\overline{a}}\phi_2 = \partial_a\phi_2 + \partial_{\overline{a}}\phi_2 = b\beta\frac{(a+\overline{a})}{V}\sin(\theta) + 2b\beta\cos(\theta). \tag{2.56}$$

Finally, using (2.53) and (2.56) we have for *the first quaternionic derivative* of the function e^{p^2} the following expression:

$$\left(e^{p^2}\right)' = \phi_1' + \phi_2' \cdot j = (a+\overline{a})\beta\gamma + \beta[-2V\sin(\theta) + (a-\overline{a})\cos(\theta)] \\ + \left[b\beta\frac{(a+\overline{a})}{V}\sin(\theta) + 2b\beta\cos(\theta)\right]\cdot j, \tag{2.57}$$

where V is defined by (2.34), β by (2.35), θ by (2.36), and γ by (2.52). Using (2.52), uncovering brackets, regrouping and uniting the summands involving the functions $\sin(\theta)$ as well as $\cos(\theta)$, we get, after some algebra, from (2.57) the following expression:

$$\begin{aligned}\left(e^{p^2}\right)' = \phi_1' + \phi_2' \cdot j &= a\beta\left[\cos(\theta) + \frac{(a-\overline{a})\sin(\theta)}{2V}\right] + \overline{a}\beta\left[\cos(\theta) + \frac{(a-\overline{a})\sin(\theta)}{2V}\right] - 2\beta V\sin(\theta) + \beta a\cos(\theta) \\ &-\beta\overline{a}\cos(\theta) + \left[\frac{\beta b(a+\overline{a})\sin(\theta)}{V} + 2b\beta\cos(\theta) + \frac{\beta b(a-\overline{a})\sin(\theta)}{V} - \frac{\beta b(a-\overline{a})\sin(\theta)}{V}\right]\cdot j \\ &= \left\{\beta(a+\overline{a}+a-\overline{a})\cos(\theta) + \beta\left[\frac{a(a-\overline{a})}{2V} + \frac{\overline{a}(a-\overline{a})}{2V} - 2V\right]\sin(\theta)\right\} \\ &+ \left\{2\beta\left[\cos(\theta) + \frac{(a-\overline{a})\sin(\theta)}{2V}\right]b + \frac{\beta\sin(\theta)}{V}b(a+\overline{a}-a+\overline{a})\right\}\cdot j \\ &= 2\left\{\beta\left[\cos(\theta) + \frac{(a-\overline{a})\sin(\theta)}{2V}\right]a - \frac{\beta\sin(\theta)}{V}b\overline{b}\right\} + 2\left\{\beta\left[\cos(\theta) + \frac{(a-\overline{a})\sin(\theta)}{2V}\right]b + \frac{\beta\sin(\theta)}{V}b\overline{a}\right\}\cdot j \\ &= 2\left(\phi_1 a - \phi_2\overline{b}\right) + 2(\phi_1 b + \phi_2\overline{a})\cdot j,\end{aligned} \tag{2.58}$$

where ϕ_1 and ϕ_2 are defined by (2.37) and (2.38), respectively, and it follows that the constituents ϕ_1' and ϕ_2' are

$$\phi_1' = 2\left(\phi_1 a - \phi_2\overline{b}\right) = 2\left\{\beta\left[\cos(\theta) + \frac{(a-\overline{a})\sin(\theta)}{2V}\right]a - \frac{\beta\sin(\theta)}{V}b\overline{b}\right\}, \tag{2.59}$$

$$\phi_2' = 2(\phi_1 b + \phi_2\overline{a}) = 2\left\{\beta\left[\cos(\theta) + \frac{(a-\overline{a})\sin(\theta)}{2V}\right]b + \frac{\beta\sin(\theta)}{V}b\overline{a}\right\}. \tag{2.60}$$

By using the rule of quaternionic multiplication (1.3), the expression (2.58) for the first derivatives of e^{p^2} becomes

$$\left(e^{p^2}\right)' = \phi_1' + \phi_2' \cdot j = 2\left(\phi_1 a - \phi_2\overline{b}\right) + 2(\phi_1 b + \phi_2\overline{a})\cdot j = 2(\phi_1 + \phi_2\cdot j)\cdot(a + b\cdot j) = 2e^{p^2}\cdot p. \tag{2.61}$$

Now we show how to compute the first derivative of the function e^{p^2}, using the Chain Rule 2.5. In accordance with (2.28) we obtain the first derivative as follows:

$$\left(e^{p^2}\right)' = \left(e^{p^2}\right)'_{p^2} \cdot (p^2)' = 2e^{p^2} \cdot p, \tag{2.62}$$

where $\left(e^{p^2}\right)'_{p^2}$ denotes the partial derivative of e^{p^2} with respect to p^2, the differentiation formulae for the exponential and power functions are the same as in complex case (see [1, 2]). We see that the expression (2.62) is the same as the expression (2.61) obtained directly by using the base formula (1.2). We also see that using Chain Rule 2.5 reduces essentially the volume of calculations.

Simplifying the formulae (2.59) and (2.60) for the further use, we can rewrite the expression (2.58) for the first derivative of e^{p^2} as follows:

$$\left(e^{p^2}\right)' = \phi_1' + \phi_2' \cdot j, \tag{2.63}$$

where

$$\phi_1' = 2\beta a\cos(\theta) + \beta\frac{(a^2 - a\bar{a} - 2b\bar{b})\sin(\theta)}{V}, \tag{2.64}$$

$$\phi_2' = 2\beta b\cos(\theta) + \beta b\frac{(a+\bar{a})\sin(\theta)}{V}. \tag{2.65}$$

Consider the second derivative of the function e^{p^2}. In accordance with (1.2) we can represent the second derivative in the Cayley–Dickson doubling form by the following expression:

$$\left(e^{p^2}\right)'' = \phi_1'' + \phi_2'' \cdot j, \tag{2.66}$$

where

$$\phi_1'' = \partial_{a,\bar{a}}\phi_1' = \partial_a\phi_1' + \partial_{\bar{a}}\phi_1', \tag{2.67}$$

$$\phi_2'' = \partial_{a,\bar{a}}\phi_2' = \partial_a\phi_2' + \partial_{\bar{a}}\phi_2'. \tag{2.68}$$

Using (2.64) we compute the partial derivatives belonging to the constituent ϕ_1'' as follows:

$$\partial_a\phi_1' = 2\partial_a[\beta a\cos(\theta)] + \partial_a\left[\frac{\beta(a^2 - a\bar{a} - 2b\bar{b})\sin(\theta)}{V}\right] = 2[\beta(1+a^2)\cos(\theta) - \beta a\sin(\theta)\partial_a\theta]$$
$$+ \frac{\{[\beta a(2 + a^2 - a\bar{a} - 2b\bar{b}) - \beta\bar{a}]\sin(\theta) + \beta(a^2 - a\bar{a} - 2b\bar{b})\cos(\theta)\partial_a\theta\}V - \beta(a^2 - a\bar{a} - 2b\bar{b})\sin(\theta)\partial_a V}{V^2},$$

$$\partial_{\bar{a}}\phi_1' = 2\partial_{\bar{a}}[\beta a\cos(\theta)] + \partial_{\bar{a}}\left[\frac{\beta(a^2 - a\bar{a} - 2b\bar{b})\sin(\theta)}{V}\right] = 2[\beta a\bar{a}\cos(\theta) - \beta a\sin(\theta)\partial_{\bar{a}}\theta]$$
$$+ \frac{\{[\beta\bar{a}(a^2 - a\bar{a} - 2b\bar{b}) - \beta a]\sin(\theta) + \beta(a^2 - a\bar{a} - 2b\bar{b})\cos(\theta)\partial_{\bar{a}}\theta\}V - \beta(a^2 - a\bar{a} - 2b\bar{b})\sin(\theta)\partial_{\bar{a}}V}{V^2}$$

whence, according to (2.67) and using (2.40), (2.41), (2.43) and (2.44), we obtain

$$\phi_1'' = \partial_a\phi_1' + \partial_{\bar{a}}\phi_1' = 2\beta(1 + 2a^2 - 2b\bar{b})\cos(\theta) + \beta\frac{(a - \bar{a} + 2a^3 - 6ab\bar{b} - 2\bar{a}b\bar{b} - 2a^2\bar{a})\sin(\theta)}{V}, \tag{2.69}$$

Analogously, using (2.65), we compute the partial derivatives belonging to the constituent ϕ_2'' as follows:

$$\partial_a\phi_2' = 2b\partial_a[\beta\cos(\theta)] + b\partial_a\left[\frac{\beta(a+\bar{a})\sin(\theta)}{V}\right] = 2b[\beta a\cos(\theta) - \beta\sin(\theta)\partial_a\theta]$$
$$+b\left\{\frac{[\beta(1 + a\bar{a} + a^2)\sin(\theta) + \beta(a+\bar{a})\cos(\theta)\partial_a\theta]V - \beta(a+\bar{a})\sin(\theta)\partial_a V}{V^2}\right\},$$

$$\partial_{\bar{a}}\phi_2' = 2b\partial_{\bar{a}}[\beta\cos(\theta)] + b\partial_{\bar{a}}\left[\frac{\beta(a+\bar{a})\sin(\theta)}{V}\right] = 2b[\beta\bar{a}\cos(\theta) - \beta\sin(\theta)\partial_{\bar{a}}\theta]$$
$$+b\left\{\frac{\left[\beta\left(1 + a\bar{a} + \bar{a}^2\right)\sin(\theta) + \beta(a+\bar{a})\cos(\theta)\partial_{\bar{a}}\theta\right]V - \beta(a+\bar{a})\sin(\theta)\partial_{\bar{a}}V}{V^2}\right\}.$$

In accordance with (2.68), the summation of these derivatives yields the following expression:

$$\phi_2'' = \partial_a\phi_2' + \partial_{\bar{a}}\phi_2' = 4\beta b(a+\bar{a})\cos(\theta) + 2\beta b\frac{\left(1 + a^2 + \bar{a}^2 - 2b\bar{b}\right)\sin(\theta)}{V}. \tag{2.70}$$

Combining (2.66), (2.69), and (2.70), we get for the second derivative of e^{p^2} the following full expression:

$$\left(e^{p^2}\right)'' = \phi_1'' + \phi_2'' \cdot j = \left\{2\beta(1 + 2a^2 - 2b\bar{b})\cos(\theta) + \beta\frac{(a - \bar{a} + 2a^3 - 2a^2\bar{a} - 6ab\bar{b} - 2\bar{a}b\bar{b})\sin(\theta)}{V}\right\}$$
$$+\left\{4\beta b(a+\bar{a})\cos(\theta) + 2\beta b\frac{\left(1 + a^2 + \bar{a}^2 - 2b\bar{b}\right)\sin(\theta)}{V}\right\} \cdot j, \tag{2.71}$$

whence

$$\phi_1'' = 2\beta(1 + 2a^2 - 2b\bar{b})\cos(\theta) + \beta\frac{(a - \bar{a} + 2a^3 - 2a^2\bar{a} - 6ab\bar{b} - 2\bar{a}b\bar{b})\sin(\theta)}{V}, \tag{2.72}$$

$$\phi_2'' = 4\beta b(a+\bar{a})\cos(\theta) + 2\beta b\frac{\left(1 + a^2 + \bar{a}^2 - 2b\bar{b}\right)\sin(\theta)}{V}. \tag{2.73}$$

Now we want to demonstrate the computation of the second derivative of the function e^{p^2} by using the above rules for finding derivatives of combinations of $\mathbb{H}$-holomorphic functions. Using (2.62), the rule for multiplying by a constant and the product and chain rules, we have

$$\left(e^{p^2}\right)'' = \left(\left(e^{p^2}\right)'\right)' = 2\left(e^{p^2} \cdot p\right)' = 2\left(e^{p^2} \cdot 2p \cdot p + e^{p^2}\right) = 2e^{p^2} \cdot (2p^2 + 1).$$

Since $p^2 = (a + b \cdot j)^2 = (a^2 - b\bar{b}) + b(a+\bar{a}) \cdot j$, we have $2p^2 + 1 = \left[2(a^2 - b\bar{b}) + 1\right] + 2b(a+\bar{a}) \cdot j$. Using the quaternionic multiplication rule (1.3) and setting in it $a_1 = 2\phi_1$, $a_2 = 2\phi_2$ (ϕ_1 and ϕ_2 are defined by (2.37) and (2.38), $2e^{p^2} = 2\phi_1 + 2\phi_2 \cdot j$) as well as $b_1 = 2(a^2 - b\bar{b}) + 1$, $b_2 = 2b(a+\bar{a})$, we obtain

$$
\begin{aligned}
\left(e^{p^2}\right)'' &= 2e^{p^2}\cdot(2p^2+1) = (2\phi_1 + 2\phi_2\cdot j)\cdot\left\{\left[2\left(a^2 - b\overline{b}\right)+1\right] + 2b(a+\overline{a})\cdot j\right\} \\
&= \left\{\left[2\beta\cos(\theta) + \tfrac{\beta(a-\overline{a})\sin(\theta)}{V}\right]\left[2\left(a^2 - b\overline{b}\right)+1\right] - \tfrac{2\beta\sin(\theta)}{V} b2\overline{b}(a+\overline{a})\right\} \\
&+ \left\{\left[2\beta\cos(\theta) + \tfrac{\beta(a-\overline{a})\sin(\theta)}{V}\right] 2b(a+\overline{a}) + \tfrac{2\beta\sin(\theta)}{V} b\left[2\left(\overline{a}^2 - b\overline{b}\right)+1\right]\right\}\cdot j \\
&= \left[2\beta\left(1+2a^2-2b\overline{b}\right)\cos(\theta) + \tfrac{\beta(a-\overline{a})\left(1+2a^2-2b\overline{b}\right)\sin(\theta)}{V} - \tfrac{4\beta b\overline{b}(a+\overline{a})\sin(\theta)}{V}\right] \\
&+ \left[4\,\beta b(a+\overline{a})\cos(\theta) + \frac{2\beta b\left(a^2-\overline{a}^2\right)\sin(\theta)}{V} + \frac{2\beta b\left(1+2\overline{a}^2-2b\overline{b}\right)\sin(\theta)}{V}\right]\cdot j \\
&= \left[2\beta\left(1+2a^2-2b\overline{b}\right)\cos(\theta) + \beta\,\frac{\left(a-\overline{a}+2a^3-2a^2\overline{a}-6ab\overline{b}-2\overline{a}b\overline{b}\right)\sin(\theta)}{V}\right] \\
&+ \left[4\beta b(a+\overline{a})\cos(\theta) + 2\beta b\,\frac{\left(1+a^2+\overline{a}^2-2b\overline{b}\right)\sin(\theta)}{V}\right]\cdot j.
\end{aligned}
$$

This expression for $\left(e^{p^2}\right)''$ coincides with the expression (2.71) obtained by using the base formula (1.2) alone. We see that by using the above rules for adequate quaternionic differentiation the calculation of derivatives is not so tedious as the direct calculation by using the expression (1.2) alone.

3 Differentiation of quaternionic power series

There exist much research (see, for example, [4], [5]), which are concerned with the study of quaternionic series. We discuss here some aspects of this issue within the framework of the theory of essentially adequate quaternionic differentiability in question.

As the rule 2.3 show, we must deal with sums of power functions multiplied by real constants. In accordance with Theorems 1.4.4 and 1.4.5 the quaternionic power functions p^l, $l = 0,1,2,\ldots$, and their derivatives of all orders are $\mathbb{H}$-holomorphic. Putting $f_l(p) = p^l$ in (2.10), we obtain the following $\mathbb{H}$-holomorphic polynomials:

$$\psi_H(p) = S_n = \textstyle\sum_{l=0}^{n} r_l p^l,\ n = 1,2,\ldots,\ r_l \in \mathbb{R},\ p \in \mathbb{H}. \tag{3.1}$$

It is natural to consider an object that is like a $\mathbb{H}$-holomorphic polynomial, but with infinitely many terms. Proceeding with (3.1) by way of increasing n, we get to the quaternionic power series:

$$\psi(p) = \mathrm{S} = \textstyle\sum_{l=0}^{\infty} r_l p^l = r_0 + r_1 p^1 + r_2 p^2 + \cdots + r_l p^l + \cdots, \quad r_l \in \mathbb{R},\ p \in \mathbb{H}, \tag{3.2}$$

which represents the Taylor series $\sum_{l=0}^{\infty} r_l (p-p_0)^l$ at $p_0 = 0$ or the Maclaurin series for $\psi(p)$.

Since all algebraic operations (including definitions of the absolute values) for quaternionic power functions are identical to those in complex analysis, we can expect that all notions related to quaternionic power series can be introduced similarly to complex those. As in complex analysis (see [7]) a series $\sum_{l=0}^{\infty} r_l p^l$ is based on a sequence (S_n) of partial sums, where $S_n = \sum_{k=0}^{n} r_k p^k$. The series is said to be convergent if the sequence of partial sums is convergent, and $\lim_{n\to\infty} S_n = S$ is called the sum S of the series. Otherwise the series is divergent. A series $\sum_{l=0}^{\infty} r_l p^l$ is said to be *absolutely convergent* if a series $\sum_{l=0}^{\infty} |r_l p^l| = \sum_{l=0}^{\infty} r_l |p^l|$ converges. A series $\sum_{l=0}^{\infty} r_l p^l$ is said to be *conditionally convergent* if it converges but not absolutely. Analogously to complex analysis [7], we call a quaternionic series $\sum_{l=0}^{\infty} r_l p^l$ *uniformly convergent* to its sum $S(p)$ on the set $T \subset \mathbb{H}$ if it converges at all $p \in T$, and, moreover, for every $\varepsilon > 0$ there exists a natural number N_ε (which depends only on ε) such that if $n \geq N_\varepsilon$, then $|S_n(p) - S(p)| < \varepsilon$ *for all* $p \in T$. An equivalent formulation [8,9] for uniform convergence is via the Cauchy criterion (a necessary and sufficient condition), which we generalize as follows: a series $\sum_{l=0}^{\infty} r_l p^l$ *converges uniformly* on T in the previous sense if and only if for every $\varepsilon > 0$ there exists a natural number N_ε such that $|S_n(p) - S_m(p)| < \varepsilon$ for all $n, m \geq N_\varepsilon$ and for all $p \in T$.

We associate each power series with a real number $R \in [0,\infty]$ called its radius of convergence. Analogously to [5], we consider a function $\psi(p)$ defined as the sum of the quaternionic power series (3.2) in its domain of convergence, which we regard as an open connected ball $B(0,R) = \{p : |p| < R\} \in \mathbb{H}$ centered at 0, such that $\psi(p): B(0,R) \to \mathbb{H}$. For simplicity, we do not consider functions with possible singularities.

We reproduce further some assertions and theorems from complex analysis [6,7,8,9] as "propositions" adapted to quaternions. At that we repeat proofs, adapting them to the quaternionic case. We begin by the following important property of convergent series.

Proposition 3.1 *(The term test). The only series* $\sum_{l=0}^{\infty} r_l p^l$ *that can converge are those whose terms approach* 0. *In other words, if* $\sum_{l=0}^{\infty} r_l p^l$ *converges, then* $r_l p^l \to 0$ *as* $l \to \infty$.

Proof. If the series $\sum_{l=0}^{\infty} r_l p^l$ converges in $B(0,R)$, then the limit of the sequence of its partial sums approaches the sum S, i. e. $S_n \to S$ as $n \to \infty$, where S_n is the $n'th$ partial sum $S_n = \sum_{k=0}^{n} r_k p^k$. Then, putting $n = l$, we have

$$\lim_{l\to\infty} r_l p^l = \lim_{l\to\infty}(S_l - S_{l-1}) = \lim_{l\to\infty} S_l - \lim_{l\to\infty} S_{l-1} = S - S = 0. \text{ Q.E.D.}$$

The contrapositive of that statement gives a test which can tell us that some series diverge. Note, however, the terms converging to 0 don't imply the series converges. The term test is only a necessary, but not a sufficient convergence test.

Proposition 3.2 (*d'Alembert's Ratio Test*). *If the series* $\sum_{l=0}^{\infty} r_l p^l$ *is a quaternionic power series with the property that*

$$\lim_{l\to\infty} \frac{|r_{l+1} p^{l+1}|}{|r_l p^l|} = L \tag{3.3}$$

(provided the limit exists), then the series is absolutely convergent if $L < 1$ *and divergent if* $L > 1$. *If* $L = 1$, *then the test is inconclusive, so we have to use some other test. If the series is convergent, then* $R = \frac{1}{L}$ *(possibly infinite) is a radius of convergence.*

Proof. Since we deal with absolute values, the proof is quite analogous to complex one [7]. We skip the details.

Proposition 3.3 (*Weierstrass M-test*). *Suppose that* $\{r_l p^l\}$, $l = 0,1,2, \dots$, *is a sequence of quaternionic power functions defined on a set* $T \subset \mathbb{H}$, *and there is a sequence of positive numbers* $\{M_l\}$ *satisfying* $\forall l \geq 0, \forall p \in T: |r_l p^l| \leq M_l$. *Suppose that* $\sum_{l=0}^{\infty} M_l < \infty$, *i. e. the series* $\sum_{l=0}^{\infty} r_l p^l$ *is majorized by the convergent series* $\sum_{l=0}^{\infty} M_l$. *Then the series* $\sum_{l=0}^{\infty} r_l p^l$ *converges absolutely and uniformly on a set* T.

Proof. Consider the sequence of partial sums $S_n(p) = \sum_{l=0}^{n} r_l p^l$. Since $\sum_{l=0}^{\infty} M_l$ converges and $M_l \geq 0$ for every l, then by Cauchy's convergence test [8,9] for every $\varepsilon > 0$ there exists a positive integer N_ε such that $\forall n > m > N_\varepsilon$ $(n, m = 1,2, \dots)$: $\sum_{l=\mathrm{m}+1}^{n} M_l < \varepsilon$. For the chosen N_ε, $\forall p \in T$, $\forall n > m > N_\varepsilon$: $|S_n(p) - S_m(p)| = |\sum_{l=\mathrm{m}+1}^{n} r_l p^l| \leq \sum_{l=\mathrm{m}+1}^{n} |r_l p^l| \leq \sum_{l=\mathrm{m}+1}^{n} M_l < \varepsilon$. Thus, the sequence of partial sums $S_n(p)$ converges uniformly on T. Then, by definition, the series $\sum_{l=0}^{\infty} r_l p^l$ converges uniformly on T. Since the series $\sum_{l=0}^{\infty} r_l p^l$ is majorized by the convergent series $\sum_{l=0}^{\infty} M_l$ of numbers $M_l \geq 0$, it converges absolutely on T as well.

Proposition 3.4 *If the terms of the power series* $\sum_{l=0}^{\infty} r_l p^l$ *are bounded at some point* $p_0 \in \mathbb{H}$:

$$\left|r_l p_0^l\right| \leq M, \tag{3.4}$$

where $M = const$, $l = 0,1,2, \dots$, *then the series converges in the open ball* $B(0, |p_0|) = \{p: |p| < |p_0|\}$. *Moreover, it converges absolutely and uniformly on any set* K *that is properly contained in* $B(0, |p_0|)$.

Proof. We adapt to the quaternionic case the proof considered in [6]. Suppose that $p_0 \neq 0$, so that $|p_0| = \zeta > 0$, otherwise the ball $B(0, \zeta)$ is empty. Let K be properly contained in $B(0, \zeta) = \{p: |p| < \zeta\}$, then there exists $q < 1$ such that $\frac{|p|}{\zeta} \leq q < 1$ for all $p \in K$. Therefore *for any* $p \in K$ and any $l \in \mathbb{N}$ we have $r_l |p^l| \leq r_l \zeta^l q^l$. However, assumption (3.4) implies that $\left|r_l p_0^l\right| = r_l \zeta^l \leq M$ so that the series $\sum_{l=0}^{\infty} r_l p^l$ is majorized by a convergent series $M \sum_{l=0}^{n} q^l$ (the geometric series $\sum_{l=0}^{n} q^l$, $q < 1$, is convergent) *for all* $p \in K$. Therefore, according to Assertion 3.3, the series $\sum_{l=0}^{\infty} r_l p^l$ converges uniformly and absolutely on K. This proves the second statement of this proposition. The first one follows from the second since any point $p \in B(0, |p_0|) = \{p: |p| < \zeta\}$ belongs to a ball $B(0, \zeta') = \{p: |p| < \zeta'\}$, with $\zeta' < \zeta$, i. e. properly contained in $B(0, |p_0|)$. Q.E.D.

In complex analysis, the following theorem (*Weierstrass*) [6] holds: If the functional series

$$f(\xi) = \sum_{l=0}^{\infty} f_l(\xi), \xi \in \mathbb{C}, \tag{3.5}$$

of functions $f_l(\xi)$ $\mathbb{C}$-holomorphic in a domain D converges uniformly on any compact subset of this domain, then (i) the sum $f(\xi)$ of this series is holomorphic in D; (ii) the series may be differentiated termwise:

$$f'(\xi) = \sum_{l=0}^{\infty} f_l'(\xi) \tag{3.6}$$

and this differentiation may be performed arbitrarily many times at any point in D.

We regard a domain D as a set G_2(see Theorem 1.4.4), which is an open disk $D(0, R) = \{\xi: |\xi| < R\}$, where $R > 0$ is radius of convergence of the series (3.5). Further, we use a quaternionic ball $B(0, R) = \{p: |p| < R\}$ as a quaternionic generalization G_4 of a complex disk $D(0, R)$. We will assume that the closed subsets in question are compact.

We see that the conditions of termwise differentiation in complex analysis are the following: (1) the functions $f_l(\xi)$ are $\mathbb{C}$-holomorphic in D; (2) the series (3.5) converges uniformly on an closed subset of D. At that we also are to deduce that (3) convergence radius of the series (3.6) is not less than convergence radius R of the series (3.5).

If analogous requirements will be fulfilled in the case of quaternionic functions and series, then this makes it valid to apply Theorem 1.4.5 and Corollary 1.4.6 to prove the possibility of termwise differentiability in the quaternionic case. Adapting all this to the quaternionic case we can formulate on $\mathbb{H}$ the following

Proposition 3.5 *Let a* $\mathbb{H}$*-holomorphic function* $\psi_H(p)$ *be represented by a convergent power series* $\psi_H(p) = \sum_{l=0}^{\infty} r_l p^l$ *in an open ball* $B(0, R) = \{p: |p| < R\}$ *with radius of convergence* $R > 0$. *If the series*

$$\psi_H(p) = \sum_{l=0}^{\infty} r_l p^l \tag{3.7}$$

converges uniformly on any closed subset $T \subset B(0, R)$, *then this series may be differentiated termwise in* $B(0, R)$, *i.e. at every point of* $B(0, R)$ *the sum function* $\psi_H(p)$ *is* $\mathbb{H}$*-differentiable and*

$$\psi_H'(p) = \sum_{l=0}^{\infty} (r_l p^l)' = \sum_{l=1}^{\infty} l r_l p^{l-1} = r_1 + 2r_2 p + 3r_3 p^2 + \cdots + l r_l p^{l-1} + \cdots. \tag{3.8}$$

Proof. It is evident that, according to Theorems 1.4.4, 1.4.5 and Corollary 1.4.5, the functions $f_l(p) = r_l p^l$ and their first derivatives $(r_l p^l)' = l r_l p^{l-1}$), also derivatives of all orders, are $\mathbb{H}$-holomorphic in $B(0, R)$. Hence requirement (1) of complex analysis is fulfilled in the quaternionic case. The $\mathbb{H}$-holomorphicity of $\psi_H(p) = \sum_{l=0}^{\infty} r_l p^l$ in a ball $B(0, R)$ follows from Theorem 1.4.4 applied to the $\mathbb{C}$-holomorphic function $\psi_C(\xi) = \sum_{l=0}^{\infty} r_l \xi^l$. Now, we prove that $\psi_H(p) = \sum_{l=0}^{\infty} r_l p^l$ converges uniformly on any closed subset $T \subset B(0, R)$. Take any $p \in B(0, R)$ and consider any positive ζ such that $|p| < \zeta = |p_0| < R$. By virtue of the convergence of the series

$$\sum_{l=0}^{\infty} r_l \zeta^l = r_0 + r_1 \zeta^1 + r_2 \zeta^2 + \cdots + r_l \zeta^l + \cdots, \ r_l, \zeta \in \mathbb{R}$$

its general term is bounded above (see Proposition 3.1):

$$r_l \zeta^l \leq M, \forall l = 0,1,2, \dots, M = const.$$

Then, according to Proposition 3.4 ($r_l |p_0|^l = r_l \zeta^l \leq M$), the series (3.7) converges absolutely and uniformly on any set $T \subset B(0, \zeta)$. Hence requirement (2) of complex analysis is also fulfilled in the quaternionic case.

Further, for the absolute value of the l'th term of the series (3.8) we have the following estimate:

$$|(r_l p^l)'| = l r_l |p^{l-1}| = l r_l \zeta^l \cdot \left|\frac{p}{\zeta}\right|^{l-1} \cdot \frac{1}{\zeta} \le \frac{M}{\zeta} \cdot l \cdot \left|\frac{p}{\zeta}\right|^{l-1}, \ l = 1,2, ...$$

where the expression for the first derivative: $(r_l p^l)' = l r_l p^{l-1}$ is used. The series

$$\sum_{l=0}^{\infty} |(r_l p^l)'| \le \frac{M}{\zeta} \sum_{l=1}^{\infty} l \left|\frac{p}{\zeta}\right|^{l-1} = \frac{M}{\zeta} \left\{1 + 2\left|\frac{p}{\zeta}\right| + 3\left|\frac{p}{\zeta}\right|^2 + \cdots + l\left|\frac{p}{\zeta}\right|^{l-1} + \cdots\right\}$$

is convergent in $B(0,\zeta)$. Given $\left|\frac{p}{\zeta}\right| < 1$, one can verify this by using d'Alembert's ratio test . Then, by Proposition 3.3, the series $\sum_{l=0}^{\infty}(r_l p^l)'$ converges absolutely and uniformly on $B(0,\zeta)$. Since the series $\sum_{l=0}^{\infty}(r_l p^l)'$ converges absolutely on $B(0,\zeta)$ and the choice of p (hence $\zeta < R$) is arbitrary, it is clear that the convergence radius $R'$of $\psi_H'(p)$ is not less than the convergence radius R of $\psi_H(p)$. This conclusion is sufficient for our objective and there is no need to prove that R' is exactly equal to R. Thus, we have established that the last condition (3) holds in the quaternionic case as well. Further, by using Corollary 1.4.6, we completely prove the Proposition 3.5. Q.E.D.

Note that this proposition as all others could be fully proved by following the known methods of complex analysis, since by virtue of commutativity of quaternionic multiplication in the case of ℍ-holomorphic functions, the algebras of ℍ-holomorphic and ℂ-holomorphic functions are identical.

Proposition 3.6 *(a Maclaurin expansion). Let a ℍ-holomorphic function $\psi_H(p)$ be represented by a sum of a convergent quaternionic power series:*

$$\psi_H(p) = \sum_{k=0}^{\infty} r_k p^k = r_0 + r_1 p^1 + r_2 p^2 + \cdots + r_k p^k + \cdots, r_k \in \mathbb{R}$$

in a ball $B(0,R) = \{p: |p| < R\}$. Then the coefficients r_k are determined uniquely as

$$r_k = \frac{\psi_H^{(k)}(0)}{k!}, \ k = 0,1,2, ..., \tag{3.9}$$

where $\psi_H^{(k)}(0)$ are the k'th derivative of $\psi_H(p)$ at $p = 0$ computed by the formula (1.2) and above rules.

Proof. Since $\psi_H(p)$ is ℍ-holomorphic in a ball $B(0,R)$, there exist all derivatives $\psi_H^{(k)}(p)$, $k = 0,1, ...$, in $B(0,R)$. Putting $p = 0$ in $\psi_H(p) = \sum_{k=0}^{\infty} r_k p^k$ we find $r_0 = \psi_H(0)$. Differentiating $\psi_H(p) = \sum_{k=0}^{\infty} r_k p^k$ termwise and using the formula $(r_k p^k)' = r_k k p^{k-1}$ we obtain

$$\psi_H'(p) = r_1 + 2r_2 p + 3r_3 p^2 + \cdots + k r_k p^{k-1} + \cdots.$$

Inserting $p = 0$ into this expression we obtain $r_1 = \psi_H'(0)$. Differentiating the last expression we get the second derivative as follows:

$$\psi_H''(p) = 2r_2 + 3 \cdot 2r_3 p + 4 \cdot 3r_4 p^2 + 5 \cdot 4r_5 p^3 + \cdots,$$

Putting $p = 0$ in this expression we obtain $r_2 = \frac{\psi_H''(0)}{2}$. Differentiating $\psi_H(p) = \sum_{k=0}^{\infty} r_k p^k$ k times we get

$$\psi_H^{(k)}(p) = k!\, r_k + \zeta_1(n) r_{k+1} p + \zeta_2(n) r_{k+2} p^2 + \cdots,$$

where coefficients $\zeta_1(n), \zeta_2(n), ...$ are some integers, which are equal to products of positive integers depending on n only. Then once again putting $p = 0$ we obtain $r_k = \frac{\psi_H^{(k)}(0)}{k!}$. Q.E.D.

Example 3.7. In complex analysis we have, for example, the following series [8]:

$$e^z = \sum_{l=0}^{\infty} \frac{z^l}{l!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots, z \in \mathbb{C},$$

$$\sin z = \sum_{l=0}^{\infty} (-1)^l \frac{z^{(2l+1)}}{(2l+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots,$$

$$\cos z = \sum_{l=0}^{\infty} (-1)^l \frac{z^{2l}}{(2l)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots,$$

Replacing z by p in these expressions, we obtain in accordance with Theorem 1.4.4 the following power series expansions about a point $p = 0$ for quaternionic ℍ-holomorphic functions e^p, $\sin p$, and $\cos p$:

$$e^p = \sum_{l=0}^{\infty} \frac{p^l}{l!} = 1 + p + \frac{p^2}{2!} + \frac{p^3}{3!} + \cdots, p \in \mathbb{H}, \tag{3.10}$$

$$\sin p = \sum_{l=0}^{\infty} (-1)^l \frac{p^{(2l+1)}}{(2l+1)!} = p - \frac{p^3}{3!} + \frac{p^5}{5!} - \frac{p^7}{7!} + \cdots, \tag{3.11}$$

$$\cos p = \sum_{l=0}^{\infty} (-1)^l \frac{p^{2l}}{(2l)!} = 1 - \frac{p^2}{2!} + \frac{p^4}{4!} - \frac{p^6}{6!} + \cdots. \tag{3.12}$$

Applying d'Alembert's ratio test (3.3) to the series $e^p = \sum_{l=0}^{\infty} r_l p^l$, we have

$$L = \lim_{l\to\infty} \frac{|r_{l+1} p^{l+1}|}{|r_l p^l|} = \lim_{l\to\infty} \frac{|p^{(l+1)}|l!}{(l+1)!|p^l|} = |p| \lim_{l\to\infty} \frac{1}{l+1} = 0 < 1.$$

Then, in accordance with Proposition 3.2, the radius of convergence of $e^p = \sum_{l=0}^{\infty} r_l p^l$ is $R = \frac{1}{L} = \infty$. Analogously, for the series $\sin p = \sum_{l=0}^{\infty} (-1)^l \frac{p^{(2l+1)}}{(2l+1)!}$ we find

$$L = \lim_{l\to\infty} \frac{|r_{l+1} p^{l+1}|}{|r_l p^l|} = \lim_{l\to\infty} \frac{(2l+1)!\,(-1)^{l+1}|p^{2(l+1)+1}|}{[2(l+1)+1]!\,(-1)^l|p^{(2l+1)}|} = -\lim_{l\to\infty} \frac{|p^2|}{(2l+3)(2l+2)} = 0 < 1,$$

whence the radius of convergence of the power series expansion for $\sin p$ is also $R = \frac{1}{L} = \infty$. The calculation for $\cos p$ gives $R = \infty$ as well. Thus, the convergence set of series expansions for the functions e^p, $\sin p$, $\cos p$ is the set of all quaternion numbers.

The verification of the uniform convergence of these series is the same as used in the proof of Proposition 3.5, since these series can be majorized for any $p \in \mathbb{H}$ by the convergent series with the arbitrary ζ^l instead of p^l, where ζ is a positive number such that $|p| < \zeta < \infty$. According to Proposition 3.5, we can differentiate termwise these series at all points $p \in \mathbb{H}$. At that we obtain at all points $p \in \mathbb{H}$ the following expressions:

$$(e^p)' = \left(\sum_{l=0}^{\infty} \frac{p^l}{l!}\right)' = \sum_{l=0}^{\infty} \left(\frac{p^l}{l!}\right)' = \sum_{k=1}^{\infty} \frac{p^{k-1}}{(k-1)!} = \sum_{l=0}^{\infty} \frac{p^l}{l!} = e^p,$$

$$(\sin p)' = \left(\sum_{l=0}^{\infty} (-1)^l \frac{p^{(2l+1)}}{(2l+1)!}\right)' = \sum_{l=0}^{\infty} (-1)^l \left(\frac{p^{(2l+1)}}{(2l+1)!}\right)' = \sum_{l=0}^{\infty} (-1)^l \frac{p^{2l}}{(2l)!} = \cos p$$

$$(\cos p)' = \left(\sum_{l=0}^{\infty} (-1)^l \frac{p^{2l}}{(2l)!}\right)' = \sum_{l=0}^{\infty} (-1)^l \left(\frac{p^{2l}}{(2l)!}\right)' = \sum_{k=1}^{\infty} (-1)^k \frac{p^{2k-1}}{(2k-1)!} = -\sum_{l=0}^{\infty} (-1)^l \frac{p^{(2l+1)}}{(2l+1)!} = -\sin p,$$

which are analogous to complex those.

Now we show that formulae (3.10), (3.11) and (3.12) can be obtained by means of Maclaurin series expansions of the quaternionic functions e^p, $\cos p$, $\sin p$. In other words, we want to show that the coefficients of series (3.10), (3.11) and (3.12) are the coefficients (3.9) of Maclaurin series expansions of these functions.

As already noted in the subsection 2.7, any quaternion $p = x + yi + zj + uk$ can be represented by a "purely complex" expression $p = x + rV$, where x and $V = \sqrt{y^2 + z^2 + u^2}$ are real values and $r = \frac{yi+zj+uk}{\sqrt{y^2+z^2+u^2}}$ is a purely imaginary unit quaternion. Therefore, Euler's formula $e^{iy} = \cos y + i \sin y$, which is valid in the complex plane $\xi = x + yi$, must be generalized to the quaternionic case $p = x + rV$ as follows: $e^{rV} = \cos V + r \sin V$. Then the known complex formulae [9] $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$ must be generalized to the quaternion case by the following expressions:

$$\cos p = \frac{e^{rp}+e^{-rp}}{2}, \tag{3.13}$$

$$\sin p = \frac{e^{rp}-e^{-rp}}{2r}, \tag{3.14}$$

where $rp = r(x + rV) = rx - V$.

Using (3.13), (3.14) and relations $= \frac{a+\bar{a}}{2}$, $= \frac{a-\bar{a}}{2i}$, $z = \frac{b+\bar{b}}{2}$, $u = \frac{b-\bar{b}}{2i}$ we obtain, after some algebra, the following expressions for the functions $\cos p$ and $\sin p$ in the Cayley–Dickson doubling form:

$$\cos p = \frac{e^{rp}+e^{-rp}}{2} = \phi_{1(\cos p)} + \phi_{2(\cos p)} \cdot j, \tag{3.15}$$

where

$$\phi_{1(\cos p)} = \frac{(e^{-V}+e^{V})\cos\frac{a+\bar{a}}{2}}{2} + \frac{(a-\bar{a})(e^{-V}-e^{V})\sin\frac{a+\bar{a}}{2}}{4V}, \tag{3.16}$$

$$\phi_{2(\cos p)} = \frac{(e^{-V}-e^{V})\sin\frac{a+\bar{a}}{2}}{2V} b, \tag{3.17}$$

and

$$\sin p = \frac{e^{rp}-e^{-rp}}{2r} = \phi_{1(\sin p)} + \phi_{2(\sin p)} \cdot j, \tag{3.18}$$

where

$$\phi_{1(\sin p)} = \frac{(e^{-V}+e^{V})\sin\frac{a+\bar{a}}{2}}{2} - \frac{(a-\bar{a})(e^{-V}-e^{V})\cos\frac{a+\bar{a}}{2}}{4V}, \tag{3.19}$$

$$\phi_{2(\sin p)} = -\frac{(e^{-V}-e^{V})\cos\frac{a+\bar{a}}{2}}{2V} b. \tag{3.20}$$

Earlier in [1], we have considered in details the function $e^p = \phi_{1(e^p)} + \phi_{2(e^p)} \cdot j$ in the Cayley–Dickson doubling form. Its constituents are the following:

$$\phi_{1(e^p)} = 2\beta_e \cos V + \frac{\beta_e(a-\bar{a})\sin V}{V}, \tag{3.21}$$

$$\phi_{2(e^p)} = \frac{2\beta_e b \sin V}{V}, \tag{3.22}$$

where $\beta_e = \frac{e^{\frac{a+\bar{a}}{2}}}{2}$.

Now we calculate the values of functions e^p, $\cos p$, and $\sin p$ at the point $p = a + bj = 0$ hence at

$$a = 0,\ \bar{a} = 0,\ b = 0,\ \bar{b} = 0, V = 0. \tag{3.23}$$

At that to eliminate the indeterminate expressions of the form $\frac{0}{0}$ we use the well-known ("first noteworthy", [8]) limit:

$$\lim_{V\to 0} \frac{\sin V}{V} = 1 \tag{3.24}$$

as well as the following limit calculated by using L'Hospital's rule [8]:

$$\lim_{V\to 0} \frac{(e^{-V}-e^{V})}{V} = \lim_{V\to 0} \frac{(e^{-V}-e^{V})'}{V'} = \lim_{V\to 0} \frac{-e^{-V}-e^{V}}{1} = -2. \tag{3.25}$$

Substituting (3.23), (3.24) and (3.25) into (3.16), (3.17), (3.19), (3.20), (3.21) and (3.22), we get the following values of the functions e^p, $\sin p$, and $\cos p$ at the point $p = 0$:

$$e^0 = 1,\ \sin 0 = 0,\ \cos 0 = 1, \tag{3.26}$$

coinciding with analogous values in real and complex areas. Further, using (3.26), we get in accordance with (3.9) the expansion coefficients with $k = 0$ b as follows:

$$r_{0(e^p)} = e^0 = 1; \quad r_{0(\sin 0)} = \sin 0 = 0; \quad r_{0(\cos p)} = \cos 0 = 1.$$

They coincide with the coefficients of the terms of degree 0 in (3.10), (3.11), and (3.12).

As shown in [1], the derivatives of all orders of the quaternionic function e^p is e^p as well. Then, according to (3.9), we get $r_{k(e^p)} = \frac{\psi_H^{(k)}(0)}{k!} = \frac{(e^p)^{(k)}(0)}{k!} = \frac{e^0}{k!} = \frac{1}{k!}$, where $k = 0,1,2, \dots$. The obtained values of coefficients coincide (the indices k and l are the same here) with the corresponding values in (3.10). The Maclaurin series expansion for the function e^p is completed. It coincide with the series (3.10) obtained by using Theorem 1.4.4.

Now, we show how the derivatives of $\cos p$ and $\sin p$ are computed. According to (1.2) we define the first quaternionic derivative of $\cos p$ as follows:

$$(\cos p)' = \phi'_{1(\cos p)} + \phi'_{2(\cos p)} \cdot j, \tag{3.27}$$

where

$$\phi'_{1(\cos p)} = \partial_a \phi_{1(\cos p)} + \partial_{\overline{a}} \phi_{1(\cos p)}, \tag{3.28}$$

$$\phi'_{2(\cos p)} = \partial_a \phi_{2(\cos p)} + \partial_{\overline{a}} \phi_{2(\cos p)}. \tag{3.29}$$

We need now to compute the partial derivatives of $\phi_{1(\cos p)}$ and $\phi_{2(\cos p)}$ with respect to each of $a, \overline{a}$. After some calculation, we have for $\partial_a \phi_{1 \cos p)}$ and $\partial_{\overline{a}} \phi_{1(\cos p)}$ the following expressions:

$$\partial_a \phi_{1(\cos p)} = \partial_a \left[\frac{(e^{-V}+e^V)\cos\frac{a+\overline{a}}{2}}{2} + \frac{(a-\overline{a})(e^{-V}-e^V)\sin\frac{a+\overline{a}}{2}}{4V} \right] = \frac{1}{2}\left[\frac{(a-\overline{a})(e^{-V}-e^V)\cos\frac{a+\overline{a}}{2}}{4V} - \frac{(e^{-V}+e^V)\sin\frac{a+\overline{a}}{2}}{2} \right]$$
$$+ \frac{1}{2}\left[\frac{(e^{-V}-e^V)\sin\frac{a+\overline{a}}{2}}{2V} + \frac{(a-\overline{a})^2(e^{-V}+e^V)\sin\frac{a+\overline{a}}{2}}{8V^2} + \frac{(a-\overline{a})(e^{-V}-e^V)\cos\frac{a+\overline{a}}{2}}{4V} + \frac{(a-\overline{a})^2(e^{-V}-e^V)\sin\frac{a+\overline{a}}{2}}{8V^3} \right],$$

$$\partial_{\overline{a}} \phi_{1(\cos p)} = \partial_{\overline{a}} \left[\frac{(e^{-V}+e^V)\cos\frac{a+\overline{a}}{2}}{2} + \frac{(a-\overline{a})(e^{-V}-e^V)\sin\frac{a+\overline{a}}{2}}{4V} \right] = \frac{1}{2}\left[-\frac{(a-\overline{a})(e^{-V}-e^V)\cos\frac{a+\overline{a}}{2}}{4V} - \frac{(e^{-V}+e^V)\sin\frac{a+\overline{a}}{2}}{2} \right]$$
$$+ \frac{1}{2}\left[-\frac{(e^{-V}-e^V)\sin\frac{a+\overline{a}}{2}}{2V} - \frac{(a-\overline{a})^2(e^{-V}+e^V)\sin\frac{a+\overline{a}}{2}}{8V^2} + \frac{(a-\overline{a})(e^{-V}-e^V)\cos\frac{a+\overline{a}}{2}}{4V} - \frac{(a-\overline{a})^2(e^{-V}-e^V)\sin\frac{a+\overline{a}}{2}}{8V^3} \right].$$

Substituting these expressions into (3.28) and reducing like terms, we get

$$\phi'_{1(\cos p)} = \partial_a \phi_{1(\cos p)} + \partial_{\overline{a}} \phi_{1(\cos p)} = -\frac{(e^{-V}+e^V)\sin\frac{a+\overline{a}}{2}}{2} + \frac{(a-\overline{a})(e^{-V}-e^V)\cos\frac{a+\overline{a}}{2}}{4V} = -\phi_{1(\sin p)}. \tag{3.30}$$

Quite analogously, we obtain for partial derivatives $\partial_a \phi_{2(\cos p)}$ and $\partial_{\overline{a}} \phi_{2(\cos p)}$ the following expressions:

$$\partial_a \phi_{2(\cos p)} = \frac{b}{2} \partial_a \left[\frac{(e^{-V}-e^V)\sin\frac{a+\overline{a}}{2}}{V} \right] = \frac{b}{2}\left[\frac{(e^{-V}-e^V)\cos\frac{a+\overline{a}}{2}}{2V} + \frac{(a-\overline{a})(e^{-V}+e^V)\sin\frac{a+\overline{a}}{2}}{4V^2} + \frac{(a-\overline{a})(e^{-V}-e^V)\sin\frac{a+\overline{a}}{2}}{4V^3} \right],$$

$$\partial_{\overline{a}} \phi_{2(\cos p)} = \frac{b}{2} \partial_{\overline{a}} \left[\frac{(e^{-V}-e^V)\sin\frac{a+\overline{a}}{2}}{V} \right] = \frac{b}{2}\left[\frac{(e^{-V}-e^V)\cos\frac{a+\overline{a}}{2}}{2V} - \frac{(a-\overline{a})(e^{-V}+e^V)\sin\frac{a+\overline{a}}{2}}{4V^2} - \frac{(a-\overline{a})(e^{-V}-e^V)\sin\frac{a+\overline{a}}{2}}{4V^3} \right].$$

Substituting these expressions into (3.29) we get the second constituent of the derivative of $\cos p$:

$$\phi'_{2(\cos p)} = \partial_a \phi_{2(\cos p)} + \partial_{\overline{a}} \phi_{2(\cos p)} = \frac{(e^{-V}-e^V)\cos\frac{a+\overline{a}}{2}}{2V} b = -\phi_{2(\sin p)}. \tag{3.31}$$

Finally, substituting (3.30) and (3.31) into (3.27), we get the following expression:

$$(\cos p)' = \phi'_{1(\cos p)} + \phi'_{2(\cos p)} \cdot j = -\phi_{1(\sin p)} - \phi_{2(\sin p)} \cdot j = -\sin p. \tag{3.32}$$

Thus, we have also shown that the derivatives of quaternionic and complex cosine functions are indeed of the same form in accordance with Corollary 1.4.6.

In an analogous fashion, we obtain for the quaternionic derivative of $\sin p$ the following expression:

$$(\sin p)' = \cos p, \tag{3.33}$$

which coincides with the corresponding expression from the complex analysis. We skip the details here.

The formulae (3.32) and (3.33) allow us to calculate further, according to (3.9), the coefficients $r_k = \frac{\psi_H^{(k)}(0)}{k!}$, $k = 1,2, \dots$, of Maclaurin series expansions for the functions $\sin p$ and $\cos p$, and then compare them with the coefficients in (3.11) and (3.12). Since the expressions for derivatives of the functions $\sin p$ and $\cos p$ are the same as in complex analysis, the using of (3.9) yields the same coefficients of Maclaurin series expansions as in (3.11) and (3.12).

To illustrate this consider the function $\sin p$. For $\sin p$ we get the coefficient of p^1 as follows: $r_{1(\sin p)} = \frac{\psi'_H(0)}{1!} = (\sin p)'_{p=0} = \cos 0 = 1$. It coincides with the coefficient of p^1 in (3.11) that equals 1. The calculation of the coefficient of p^2 yields $r_{2(\sin p)} = \frac{\psi''_H(0)}{2!} = \frac{(\sin p)''_{p=0}}{2} = \frac{(\cos p)'_{p=0}}{2} = -\frac{\sin 0}{2} = 0$. It coincides with the coefficient of p^2 in (3.11) that is equal to 0. The coefficient of p^3 is calculated as follows: $r_{3(\sin p)} = \frac{\psi_H^{(3)}(0)}{3!} = \frac{(\sin p)^{(3)}_{p=0}}{3!} = -\frac{\cos 0}{3!} = -\frac{1}{3!}$. It coincides with the coefficient of p^3 in (3.11) that is equal to $-\frac{1}{3!}$. In this way one could go on.

Note that representing a function by the Maclaurin series is unique, since the coefficients of series are determined uniquely in accordance with the formula (3.9), by using the derivatives of the functions in question. Therefore, in problems of representing a quaternionic function by a series, the answer does not depend on the method adopted for this purpose.

4 Conclusions

We have established the base rules for the quaternionic differentiation of combinations of H-holomorphic functions, including differentiation of quaternionic power series, which are similar to rules for complex differentiation. Thus we can say that the creation of quaternionic analysis fully similar to complex one is in principle possible.

We have also established that among the quaternionic functions, for which multiplication is in general non-commutative, there exist the whole class of functions, called H-holomorphic, for which the quaternionic multiplication behave as commutative. One can always directly verify the fact that replacing a complex variable as a single whole by a quaternionic variable in expressions for two arbitrary complex holomorphic functions yields (without change of a functional dependence form) two quaternionic functions, whose quaternionic multiplication behave as commutative.

This fact exists in reality independently of whether we like it or not. The second similar fact is the existence of equations (1.5), which is also independent of anyone's viewpoint on quaternionic analysis. The theory of essentially adequate quaternionic differentiability in question is, as far as we know, the only theory, which explains these facts. This is a sufficient ground for believing that the presented theory is true. The explanation of these facts represents, so to say, an "external justification" of the theory in question.

The presented theory, in a sense, possesses also an "internal perfection", since this theory follows in essence from one general idea, underlying the concept of essentially adequate differentiability [1]: "each point of any real line is at the same time a point of some plane and space as a whole, and therefore any characterization of differentiability at a point must be the same regardless of whether we think of that point as a point on the real axis or a point in the complex plane, or a point in three-dimensional space".

The derivative in real and complex analysis plays the role of a measure of "transformation" ("deformation", dilatation) at a point of a line and a plane caused ultimately by some physical field. Indeed, if any function is to be interpreted as a "transformation" ("deformation") at a point of space, then its derivative, defined as a limit of a quotient of a "transformed" linear segment and an initial "non-transformed" linear segment, represents some measure of "transformation" ("deformation") at a point of space. Since a physical field strength (or a vector of "deformation " at a point) is always unambiguous, we require an unambiguity of the derivative, hence an equality of the left and the right quaternionic derivatives. However this requirement is relevant only to "deformation" of space at a point, not to rotations of vectors in space, which are represented by the non-commutative quaternion algebra. Therefore, there is no contradiction. Moreover, this requirement leads ultimately to the definition of the full quaternionic derivative [1,2], uniting the left and right quaternionic derivatives, and having symmetry in variables as a result. Such a symmetry represents undoubtedly the symmetry of physical space [1].

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Appendix A

We demonstrate here that the second derivative of the function e^{p^2}: $\left(e^{p^2}\right)'' = \phi_1'' + \phi_2'' \cdot j$, where ϕ_1'' and ϕ_2'' are defined by (2.72) and (2.73), is ℍ-holomorphic, i. e. functions ϕ_1'' and ϕ_2'' satisfy equations (1.1). In subsection 2.7 we have computed partial derivatives of intermediate functions V, β and θ, defined by (2.34), (2.35), and (2.36), with respect to variables a and $\bar{a}$ (see formulae (2.40), (2.41), (2.42), (2.43), (2.44), (2.45)). Now we need to compute partial derivatives of these functions with respect to variables b and $\bar{b}$. The computation yields:

$$\partial_b V = \frac{\bar{b}}{2V}, \tag{A1}$$

$$\partial_b \theta = \frac{(a+\bar{a})\bar{b}}{2V}, \tag{A2}$$

$$\partial_b \beta = -\beta\bar{b}, \tag{A3}$$

$$\partial_{\bar{b}} V = \frac{b}{2V}, \tag{A4}$$

$$\partial_{\bar{b}} \theta = \frac{(a+\bar{a})b}{2V}, \tag{A5}$$

$$\partial_{\bar{b}} \beta = -\beta b. \tag{A6}$$

After quite tedious calculation, the derivatives of ϕ_1'' and ϕ_2'' needed to verify equations (1.1-1) and (1.1-3) are the following:

$$\partial_a\phi_1'' = 2\partial_a\left[\beta\left(1+2a^2-2b\overline{b}\right)\cos(\theta)\right] + \partial_a\left[\beta\,\frac{\left(a-\overline{a}+2a^3-2a^2\overline{a}-6ab\overline{b}-2\overline{a}b\overline{b}\right)\sin(\theta)}{V}\right] \quad \text{(A7)}$$

$$= 2\beta\left[\left(5a+2a^3-2ab\overline{b}\right)\cos(\theta) - \left(1+2a^2-2b\overline{b}\right)\sin(\theta)\,\partial_a\theta\right]$$

$$+\frac{\left[\beta\left(7a^2-5a\overline{a}+2a^4-6a^2b\overline{b}-2a\overline{a}b\overline{b}-2a^3\overline{a}+1-6b\overline{b}\right)\sin(\theta)+\beta\left(a-\overline{a}+2a^3-6ab\overline{b}-2\overline{a}b\overline{b}-2a^2\overline{a}\right)\cos(\theta)\partial_a\theta\right]V}{V^2}$$

$$-\frac{\beta\left(a-\overline{a}+2a^3-6ab\overline{b}-2\overline{a}b\overline{b}-2a^2\overline{a}\right)\sin(\theta)\partial_a V}{V^2},$$

$$\partial_b\phi_2'' = \partial_b\left[4\beta b(a+\overline{a})\cos(\theta) + \frac{2\beta b\left(1+a^2+\overline{a}^2-2b\overline{b}\right)\sin(\theta)}{V}\right] \quad \text{(A8)}$$

$$= 4\beta(a+\overline{a})\left[\left(1-b\overline{b}\right)\cos(\theta) - b\sin(\theta)\partial_b\theta\right] - \frac{2\beta b\left(1+a^2+\overline{a}^2-2b\overline{b}\right)\sin(\theta)\partial_b V}{V^2}$$

$$+\frac{2\left\{\left[\beta\left(1-b\overline{b}\right)\left(1+a^2+\overline{a}^2-2b\overline{b}\right)-2\beta b\overline{b}\right]\sin(\theta)+\beta b\left(1+a^2+\overline{a}^2-2b\overline{b}\right)\cos(\theta)\partial_b\theta\right\}V}{V^2},$$

where V, β, and θ are defined by formulae (2.34), (2.35). and (2.36), respectively. We have also used (2.42) and (A3).

Given $V=\overline{V}$, $\beta=\overline{\beta}$, $\theta=\overline{\theta}$, and (A6), we also obtain

$$\partial_{\overline{b}}\overline{\phi_2''} = \partial_{\overline{b}}\left[4\beta\overline{b}(a+\overline{a})\cos(\theta) + \frac{2\beta\overline{b}\left(1+a^2+\overline{a}^2-2b\overline{b}\right)\sin(\theta)}{V}\right] \quad \text{(A9)}$$

$$= 4\beta(a+\overline{a})\left[\left(1-b\overline{b}\right)\cos(\theta) - \overline{b}\sin(\theta)\partial_{\overline{b}}\theta\right] - \frac{2\beta\overline{b}\left(1+a^2+\overline{a}^2-2b\overline{b}\right)\sin(\theta)\partial_{\overline{b}} V}{V^2}$$

$$+\frac{2\left\{\left[\beta\left(1-b\overline{b}\right)\left(1+a^2+\overline{a}^2-2b\overline{b}\right)-2\beta b\overline{b}\right]\sin(\theta)+\beta\overline{b}\left(1+a^2+\overline{a}^2-2b\overline{b}\right)\cos(\theta)\partial_{\overline{b}}\theta\right\}V}{V^2}.$$

Substituting (A1) and (A2) into (A8), (A4) and (A5) into (A9), we get the following expressions:

$$\partial_b\phi_2'' = 4\beta(a+\overline{a})\left[\left(1-b\overline{b}\right)\cos(\theta) - \frac{b\overline{b}(a+\overline{a})}{2V}\sin(\theta)\right] - \frac{\beta b\overline{b}\left(1+a^2+\overline{a}^2-2b\overline{b}\right)\sin(\theta)}{V^3} \quad \text{(A10)}$$

$$+\frac{2\left[\beta\left(1-b\overline{b}\right)\left(1+a^2+\overline{a}^2-2b\overline{b}\right)-2\beta b\overline{b}\right]\sin(\theta)+\frac{\beta b\overline{b}(a+\overline{a})\left(1+a^2+\overline{a}^2-2b\overline{b}\right)\cos(\theta)}{V}}{V},$$

$$\partial_{\overline{b}}\overline{\phi_2''} = 4\beta(a+\overline{a})\left[\left(1-b\overline{b}\right)\cos(\theta) - \frac{b\overline{b}(a+\overline{a})}{2V}\sin(\theta)\right] - \frac{\beta b\overline{b}\left(1+a^2+\overline{a}^2-2b\overline{b}\right)\sin(\theta)}{V^3} \quad \text{(A11)}$$

$$+\frac{2\left[\beta\left(1-b\overline{b}\right)\left(1+a^2+\overline{a}^2-2b\overline{b}\right)-2\beta b\overline{b}\right]\sin(\theta)+\frac{\beta b\overline{b}(a+\overline{a})\left(1+a^2+\overline{a}^2-2b\overline{b}\right)\cos(\theta)}{V}}{V}.$$

Comparing expressions (A.10) and (A11) we see that they are the same. Therefore, the equation $\partial_b\phi_2'' = \partial_{\overline{b}}\overline{\phi_2''}$ (see (2.6)) is satisfied.

We will need, for further calculations, to have the expressions for intermediate functions V, β and θ and their derivatives after transition $a=\overline{a}=x$. Applying the restriction to $a=\overline{a}=x$ to formulae (2.34), (2.35), (2.36), (2.40), (2.41), (2.42), (2.43), (2.44), (2.45), (A1), (A2), (A3), (A4), (A5), and (A6), we obtain, respectively, the following relations:

1) $(V| = |b|$, 2) $(\beta| = e^{x^2-|b|^2} = \beta_3$, 3) $(\theta| = 2x|b| = \theta_3$, 4) $(\partial_a V| = 0$, 5) $(\partial_a\theta| = |b|$, 6) $(\partial_a\beta| = \beta_3 x$, (A12)

7) $(\partial_{\overline{a}}V| = 0$, 8) $(\partial_{\overline{a}}\theta| = |b|$, 9) $(\partial_{\overline{a}}\beta| = \beta_3 x$, 10) $(\partial_b V| = \frac{\overline{b}}{2|b|}$, 11) $(\partial_b\theta| = \frac{\overline{b}x}{|b|}$, 12) $(\partial_b\beta| = -\beta_3\overline{b}$,

13) $(\partial_{\overline{b}}V| = \frac{b}{2|b|}$, 14) $(\partial_{\overline{b}}\theta| = \frac{bx}{|b|}$, 15) $(\partial_{\overline{b}}\beta| = -\beta_3 b$,

which must be introduced into expressions for partial derivatives by means of replacing $V\to(V|$, $\theta\to(\theta|$, $\beta\to(\beta|$, $\partial_a\theta\to(\partial_a\theta|$, $\partial_a\beta\to(\partial_a\beta|$, $\partial_a V\to 0$, and so on. Substituting $a=\overline{a}=x$ and (A12 -1,2,3,4,5,6,) into (A7) we get

$$(\partial_a\phi_1''| = 2\beta_3\left[(5x+2x^3-2x|b|^2)\cos(\theta_3) - (1+2x^2-2|b|^2)\,|b|\sin(\theta_3)\right] \quad \text{(A13)}$$

$$+\frac{\left\{\beta_3\left(7x^2-5x^2+2x^4-6x^2|b|^2-2x^2|b|^2-2x^4+1-6|b|^2\right)\sin(\theta_3)+\beta_3\left(2x^3-6x|b|^2-2x|b|^2-2x^3\right)|b|\cos(\theta_3)\right\}|b|}{|b|^2}$$

$$= (10\beta_3 x - 12\beta_3 x|b|^2 + 4\beta_3 x^3)\cos(\theta_3) + \frac{\left(\beta_3+2\beta_3x^2-12\beta_3x^2|b|^2-8\beta_3|b|^2+4\beta_3|b|^4\right)\sin(\theta_3)}{|b|}.$$

Given the transition $a=\overline{a}=x$ and (A12-1,2,3,10,11,12) in (A10) we now obtain the expression for the $(\partial_b\phi_2''|$ as follows:

$$(\partial_b\phi_2''| = 8\beta_3 x\left[(1-|b|^2)\cos(\theta_3) - x|b|\sin(\theta_3)\right] - \frac{\beta_3\left(1+2x^2-2|b|^2\right)\sin(\theta_3)}{|b|} \quad \text{(A14)}$$

$$+\frac{2\left[\beta_3\left(1-|b|^2\right)\left(1+2x^2-2|b|^2\right)-2\beta_3|b|^2\right]\sin(\theta_3)+2\beta_3x|b|\left(1+2x^2-2|b|^2\right)\cos(\theta_3)}{|b|}$$

$$= (10\beta_3 x - 12\beta_3 x|b|^2 + 4\beta_3 x^3)\cos(\theta_3) + \frac{\left(\beta_3+2\beta_3x^2-12\beta_3x^2|b|^2-8\beta_3|b|^2+4\beta_3|b|^4\right)\sin(\theta_3)}{|b|}.$$

We see that (A13) and (A14) coincide, i. e. equation (1.1-3): $(\partial_a\phi_1''| = (\partial_b\phi_2''|$, holds for the constituents of the second derivative of the function e^{p^2}. Since the equation $\partial_b\phi_2 = \partial_{\overline{b}}\overline{\phi}_2$ is valid, equation (1.1-1), i. e. $(\partial_a\phi_1''| = (\partial_{\overline{b}}\overline{\phi}_2|$, is also satisfied.

To verify equation (1.1-2) we need to compute the partial derivatives $\partial_a\phi_2''$ and $\partial_{\overline{b}}\overline{\phi_1''}$. They are computed as follows:

$$\partial_a\phi_2'' = 4b\partial_a\left[\beta(a+\overline{a})\cos(\theta)\right] + 2b\partial_a\left[\frac{\beta\left(1+a^2+\overline{a}^2-2b\overline{b}\right)\sin(\theta)}{V}\right] \quad \text{(A15)}$$

$$= 4b[\beta(1 + a^2 + a\overline{a})\cos(\theta) - \beta(a + \overline{a})\sin(\theta)\partial_a\theta]$$
$$+2b\left\{\frac{\left[\beta\left(3a+a^3+a\overline{a}^2-2ab\overline{b}\right)\sin(\theta)+\beta\left(1+a^2+\overline{a}^2-2b\overline{b}\right)\cos(\theta)\partial_a\theta\right]V-\beta\left(1+a^2+\overline{a}^2-2b\overline{b}\right)\sin(\theta)\partial_a V}{V^2}\right\},$$

$$\partial_{\overline{b}}\overline{\phi_1''} = \partial_{\overline{b}}\left[2\beta\left(1 + 2\overline{a}^2 - 2b\overline{b}\right)\cos(\theta) + \beta\,\frac{\left(\overline{a}-a+2\overline{a}^3-2\overline{a}^2a-6\overline{a}b\overline{b}-2ab\overline{b}\right)\sin(\theta)}{V}\right] \tag{A16}$$
$$= -2\beta\left(3b + 2\overline{a}^2b - 2b^2\overline{b}\right)\cos(\theta) - 2\beta\left(1 + 2\overline{a}^2 - 2b\overline{b}\right)\sin(\theta)\partial_{\overline{b}}\theta$$
$$-\frac{\beta\left(\overline{a}-a+2\overline{a}^3-2\overline{a}^2a-6\overline{a}b\overline{b}-2ab\overline{b}\right)\sin(\theta)\partial_{\overline{b}}V}{V^2}$$
$$+\frac{-\beta\left(7\overline{a}b+ab+2\overline{a}^3b-6\overline{a}b^2\overline{b}-2ab^2\overline{b}-2\overline{a}^2ab\right)\sin(\theta)+\beta\left(\overline{a}-a+2\overline{a}^3-2\overline{a}^2a-6\overline{a}b\overline{b}-2ab\overline{b}\right)\cos(\theta)\partial_{\overline{b}}\theta}{V},$$

where $\partial_a\theta$, $\partial_a V$, $\partial_{\overline{b}}\theta$ and $\partial_{\overline{b}}V$ are defined by (2.41), (2.40), (A5) and (A4), respectively. To obtain complex conjugation $\overline{\phi_1''}$ of the function ϕ_1'' we have used the relations $V = \bar{V}$, $\beta = \bar{\beta}$, $\theta = \bar{\theta}$, following from (2.34), (2.35), (2.36). We have also used the relations (2.42) and (A6).

Substituting $a = \overline{a} = x$ and expressions (A12-1,2,3,4,5) into (A15) as well as (A12-1,2,3,13,14) into (A16) we get after the transition $a = \overline{a} = x$ the following expressions for derivatives:

$$(\partial_a\phi_2''| = 4b[\beta_3(1 + x^2 + x^2)\cos(\theta_3) - 2\beta_3 x|b|\sin(\theta_3)] \tag{A17}$$
$$+2b\left[\frac{\beta_3(3x+x^3+x^3-2x|b|^2)\sin(\theta_3)+\beta_3(1+x^2+x^2-2|b|^2)|b|\cos(\theta_3)}{|b|}\right]$$
$$= (6\beta_3 b + 12\beta_3 bx^2 - 4\beta_3 b|b|^2)\cos(\theta_3) + \frac{(6\beta_3 bx+4\beta_3 bx^3-12\beta_3 bx|b|^2)\sin(\theta_3)}{|b|},$$

$$(\,\partial_{\overline{b}}\overline{\phi_1''}| = -2\beta_3(3b + 2x^2b - 2b|b|^2)\cos(\theta_3) - 2\beta_3\frac{xb}{|b|}(1 + 2x^2 - 2|b|^2)\sin(\theta_3) \tag{A18}$$
$$-\frac{\beta_3(2x^3-2x^3-6x|b|^2-2x|b|^2)b\sin(\theta_3)}{2|b|^3}$$
$$+\frac{-\beta_3(7xb+xb+2x^3b-6xb|b|^2-2xb|b|^2-2x^3b)\sin(\theta_3)+\frac{\beta_3 xb(2x^3-2x^3-6x|b|^2-2x|b|^2)\cos(\theta_3)}{|b|}}{|b|}$$
$$= -(6\beta_3 b + 12\beta_3 bx^2 - 4\beta_3 b|b|^2)\cos(\theta_3) - \frac{(6\beta_3 bx+4\beta_3 bx^3-12\beta_3 bx|b|^2)\sin(\theta_3)}{|b|}.$$

We see that obtained expressions (A17) and (A18) represent the results equal in absolute value and opposite in sign, i. e. equation (1.1-2): $(\partial_a\phi_2''| = -(\,\partial_{\overline{b}}\overline{\phi_1''}|$ holds for the constituents of the second derivative of the function e^{p^2}.

To verify equation (1.1-4) we need to compute the partial derivatives $\partial_{\overline{a}}\phi_2''$ and $\partial_{\overline{b}}\phi_1''$. They are computed as follows:

$$\partial_{\overline{a}}\phi_2'' = \partial_{\overline{a}}\left[4\beta b(a + \overline{a})\cos(\theta) + 2\beta b\,\frac{\left(1+a^2+\overline{a}^2-2b\overline{b}\right)\sin(\theta)}{V}\right] \tag{A19}$$
$$= 4\beta b\left(1 + a\overline{a} + \overline{a}^2\right)\cos(\theta) - 4\beta b(a + \overline{a})\sin(\theta)\partial_{\overline{a}}\theta$$
$$+2b\left\{\frac{\left[\beta\left(3\overline{a}+a^2\overline{a}+\overline{a}^3-2\overline{a}b\overline{b}\right)\sin(\theta)+\beta\left(1+a^2+\overline{a}^2-2b\overline{b}\right)\cos(\theta)\partial_{\overline{a}}\theta\right]V-\beta\left(1+a^2+\overline{a}^2-2b\overline{b}\right)\sin(\theta)\partial_{\overline{a}}V}{V^2}\right\},$$

$$\partial_{\overline{b}}\phi_1'' = \partial_{\overline{b}}\left[2\beta\left(1 + 2a^2 - 2b\overline{b}\right)\cos(\theta) + \beta\,\frac{(a-\overline{a}+2a^3-6ab\overline{b}-2\overline{a}b\overline{b}-2a^2\overline{a})\sin(\theta)}{V}\right] \tag{A20}$$
$$= 2\left[-\beta\left(3b + 2a^2b - 2bb\overline{b}\right)\cos(\theta) - \beta\left(1 + 2a^2 - 2b\overline{b}\right)\sin(\theta)\partial_{\overline{b}}\theta\right] - \frac{\beta(a-\overline{a}+2a^3-6ab\overline{b}-2\overline{a}b\overline{b}-2a^2\overline{a})\sin(\theta)\partial_{\overline{b}}V}{V^2}$$
$$+\frac{-\beta b(a-\overline{a}+2a^3-6ab\overline{b}-2\overline{a}b\overline{b}-2a^2\overline{a})+\beta(-6ab-2\overline{a}b)\sin(\theta)+\beta(a-\overline{a}+2a^3-6ab\overline{b}-2\overline{a}b\overline{b}-2a^2\overline{a})\cos(\theta)\partial_{\overline{b}}\theta}{V}.$$

Substituting $a = \overline{a} = x$ and expressions (A12-1,2,3,7,8) into (A19) as well as expressions (A12-1,2,3,13,14) into (A20), we have after the transition $a = \overline{a} = x$ the following expressions:

$$(\partial_{\overline{a}}\phi_2''| = 4\beta_3 b(1 + 2x^2)\cos(\theta_3) - 8\beta bx\,|b|\sin(\theta_3) \tag{A21}$$
$$+\frac{2b[\beta(3x+2x^3-2x|b|^2)\sin(\theta_3)+\beta(1+2x^2-2|b|^2)|b|\cos(\theta_3)]}{|b|}$$
$$= (6\beta_3 b + 12\beta_3 bx^2 - 4\beta_3 b|b|^2)\cos(\theta_3) + \frac{(6\beta_3 bx-12\beta_3 bx|b|^2+4\beta_3 bx^3)\sin(\theta_3)}{|b|}$$

and

$$(\partial_{\overline{b}}\phi_1''| = -2\beta_3(3b + 2x^2b - 2b|b|^2)\cos(\theta_3) - \frac{2\beta_3 bx\left(1+2x^2-2|b|^2\right)\sin(\theta_3)}{|b|} \tag{A22}$$
$$+\frac{4\beta_3 bx\sin(\theta_3)}{|b|} + \frac{(8\beta_3 bx|b|^2-8\beta_3 bx)\sin(\theta_3)}{|b|} - 8\beta_3 bx^2\cos(\theta_3)$$
$$= -(6\beta_3 b + 12\beta_3 bx^2 - 4\beta_3 b|b|^2)\cos(\theta_3) - \frac{(6\beta_3 bx-12\beta_3 bx|b|^2+4\beta_3 bx^3)\sin(\theta_3)}{|b|}.$$

We see that obtained expressions (A21) and (A22) represent the results equal in absolute value and opposite in sign, i. e. equation (1.1-4): $(\partial_{\overline{a}}\phi_2''| = -(\partial_{\overline{b}}\phi_1''|$ holds for the constituents of the second derivative of the function e^{p^2}.

Thus, we have shown by direct computation that the second derivative of the function e^{p^2} is $\mathbb{H}$-holomorphic. It was evident from the very beginning by virtue of Theorem 1.4.5.