Question 472: Some Series

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abstract

This note presents some numerical series

1. Introduction

- The natural numbers:

$$\mathbb{N} = \{1, 2, 3, 4 ...\} \quad (1)$$

- The factorial number:

$$n! = 1 \cdot 2 \cdot 3 \cdots n \quad , n \in \mathbb{N} \quad , 0! = 1 \quad (2)$$

- The binomial number:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad (3)$$

$$\binom{2n}{n} = \frac{(2n)!}{n!^2} \quad (4)$$

- The pochhammer symbol:

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) \quad , n \in \mathbb{N} \quad ; (a)_0 = 1 \quad (5)$$

- Identity:

$$\frac{(1/2)_n}{n!} = \binom{2n}{n} 2^{-2n} \quad , n \in \mathbb{N} \cup \{0\} \quad (6)$$

- The hypergeometric function:

$$ {}_2F_1\left(a,b;c;x\right)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n\,n!}x^n \quad ,|x|<1 \tag{7}$$

- The number h_n :

$$ h_n=\sum_{k=1}^{n}\frac{1}{2k-1}=1+\frac{1}{3}+\frac{1}{5}+...+\frac{1}{2n-1} \tag{8}$$

- Complex numbers: $z=x+iy\in\mathbb{C}; x,y\in\mathbb{R}; x=\operatorname{Re}(z), y=\operatorname{Im}(z); i=\sqrt{-1}$.

2. Some Series

Entry 1: If $0\le\theta<\pi/3$, then

$$ i\theta\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\cos\theta-i\sin\theta\right)=-4\sum_{n=1}^{\infty}\binom{2n}{n}^2 2^{-4n}h_n\left(1-\cos\theta-i\sin\theta\right)^n \tag{9}$$

Examples: $\theta=\pi/4,\pi/6,\pi/8,\pi/12$.

$$ i\pi\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)=-16\sum_{n=1}^{\infty}\binom{2n}{n}^2 2^{-4n}h_n\left(1-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)^n \tag{10}$$

$$ i\pi\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\frac{\sqrt{3}}{2}-\frac{i}{2}\right)=-24\sum_{n=1}^{\infty}\binom{2n}{n}^2 2^{-5n}h_n\left(2-\sqrt{3}-i\right)^n \tag{11}$$

$$ i\pi\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\frac{\sqrt{2+\sqrt{2}}}{2}-\frac{i\sqrt{2-\sqrt{2}}}{2}\right)= \\ =-32\sum_{n=1}^{\infty}\binom{2n}{n}^2 2^{-5n}h_n\left(2-\sqrt{2+\sqrt{2}}-i\sqrt{2-\sqrt{2}}\right)^n \tag{12}$$

$$ i\pi\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\frac{\sqrt{6}+\sqrt{2}}{4}-i\frac{\sqrt{6}-\sqrt{2}}{4}\right)= \\ =-48\sum_{n=1}^{\infty}\binom{2n}{n}^2 2^{-6n}h_n\left(4-\sqrt{2}-\sqrt{6}-i\left(\sqrt{6}-\sqrt{2}\right)\right)^n \tag{13}$$

Entry 2: If $0 \le \theta < \pi / 3$, then

$$\theta \sum_{n=0}^{\infty} \binom{2n}{n}^2 2^{-4n} X_n = -4 \sum_{n=1}^{\infty} \binom{2n}{n}^2 2^{-4n} h_n Y_n \tag{14}$$

$$\theta \sum_{n=0}^{\infty} \binom{2n}{n}^2 2^{-4n} Y_n = 4 \sum_{n=1}^{\infty} \binom{2n}{n}^2 2^{-4n} h_n X_n \tag{15}$$

where

$$X_n + i Y_n = \left(1 - \cos\theta - i \sin\theta\right)^n \quad , n \in \mathbb{N} \cup \{0\} \tag{16}$$

$$X_n = \operatorname{Re}\left(\left(1 - \cos\theta - i \sin\theta\right)^n\right) \quad , n \in \mathbb{N} \cup \{0\} \tag{17}$$

$$Y_n = \operatorname{Im}\left(\left(1 - \cos\theta - i \sin\theta\right)^n\right) \quad , n \in \mathbb{N} \cup \{0\} \tag{18}$$

$$X_n = \frac{1}{2}\left(\left(1 - \cos\theta - i \sin\theta\right)^n + \left(1 - \cos\theta + i \sin\theta\right)^n\right) \quad , n \in \mathbb{N} \cup \{0\} \tag{19}$$

$$Y_n = \frac{1}{2i}\left(\left(1 - \cos\theta - i \sin\theta\right)^n - \left(1 - \cos\theta + i \sin\theta\right)^n\right) \quad , n \in \mathbb{N} \cup \{0\} \tag{20}$$

$$\begin{cases} X_{n+1} = \left(1 - \cos\theta\right) X_n + \left(\sin\theta\right) Y_n \\ Y_{n+1} = -\left(\sin\theta\right) X_n + \left(1 - \cos\theta\right) Y_n \\ X_0 = 1, Y_0 = 0 \end{cases} \tag{21}$$

Example: $\theta = \pi / 4$.

$$\pi \sum_{n=0}^{\infty} \binom{2n}{n}^2 2^{-4n} X_n = -16 \sum_{n=1}^{\infty} \binom{2n}{n}^2 2^{-4n} h_n Y_n \tag{22}$$

$$\pi \sum_{n=0}^{\infty} \binom{2n}{n}^2 2^{-4n} Y_n = 16 \sum_{n=1}^{\infty} \binom{2n}{n}^2 2^{-4n} h_n X_n \tag{23}$$

where

$$X_{n+1} = \left(1 - \frac{1}{\sqrt{2}}\right) X_n + \frac{1}{\sqrt{2}} Y_n \ , \ Y_{n+1} = -\frac{1}{\sqrt{2}} X_n + \left(1 - \frac{1}{\sqrt{2}}\right) Y_n \ , \ X_0 = 1, Y_0 = 0 \tag{24}$$

Entry 3:

$$\pi\sum_{n=0}^{\infty}\binom{2n}{n}^2 2^{-5n}\left(a_n+b_n\sqrt{2}\right)=-16\sum_{n=1}^{\infty}\binom{2n}{n}^2 2^{-5n}h_n\left(c_n+d_n\sqrt{2}\right) \tag{25}$$

$$\pi\sum_{n=0}^{\infty}\binom{2n}{n}^2 2^{-5n}\left(c_n+d_n\sqrt{2}\right)=16\sum_{n=1}^{\infty}\binom{2n}{n}^2 2^{-5n}h_n\left(a_n+b_n\sqrt{2}\right) \tag{26}$$

where

$$\begin{cases} a_{n+1}=2a_n-2b_n+2d_n \\ b_{n+1}=-a_n+2b_n+c_n \\ c_{n+1}=-2b_n+2c_n-2d_n \\ d_{n+1}=-a_n-c_n+2d_n \\ a_0=1, b_0=c_0=d_0=0 \end{cases} \tag{27}$$

Let $\alpha=\frac{1+i}{\sqrt{2}}, \beta=\frac{-1+i}{\sqrt{2}}$, then

$$a_n=\frac{2^n}{4}\left((1-\alpha)^n+(1+\alpha)^n+(1-\beta)^n+(1+\beta)^n\right) \quad ,n\in\mathbb{N}\cup\{0\} \tag{28}$$

$$b_n=\frac{2^n(1-i)}{8}\alpha\left((1-\alpha)^n-(1+\alpha)^n-(1-\beta)^n+(1+\beta)^n\right) \quad ,n\in\mathbb{N}\cup\{0\} \tag{29}$$

$$c_n=-\frac{2^n i}{4}\left((1-\alpha)^n+(1+\alpha)^n-(1-\beta)^n-(1+\beta)^n\right) \quad ,n\in\mathbb{N}\cup\{0\} \tag{30}$$

$$d_n=-\frac{2^n(1+i)}{8}\alpha\left((1-\alpha)^n-(1+\alpha)^n+(1-\beta)^n-(1+\beta)^n\right) \quad ,n\in\mathbb{N}\cup\{0\} \tag{31}$$

Entry 4: If $k\in\mathbb{N}$, then

$$\pi=\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+...+\sqrt{2}}}}}_{k-radicals}\left(\frac{2^k}{2^k-1}+2^k\sum_{n=1}^{\infty}\frac{\left(2^{-1}-2^{-k-1}\right)_n}{n!\left(n\,2^{k+1}+2^k-1\right)}\right) \tag{32}$$

$$\pi=\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+...+\sqrt{2}}}}}_{k-radicals}\left(\frac{2^k}{2^k+1}+2^k\sum_{n=1}^{\infty}\frac{\left(2^{-1}+2^{-k-1}\right)_n}{n!\left(n\,2^{k+1}+2^k+1\right)}\right) \tag{33}$$

$$\pi = 2^k \sqrt{2 - \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + ... + \sqrt{2}}}}}_{(k-1)-radicals}} \left(1 + 2^{-k-1} \sum_{n=1}^{\infty} \frac{\left(2^{-k-1}\right)_n}{n!\left(n + 2^{-k-1}\right)}\right) \tag{34}$$

Entry 5:

$$\ln\left(\frac{4}{3}\right) \sum_{n=0}^{\infty} \binom{4n}{2n}^2 (-768)^{-n} + \frac{\pi}{48\sqrt{3}} \sum_{n=0}^{\infty} \binom{4n+2}{2n+1}^2 (-768)^{-n} = -8 \sum_{n=1}^{\infty} \binom{4n}{2n}^2 (-768)^{-n} h_{2n} \tag{35}$$

$$-\ln\left(\frac{4}{3}\right) \sum_{n=0}^{\infty} \binom{4n+2}{2n+1}^2 (-768)^{-n} + \frac{16\pi}{\sqrt{3}} \sum_{n=0}^{\infty} \binom{4n}{2n}^2 (-768)^{-n} = -2^{11}\, 3 \sum_{n=1}^{\infty} \binom{4n-2}{2n-1}^2 (-768)^{-n} h_{2n-1} \tag{36}$$

Entry 6: If $a = \frac{3\sqrt{3}}{3 + \sqrt{2}\sqrt[4]{27}}$, then

$$\pi = 3\sqrt{3} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{2^{-2n} a^{2n+1}}{3n+2} \, {}_2F_1\left(\frac{1}{6}, n + \frac{2}{3}; n + \frac{5}{3}; 1\right) \tag{37}$$

$$\pi = \frac{6\sqrt{3}}{5} \sum_{n=0}^{\infty} \binom{2n}{n} 2^{-2n} a^{2n+1} \, {}_2F_1\left(\frac{5}{6}, -n + \frac{1}{3}; \frac{11}{6}; 1\right) \tag{38}$$

Entry 7:

$$\sum_{n=0}^{\infty} \left(\frac{(1/4)_n}{n!}\right)^2 \frac{2^{-n}}{2n+1} = \sqrt{2} \sum_{n=0}^{\infty} \left(\frac{(3/4)_n}{n!}\right)^2 f_n \tag{39}$$

$$\sum_{n=0}^{\infty} \left(\frac{(3/4)_n}{n!}\right)^2 \frac{(2n+5)\, 2^{-n-1}}{(2n+1)(2n+5)} = \sqrt{2} \sum_{n=0}^{\infty} \left(\frac{(1/4)_n}{n!}\right)^2 f_n \tag{40}$$

where

$$f_0 = \frac{1}{4} + \frac{\pi}{8}, \; f_n = -\frac{2^{-n-2}}{n+1} + \frac{2n-1}{2n+2} f_{n-1} \quad , n \in \mathbb{N} \tag{41}$$

Entry 8:

$$\sum_{n=0}^{\infty}\left(\frac{(3/4)_n}{n!}\right)^2\frac{2^{-n}}{2n+1}=\sqrt{2}\sum_{n=0}^{\infty}\left(\frac{(1/4)_n}{n!}\right)^2 f_n \tag{42}$$

where

$$f_0=\frac{\pi}{4},\ f_n=-\frac{2^{-n-1}}{n}+\frac{2n-1}{2n}f_{n-1}\quad ,n\in\mathbb{N} \tag{43}$$

Entry 9: If $n\in\mathbb{N}$, then

$$\sum_{k=0}^{\infty}\frac{\left(2^{-n-1}\right)_k}{k!\left(k+2^{-n-1}\right)^2}=\frac{2\pi}{\sqrt{2-\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{(n-1)-radicals}}}\sum_{k=1}^{\infty}\frac{1-2^{-n-1}}{k\left(k+2^{-n-1}-1\right)} \tag{44}$$

$$\sum_{k=0}^{\infty}\frac{\left(2^{-1}-2^{-n-1}\right)_k}{k!\left(k+2^{-1}-2^{-n-1}\right)^2}=\frac{\pi}{\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n-radicals}}\sum_{k=1}^{\infty}\frac{1+2^{-n}}{k\left(k+2^{-1}-2^{-n-1}-1\right)} \tag{45}$$

$$\sum_{k=0}^{\infty}\frac{\left(2^{-1}+2^{-n-1}\right)_k}{k!\left(k+2^{-1}+2^{-n-1}\right)^2}=\frac{\pi}{\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n-radicals}}\sum_{k=1}^{\infty}\frac{1-2^{-n}}{k\left(k+2^{-1}+2^{-n-1}-1\right)} \tag{46}$$

Entry 10:

$$\frac{\pi}{6}+\frac{1}{\sqrt{3}}\sum_{n=1}^{\infty}\left(\frac{(1/2)_n}{n!}\right)^2\frac{(2/3)^n}{2n+1}\,{}_2F_1\left(n+1,n+\frac{1}{2};n+\frac{3}{2};-\frac{1}{3}\right)=\sum_{n=0}^{\infty}\frac{(1/4)_n(3/4)_n}{(n!)^2}f_n \tag{47}$$

$$\frac{\pi}{6}+\frac{1}{\sqrt{3}}\sum_{n=1}^{\infty}\left(\frac{(1/2)_n}{n!}\right)^2\frac{2^{-n}}{2n+1}\,{}_2F_1\left(1,\frac{1}{2};n+\frac{3}{2};-\frac{1}{3}\right)=\sum_{n=0}^{\infty}\frac{(1/4)_n(3/4)_n}{(n!)^2}f_n \tag{48}$$

$$\frac{\pi}{6}+\frac{\sqrt{3}}{4}\sum_{n=1}^{\infty}\left(\frac{(1/2)_n}{n!}\right)^2\frac{2^{-n}}{2n+1}\,{}_2F_1\left(1,n+1;n+\frac{3}{2};\frac{1}{4}\right)=\sum_{n=0}^{\infty}\frac{(1/4)_n(3/4)_n}{(n!)^2}f_n \tag{49}$$

$$\frac{\pi}{6}+\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{(1/2)_n}{n!}\right)^2\frac{2^{-n}}{2n+1}\,{}_2F_1\left(\frac{1}{2},n+\frac{1}{2};n+\frac{3}{2};\frac{1}{4}\right)=\sum_{n=0}^{\infty}\frac{(1/4)_n(3/4)_n}{(n!)^2}f_n \qquad (50)$$

where

$$f_0=\frac{\ln 3}{2},\; f_n=-\frac{(8n-1)3^{-2n}}{8n^2-4n}+\frac{(4n-1)(4n-3)}{8n(2n-1)}f_{n-1}\quad ,n\in\mathbb{N} \qquad (51)$$

Entry 11: If $n>0$, then

$$\pi=-12n\sum_{k=1}^{\infty}\frac{k!}{k(n)_{k+1}}\left(\frac{1}{3+\sqrt{3}}\right)^k\operatorname{Im}\left((1-i)^k\right)+ +12n\sum_{k=1}^{\infty}\frac{(-1)^k}{k(k+n)}\operatorname{Im}\left(\left(\frac{1+\sqrt{3}}{2}+i\frac{3+\sqrt{3}}{2}\right)^{-k}\right) \qquad (52)$$

$$\pi=-8n\sum_{k=1}^{\infty}\frac{k!}{k(n)_{k+1}}\left(\frac{1}{2+\sqrt{2}}\right)^k\operatorname{Im}\left((1-i)^k\right)+ +8n\sum_{k=1}^{\infty}\frac{(-1)^k}{k(k+n)}\operatorname{Im}\left(\left(\frac{\sqrt{2}}{2}+i\frac{2+\sqrt{2}}{2}\right)^{-k}\right) \qquad (53)$$

$$\pi=-6n\sum_{k=1}^{\infty}\frac{k!}{k(n)_{k+1}}\left(\frac{1}{1+\sqrt{3}}\right)^k\operatorname{Im}\left((1-i)^k\right)+ +6n\sum_{k=1}^{\infty}\frac{(-1)^k}{k(k+n)}\operatorname{Im}\left(\left(\frac{\sqrt{3}-1}{2}+i\frac{\sqrt{3}+1}{2}\right)^{-k}\right) \qquad (54)$$

$$\pi=-12n\sum_{k=1}^{\infty}\frac{k!}{k(n)_{k+1}}\operatorname{Im}\left(\left(\frac{2+\sqrt{3}}{2}+i\frac{3+2\sqrt{3}}{2}\right)^{-k}\right)+ +12n\sum_{k=1}^{\infty}\frac{(-1)^k}{k(k+n)}\operatorname{Im}\left(\left(\frac{\sqrt{3}}{2}+i\frac{3+2\sqrt{3}}{2}\right)^{-k}\right) \qquad (55)$$

$$\pi = -12n\sum_{k=1}^{\infty}\frac{k!}{k(n)_{k+1}}\operatorname{Im}\left(\left(\frac{3+\sqrt{3}}{2}+i\frac{\sqrt{3}+1}{2}\right)^{-k}\right)+ \\ +12n\sum_{k=1}^{\infty}\frac{(-1)^k}{k(k+n)}\left(\frac{1}{\sqrt{3}+1}\right)^k\operatorname{Im}\left((1-i)^k\right) \qquad (56)$$

Entry 12: If $n \in \mathbb{N} \cup \{0\}$, then

$$\pi = \frac{2n!}{(1/2)_n}\sum_{k=0}^{\infty}\left(-n+\frac{1}{2}\right)_k\frac{1}{(2k+1)k!} \qquad (57)$$

Entry 13: If $p,q \in \mathbb{N} \cup \{0\}$, then

$$\frac{1}{\pi} = \frac{(1/2)_{p+q}}{2p!q!}\,{}_4F_3\left(\frac{5}{4},\frac{1}{2},-p+\frac{1}{2},-q+\frac{1}{2};\frac{1}{4},p+1,q+1;-1\right) \qquad (58)$$

Remark: ${}_4F_3$ is the generalized hypergeometric function.

Entry 14: If $n \in \mathbb{N}, p,q \in \mathbb{N} \cup \{0\}$, then

$$\pi = \frac{n!(p+q+n-1)!}{(1/2)_{p+n}(1/2)_{q+n}}\,{}_4F_3\left(\frac{n}{2}+1,n,-p+\frac{1}{2},-q+\frac{1}{2};\frac{n}{2},p+n+\frac{1}{2},q+n+\frac{1}{2};-1\right) \qquad (59)$$

Remark: ${}_4F_3$ is the generalized hypergeometric function.

References

[1] Bruce C.Berndt : Ramanujan’s Notebooks Part II , Springer Verlag, 1989.