An Intuitive Approach to the Coherent and Squeezed States of the Quantum Harmonic Oscillator

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Preface

The purpose of this work is to introduce, in a simple, intuitive way, the coherent and squeezed states of the quantum harmonic oscillator (QHO), through a series of exercises, which are solved in detail.

Starting from the application of a spatial translation to the ground state of the QHO, we introduce the spatial and momentum translations, focusing on their application to the QHO, which leads us to the displacement operator.

Next, we introduce the coherent states and examine their basic aspects.

We then proceed to give a simple and purely intuitive introduction to the squeezed states and we conclude by identifying the coherent states as states of minimum energy expectation value compared to the respective squeezed states.

The reader is assumed to have a basic knowledge of the postulates and the mathematical formalism of quantum mechanics, including the Dirac notation and the ladder operator method of the QHO.

I. Preliminaries

Applying a spatial translation to the ground state of the quantum harmonic oscillator (QHO)

1) At time $t=0$*, the wave function of a QHO is* $\psi(x)=\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)\psi_0(x)$*,*

where $\psi_0(x)$ *is the ground-state wave function,* $\hat{p}=-i\hbar\frac{d}{dx}$ *is the momentum operator (in the position representation), and* x_1 *is an eigenvalue of the position operator (be careful, it is not the position operator).*

i) Show that the wave function $\psi(x)$ *is normalized.*

ii) Calculate the position and momentum expectation values of the QHO at time $t=0$*.*

iii) Calculate the energy expectation value of the QHO at time $t\geq 0$*.*

It is given that

$$\psi_0(x)=\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)$$

and

$$\int_{-\infty}^{\infty}dx\exp\left(-ax^2-2bx\right)=\sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{a}\right),\ a>0.$$

For the integral, see, for instance,
https://en.wikipedia.org/wiki/Gaussian_integral

Solution

i) Using the Taylor expansion of the operator $\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)$, i.e.

$$\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)=\sum_{n=0}^{\infty}\frac{\left(-\frac{i\hat{p}x_1}{\hbar}\right)^n}{n!},$$

the wave function $\psi(x)$ is written as

$$\psi(x)=\left(\sum_{n=0}^{\infty}\frac{\left(-\frac{i\hat{p}x_1}{\hbar}\right)^n}{n!}\right)\psi_0(x)=\sum_{n=0}^{\infty}\frac{\left(-\frac{i\hat{p}x_1}{\hbar}\right)^n}{n!}\psi_0(x)$$

Since x_1 is a number (not an operator), the momentum operator commutes with x_1, and with i too, and thus

$$\left(-\frac{i\hat{p}x_1}{\hbar}\right)^n=\left(-\frac{ix_1}{\hbar}\right)^n\hat{p}^n$$

Therefore, the wave function $\psi(x)$ is written as

$$\psi(x)=\sum_{n=0}^{\infty}\frac{\left(-\frac{ix_1}{\hbar}\right)^n}{n!}\hat{p}^n\psi_0(x)=\sum_{n=0}^{\infty}\frac{\left(-\frac{ix_1}{\hbar}\right)^n}{n!}\left(-i\hbar\frac{d}{dx}\right)^n\psi_0(x)=$$
$$=\sum_{n=0}^{\infty}\frac{\left(-\frac{ix_1}{\hbar}\right)^n(-i\hbar)^n}{n!}\frac{d^n}{dx^n}\psi_0(x)=\sum_{n=0}^{\infty}\frac{\left(i^2x_1\right)^n}{n!}\psi_0^{(n)}(x)=$$
$$=\sum_{n=0}^{\infty}\frac{(-x_1)^n}{n!}\psi_0^{(n)}(x)=\sum_{n=0}^{\infty}\frac{\psi_0^{(n)}(x)}{n!}(-x_1)^n=\psi_0(x-x_1)$$

Remember that the Taylor series of a (proper) function $f(x)$ about x' is written as $f(x)=\sum_{m=0}^{\infty}\frac{f^{(m)}(x')}{m!}(x-x')$

That is

$$\psi(x)=\psi_0(x-x_1) \quad (1)$$

Using (1), we have

$$\int_{-\infty}^{\infty}dx\left|\psi(x)\right|^2=\int_{-\infty}^{\infty}dx\left|\psi_0(x-x_1)\right|^2 \quad (2)$$

Changing the integration variable to

$y=x-x_1$,

we have

$$dx = dy$$

and

$$\lim_{x\to\pm\infty} y = \pm\infty$$

Thus, (2) becomes

$$\int_{-\infty}^{\infty} dx \left|\psi(x)\right|^2 = \int_{-\infty}^{\infty} dy \left|\psi_0(y)\right|^2 = 1 ,$$

because the ground-state wave function is normalized.

Therefore, the wave function $\psi(x)$ is also normalized.

Another way of showing that $\psi(x)$ is normalized is by observing that the operator $\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)$ is unitary.

Indeed, the Hermitian conjugate of $\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)$ is

$$\left(\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)\right)^\dagger = \left(\sum_{n=0}^{\infty}\frac{\left(-\frac{i\hat{p}x_1}{\hbar}\right)^n}{n!}\right)^\dagger \underbrace{=}_{\left(\sum_i \hat{A}_i\right)^\dagger = \sum_i \hat{A}_i^\dagger} \sum_{n=0}^{\infty}\left(\frac{\left(-\frac{i\hat{p}x_1}{\hbar}\right)^n}{n!}\right)^\dagger = \sum_{n=0}^{\infty}\frac{\left(\left(-\frac{i\hat{p}x_1}{\hbar}\right)^n\right)^\dagger}{n!}$$

That is

$$\left(\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)\right)^\dagger = \sum_{n=0}^{\infty}\frac{\left(\left(-\frac{i\hat{p}x_1}{\hbar}\right)^n\right)^\dagger}{n!} \qquad (3)$$

But

$$\left(\left(-\frac{i\hat{p}x_1}{\hbar}\right)^n\right)^\dagger = \left(\underbrace{\left(-\frac{i\hat{p}x_1}{\hbar}\right)\cdots\left(-\frac{i\hat{p}x_1}{\hbar}\right)}_{n\text{ times}}\right)^\dagger = \left(\left(\underbrace{\left(-\frac{i\hat{p}x_1}{\hbar}\right)\cdots\left(-\frac{i\hat{p}x_1}{\hbar}\right)}_{(n-1)\text{ times}}\right)\left(-\frac{i\hat{p}x_1}{\hbar}\right)\right)^\dagger =$$

$$= \left(-\frac{i\hat{p}x_1}{\hbar}\right)^\dagger \left(\underbrace{\left(-\frac{i\hat{p}x_1}{\hbar}\right)\cdots\left(-\frac{i\hat{p}x_1}{\hbar}\right)}_{(n-1)\text{ times}}\right)^\dagger = \left(-\frac{i\hat{p}x_1}{\hbar}\right)^\dagger \left(\left(-\frac{i\hat{p}x_1}{\hbar}\right)^{n-1}\right)^\dagger$$

That is

$$\left(\left(-\frac{i\hat{p}x_1}{\hbar}\right)^n\right)^\dagger = \left(-\frac{i\hat{p}x_1}{\hbar}\right)^\dagger \left(\left(-\frac{i\hat{p}x_1}{\hbar}\right)^{n-1}\right)^\dagger$$

Using repeatedly, n times, the previous relation, we obtain

$$\left(\left(-\frac{i\hat{p}x_1}{\hbar}\right)^n\right)^\dagger = \left(-\frac{i\hat{p}x_1}{\hbar}\right)^\dagger \left(\left(-\frac{i\hat{p}x_1}{\hbar}\right)^{n-1}\right)^\dagger = \underbrace{\left(-\frac{i\hat{p}x_1}{\hbar}\right)^\dagger \left(-\frac{i\hat{p}x_1}{\hbar}\right)^\dagger}_{\text{2 times}} \left(\left(-\frac{i\hat{p}x_1}{\hbar}\right)^{n-2}\right)^\dagger =$$

$$= \underbrace{\left(-\frac{i\hat{p}x_1}{\hbar}\right)^\dagger \left(-\frac{i\hat{p}x_1}{\hbar}\right)^\dagger \left(-\frac{i\hat{p}x_1}{\hbar}\right)^\dagger}_{\text{3 times}} \left(\left(-\frac{i\hat{p}x_1}{\hbar}\right)^{n-3}\right)^\dagger = \ldots =$$

$$= \underbrace{\left(-\frac{i\hat{p}x_1}{\hbar}\right)^\dagger \ldots \left(-\frac{i\hat{p}x_1}{\hbar}\right)^\dagger}_{n \text{ times}} \left(\underbrace{\left(-\frac{i\hat{p}x_1}{\hbar}\right)^{n-n}}_{\left(-\frac{i\hat{p}x_1}{\hbar}\right)^0 = 1}\right)^\dagger = \left(\left(-\frac{i\hat{p}x_1}{\hbar}\right)^\dagger\right)^n$$

That is

$$\left(\left(-\frac{i\hat{p}x_1}{\hbar}\right)^n\right)^\dagger = \left(\left(-\frac{i\hat{p}x_1}{\hbar}\right)^\dagger\right)^n \qquad (4)$$

By means of (4), (3) becomes

$$\left(\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)\right)^\dagger = \sum_{n=0}^{\infty} \frac{\left(\left(-\frac{i\hat{p}x_1}{\hbar}\right)^n\right)^\dagger}{n!} = \sum_{n=0}^{\infty} \frac{\left(\left(-\frac{i\hat{p}x_1}{\hbar}\right)^\dagger\right)^n}{n!}$$

Since the momentum operator is Hermitian,

$$\left(-\frac{i\hat{p}x_1}{\hbar}\right)^\dagger = \left(\frac{i\hat{p}x_1}{\hbar}\right)$$

Thus

$$\left(\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)\right)^\dagger = \sum_{n=0}^{\infty} \frac{\left(\frac{i\hat{p}x_1}{\hbar}\right)^n}{n!} = \exp\left(\frac{i\hat{p}x_1}{\hbar}\right)$$

That is

$$\left(\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)\right)^\dagger = \exp\left(\frac{i\hat{p}x_1}{\hbar}\right) \tag{5}$$

Using (5) we obtain

$$\left(\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)\right)^\dagger \exp\left(-\frac{i\hat{p}x_1}{\hbar}\right) = \exp\left(\frac{i\hat{p}x_1}{\hbar}\right)\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)$$

Since the commutator of $\frac{i\hat{p}x_1}{\hbar}$ and $-\frac{i\hat{p}x_1}{\hbar}$ is zero, applying the property

$$\exp\left(\hat{A}+\hat{B}\right) = \exp\left(\hat{A}\right)\exp\left(\hat{B}\right)\exp\left(-\frac{1}{2}\left[\hat{A},\hat{B}\right]\right)$$

yields

$$\exp\left(\frac{i\hat{p}x_1}{\hbar}\right)\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right) = \exp\left(\frac{i\hat{p}x_1}{\hbar} - \frac{i\hat{p}x_1}{\hbar}\right) = \exp 0 = 1$$

In the same way,

$$\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)\left(\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)\right)^\dagger = \exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)\exp\left(\frac{i\hat{p}x_1}{\hbar}\right) = \exp\left(-\frac{i\hat{p}x_1}{\hbar} + \frac{i\hat{p}x_1}{\hbar}\right) = 1$$

Thus

$$\left(\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)\right)^\dagger \exp\left(-\frac{i\hat{p}x_1}{\hbar}\right) = \exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)\left(\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)\right)^\dagger = 1$$

Therefore, the operator $\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)$ is unitary, and thus it preserves the norms of the states on which it acts.

Then, in the state space, we have

$$\left\| |\psi\rangle \right\| = \left\| \exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)|0\rangle \right\| = \left\| |0\rangle \right\|$$

Thus

$$\left\| |\psi\rangle \right\|^2 = \left\| |0\rangle \right\|^2 \Rightarrow \langle\psi|\psi\rangle = \langle 0|0\rangle$$

Using the completeness of the position eigenstates, we have

$$\langle\psi|\left(\underbrace{\int_{-\infty}^{\infty}dx|x\rangle\langle x|}_{1}\right)|\psi\rangle=\langle 0|\left(\underbrace{\int_{-\infty}^{\infty}dx|x\rangle\langle x|}_{1}\right)|0\rangle\Rightarrow$$

$$\Rightarrow\int_{-\infty}^{\infty}dx\underbrace{\langle\psi|x\rangle}_{\langle x|\psi\rangle^*}\langle x|\psi\rangle=\int_{-\infty}^{\infty}dx\underbrace{\langle 0|x\rangle}_{\langle x|0\rangle^*}\langle x|0\rangle\Rightarrow\int_{-\infty}^{\infty}dx\left|\langle x|\psi\rangle\right|^2=\int_{-\infty}^{\infty}dx\left|\langle x|0\rangle\right|^2$$

Substituting $\langle x|0\rangle=\psi_0(x)$ and $\langle x|\psi\rangle=\psi(x)$, and using that the ground state is normalized, i.e. $\int_{-\infty}^{\infty}dx\left|\psi_0(x)\right|^2=1$, we obtain

$$\int_{-\infty}^{\infty}dx\left|\psi(x)\right|^2=1$$

ii) At time $t=0$, the wave function of the QHO is the wave function (1). Thus, its position expectation value is written as

$$\langle x\rangle_0=\int_{-\infty}^{\infty}dx\psi_0^*(x-x_1)x\psi_0(x-x_1) \qquad (6)$$

Changing again the integration variable to

$y=x-x_1$,

we have

$x=y+x_1$,

$dx=dy$

and

$\lim_{x\to\pm\infty}y=\pm\infty$

Thus (6) becomes

$$\langle x\rangle_0=\int_{-\infty}^{\infty}dy\psi_0^*(y)(y+x_1)\psi_0(y)=\int_{-\infty}^{\infty}dy\psi_0^*(y)y\psi_0(y)+x_1\int_{-\infty}^{\infty}dy\left|\psi_0(y)\right|^2$$

The integral $\int_{-\infty}^{\infty}dy\psi_0^*(y)y\psi_0(y)$ is the position expectation value of the QHO in the ground state, which is zero.

We remind that the position expectation value of the QHO in an energy eigenstate is zero.

Also, $\int_{-\infty}^{\infty} dy \left|\psi_0(y)\right|^2 = 1$, since the ground-state wave function is normalized.

Thus

$$\langle x \rangle_0 = x_1 \tag{7}$$

Likewise, the momentum expectation value of the QHO at $t = 0$ is

$$\langle p \rangle_0 = \int_{-\infty}^{\infty} dx \psi_0^*(x - x_1)\left(-i\hbar \frac{d}{dx}\right)\psi_0(x - x_1)$$

Doing again the variable change $y = x - x_1$, the previous equation becomes

$$\langle p \rangle_0 = \int_{-\infty}^{\infty} dy \psi_0^*(y)\left(-i\hbar \frac{d}{dy}\right)\psi_0(y)$$

The integral $\int_{-\infty}^{\infty} dy \psi_0^*(y)\left(-i\hbar \frac{d}{dy}\right)\psi_0(y)$ is the momentum expectation value of the QHO in the ground state, which is zero.

We remind that the momentum expectation value of the QHO in an energy eigenstate is also zero.

Thus

$$\langle p \rangle_0 = 0 \tag{8}$$

We see that the operator $\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)$, acting on the ground state of the QHO, changes (translates) the position expectation value by x_1 while leaving unchanged the momentum expectation value.

This is easily generalized to an arbitrary, but bound, state – not necessarily an energy eigenstate – of a one-dimensional quantum system – not necessarily of the QHO. The constraint that the state on which the operator $\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)$ acts should be bound is necessary so that it is normalizable and the respective integrals are finite.

Moreover, it can be shown – see the following exercise – that if the operator $\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)$ acts on a position eigenstate $|x\rangle$, it yields the position eigenstate $|x+x_1\rangle$, i.e.

$$\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)|x\rangle = |x+x_1\rangle$$

Due to its property to translate the position, or the position expectation value, by its argument x_1, the operator $\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)$ is a spatial translation operator or, simply, a translation operator.

Obviously, for each position eigenvalue x_1, we can define a spatial translation operator

$$\hat{T}_{x_1} \equiv \exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)$$

iii) Since the Hamiltonian of the QHO is time independent, the Ehrenfest theorem gives for the time evolution of the QHO energy expectation value

$$\frac{d\langle E\rangle_t}{dt} = \frac{i}{\hbar}\left\langle\left[\hat{H},\hat{H}\right]\right\rangle_t = 0$$

Thus

$$\langle E\rangle_t = \langle E\rangle_0 \qquad (9)$$

At $t=0$, the state of the QHO is described by the wave function $\psi(x)$.

Thus, the energy expectation value of the QHO at $t=0$ is

$$\langle E\rangle_0 = \int_{-\infty}^{\infty} dx\psi^*(x)\hat{H}(x)\psi(x) \qquad (10)$$

where $\hat{H}(x)$ is the Hamiltonian of the QHO in the position representation, i.e.

$$\hat{H}(x) = \frac{\left(-i\hbar\frac{d}{dx}\right)^2}{2m} + \frac{1}{2}m\omega^2x^2$$

Substituting (1) and the previous Hamiltonian into (10), we obtain

$$\langle E\rangle_0 = \int_{-\infty}^{\infty} dx\psi_0^*(x-x_1)\left(\frac{\left(-i\hbar\frac{d}{dx}\right)^2}{2m}+\frac{1}{2}m\omega^2x^2\right)\psi_0(x-x_1) =$$

$$= \int_{-\infty}^{\infty} dx\psi_0^*(x-x_1)\frac{\left(-i\hbar\frac{d}{dx}\right)^2}{2m}\psi_0(x-x_1)+\frac{1}{2}m\omega^2\int_{-\infty}^{\infty} dx\psi_0^*(x-x_1)x^2\psi_0(x-x_1)$$

That is

$$\langle E\rangle_0 = \int_{-\infty}^{\infty} dx\psi_0^*(x-x_1)\frac{\left(-i\hbar\frac{d}{dx}\right)^2}{2m}\psi_0(x-x_1)+\frac{1}{2}m\omega^2\int_{-\infty}^{\infty} dx\psi_0^*(x-x_1)x^2\psi_0(x-x_1) \quad (11)$$

We'll calculate the two integrals separately.
Changing again the integration variable to

$y = x - x_1$,

we have

$x = y + x_1$,

$dx = dy$

and

$\lim_{x\to\pm\infty} y = \pm\infty$

Thus, the first integral on the right-hand side of (11) is written as

$$\int_{-\infty}^{\infty} dx\psi_0^*(x-x_1)\frac{\left(-i\hbar\frac{d}{dx}\right)^2}{2m}\psi_0(x-x_1) = \int_{-\infty}^{\infty} dy\psi_0^*(y)\frac{\left(-i\hbar\frac{d}{dy}\right)^2}{2m}\psi_0(y) \quad (12)$$

The second integral on the right-hand side of (11) is written as

$$\int_{-\infty}^{\infty} dx\psi_0^*(x-x_1)x^2\psi_0(x-x_1) = \int_{-\infty}^{\infty} dy\psi_0^*(y)(y+x_1)^2\psi_0(y) =$$

$$= \int_{-\infty}^{\infty} dy\psi_0^*(y)(y^2+2x_1y+x_1^2)\psi_0(y) =$$

$$= \int_{-\infty}^{\infty} dy\psi_0^*(y)y^2\psi_0(y) + 2x_1\int_{-\infty}^{\infty} dy\psi_0^*(y)y\psi_0(y) + x_1^2\int_{-\infty}^{\infty} dy\psi_0^*(y)\psi_0(y)$$

The integral $\int_{-\infty}^{\infty} dy\psi_0^*(y)y\psi_0(y)$ is the position expectation value in the ground state of the QHO, which is zero.

Also, since the ground-state wave function is normalized, the integral $\int_{-\infty}^{\infty} dy\psi_0^*(y)\psi_0(y)$ is 1.

Thus

$$\int_{-\infty}^{\infty} dx\psi_0^*(x-x_1)x^2\psi_0(x-x_1) = \int_{-\infty}^{\infty} dy\psi_0^*(y)y^2\psi_0(y) + x_1^2 \tag{13}$$

Substituting the integrals (12) and (13) into (11), we obtain

$$\langle E\rangle_0 = \int_{-\infty}^{\infty} dy\psi_0^*(y)\frac{\left(-i\hbar\frac{d}{dy}\right)^2}{2m}\psi_0(y) + \frac{1}{2}m\omega^2\left(\int_{-\infty}^{\infty} dy\psi_0^*(y)y^2\psi_0(y) + x_1^2\right) =$$

$$= \int_{-\infty}^{\infty} dy\psi_0^*(y)\frac{\left(-i\hbar\frac{d}{dy}\right)^2}{2m}\psi_0(y) + \int_{-\infty}^{\infty} dy\psi_0^*(y)\frac{1}{2}m\omega^2y^2\psi_0(y) + \frac{1}{2}m\omega^2x_1^2 =$$

$$= \int_{-\infty}^{\infty} dy\psi_0^*(y)\left(\frac{\left(-i\hbar\frac{d}{dy}\right)^2}{2m} + \frac{1}{2}m\omega^2y^2\right)\psi_0(y) + \frac{1}{2}m\omega^2x_1^2 =$$

$$= \int_{-\infty}^{\infty} dy\psi_0^*(y)\hat{H}(y)\psi_0(y) + \frac{1}{2}m\omega^2x_1^2$$

The integral $\int_{-\infty}^{\infty} dy\psi_0^*(y)\hat{H}(y)\psi_0(y)$ is the energy expectation value of the QHO in the ground state, which, obviously, is equal to the ground-state energy, i.e. $\frac{\hbar\omega}{2}$.

Thus

$$\langle E\rangle_0 = \frac{\hbar\omega}{2} + \frac{1}{2}m\omega^2 {x_1}^2 \qquad (14)$$

We see that the action of the spatial translation operator $\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)$ on the ground state of the QHO increases the energy expectation value by the positive amount $\frac{1}{2}m\omega^2 {x_1}^2$, which is the energy of a classical harmonic oscillator with amplitude $|x_1|$.

By means of (14), (9) is written as

$$\langle E\rangle_t = \frac{\hbar\omega}{2} + \frac{1}{2}m\omega^2 {x_1}^2 \quad (15)$$

This is the energy expectation value of the QHO at time $t \geq 0$.

More on spatial translation operators

2) i) If $\hat{T}_{x_1} \equiv \exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)$ *is a spatial translation operator, show that* $\left[\hat{x}, \hat{T}_{x_1}\right] = x_1 \hat{T}_{x_1}$.

ii) Using the previous commutator, show that the action of $\hat{T}_{x_1}$ *on a position eigenstate yields a position eigenstate with eigenvalue translated by* x_1*, i.e.* $\hat{T}_{x_1}|x\rangle = |x + x_1\rangle$.

iii) Using the Baker-Campbell-Hausdorff formula

$$\exp\left(\hat{A}\right)\hat{B}\exp\left(-\hat{A}\right) = \hat{B} + \left[\hat{A},\hat{B}\right] + \frac{1}{2}\left[\hat{A},\left[\hat{A},\hat{B}\right]\right] + \frac{1}{3!}\left[\hat{A},\left[\hat{A},\left[\hat{A},\hat{B}\right]\right]\right] + \ldots + \frac{1}{n!}\underbrace{\left[\hat{A},\left[\hat{A},\ldots\left[\hat{A},\hat{B}\right]\ldots\right]\right]}_{n\ \hat{A}\text{'s or } n \text{ commutators}}$$

show that, for an infinitely many times differentiable function f, $\hat{T}_{x_1}^{\dagger} f\left(\hat{x}\right)\hat{T}_{x_1} = f\left(\hat{x} + x_1\right)$.

iv) A QHO is in the state $\hat{T}_{x_1}|\psi\rangle$*, with* $|\psi\rangle$ *being an arbitrary state of the QHO. Using the previous property of* $\hat{T}_{x_1}$*, express the expectation values of the position, momentum, and energy, as well as the position and momentum uncertainties, and the position-momentum uncertainty product, in the translated state* $\hat{T}_{x_1}|\psi\rangle$ *in terms of the respective quantities in the untranslated state* $|\psi\rangle$.

Solution

i) Using the Taylor expansion of $\hat{T}_{x_1}$, the commutator $\left[\hat{x},\hat{T}_{x_1}\right]$ is written as

$$\left[\hat{x},\hat{T}_{x_1}\right]=\left[\hat{x},\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)\right]=\left[\hat{x},\sum_{n=0}^{\infty}\frac{\left(-\frac{i\hat{p}x_1}{\hbar}\right)^n}{n!}\right]=\sum_{n=0}^{\infty}\frac{1}{n!}\left[\hat{x},\left(-\frac{i\hat{p}x_1}{\hbar}\right)^n\right]=$$

$$=\sum_{n=0}^{\infty}\frac{\left(-\frac{ix_1}{\hbar}\right)^n}{n!}\left[\hat{x},\hat{p}^n\right]$$

In the last equality, we used that the quantity $-\frac{ix_1}{\hbar}$ is a number, and thus it commutes with the momentum operator.

We remind that the argument x_1 of the spatial translation operator $\hat{T}_{x_1}$ is a real number, not an operator. You can think of it as being an eigenvalue of the position operator.

Thus

$$\left[\hat{x},\hat{T}_{x_1}\right]=\sum_{n=0}^{\infty}\frac{\left(-\frac{ix_1}{\hbar}\right)^n}{n!}\left[\hat{x},\hat{p}^n\right] \tag{1}$$

We need to calculate the commutator $\left[\hat{x},\hat{p}^n\right]$.

The result of a commutator is independent of the representation we choose to calculate it.

Since the momentum operator is to the nth power, we choose to calculate the commutator in the momentum representation, where the momentum operator is a scalar variable, and thus we can handle the exponential much easier than if it was an operator.

We remind that in the momentum representation, $\hat{x}=i\hbar\frac{d}{dp}$ and $\hat{p}=p$.

Choosing an arbitrary wave function $\phi(p)$ (in the momentum representation), the action of $\left[\hat{x},\hat{p}^n\right]$ on $\phi(p)$ yields

$$\left[i\hbar\frac{d}{dp},p^n\right]\phi(p)=\left(i\hbar\frac{d}{dp}p^n-p^n i\hbar\frac{d}{dp}\right)\phi(p)=i\hbar\left(\frac{d}{dp}\left(p^n\phi(p)\right)-p^n\frac{d\phi(p)}{dp}\right)=$$
$$=i\hbar\left(\left(p^n\right)'\phi(p)+p^n\phi'(p)-p^n\phi'(p)\right)=i\hbar np^{n-1}\phi(p)$$

Here, the prime denotes differentiation with respect to p .

That is

$$\left[i\hbar\frac{d}{dp},p^n\right]\phi(p)=i\hbar np^{n-1}\phi(p)$$

Since the wave function $\phi(p)$ is arbitrary,

$$\left[i\hbar\frac{d}{dp},p^n\right]=i\hbar np^{n-1}$$

or, in representation-free form,

$$\left[\hat{x},\hat{p}^n\right]=i\hbar n\hat{p}^{n-1} \qquad (2)$$

with $n\geq 1$. If $n=0$, the commutator (2) is zero.
Substituting into (1) yields

$$\left[\hat{x},\hat{T}_{x_1}\right]=\sum_{n=1}^{\infty}\frac{\left(-\frac{ix_1}{\hbar}\right)^n}{n!}i\hbar n\hat{p}^{n-1}=\sum_{n=1}^{\infty}\frac{\left(-\frac{ix_1}{\hbar}\right)^n}{(n-1)!}i\hbar\hat{p}^{n-1}=\sum_{n=1}^{\infty}\frac{\left(-\frac{ix_1}{\hbar}\right)^{n-1}\left(\frac{-ix_1}{\hbar}i\hbar\right)}{(n-1)!}\hat{p}^{n-1}=$$
$$=\sum_{n=1}^{\infty}\frac{\left(-\frac{ix_1}{\hbar}\right)^{n-1}x_1}{(n-1)!}\hat{p}^{n-1}=x_1\sum_{n=1}^{\infty}\frac{\left(-\frac{ix_1}{\hbar}\right)^{n-1}}{(n-1)!}\hat{p}^{n-1}\underset{n'=n-1}{\equiv}x_1\sum_{n'=0}^{\infty}\frac{\left(-\frac{ix_1}{\hbar}\right)^{n'}}{n'!}\hat{p}^{n'}=x_1\sum_{n'=0}^{\infty}\frac{\left(-\frac{i\hat{p}x_1}{\hbar}\right)^{n'}}{n'!}=$$
$$=x_1\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)=x_1\hat{T}_{x_1}$$

That is

$$\left[\hat{x},\hat{T}_{x_1}\right]=x_1\hat{T}_{x_1} \qquad (3)$$

ii) Using the previous commutator, we obtain

$$\hat{x}\hat{T}_{x_1}-\hat{T}_{x_1}\hat{x}=x_1\hat{T}_{x_1}\Rightarrow\hat{x}\hat{T}_{x_1}=\hat{T}_{x_1}\hat{x}+x_1\hat{T}_{x_1}$$

Thus, the action of the operator $\hat{x}\hat{T}_{x_1}$ on a position eigenstate $|x\rangle$ yields

$$\hat{x}\hat{T}_{x_1}|x\rangle = \left(\hat{T}_{x_1}\hat{x} + x_1\hat{T}_{x_1}\right)|x\rangle = \hat{T}_{x_1}\hat{x}|x\rangle + x_1\hat{T}_{x_1}|x\rangle = \hat{T}_{x_1}x|x\rangle + x_1\hat{T}_{x_1}|x\rangle$$

Since x is a number – an eigenvalue of the position operator – it commutes with $\hat{T}_{x_1}$, and thus

$$\hat{x}\hat{T}_{x_1}|x\rangle = x\hat{T}_{x_1}|x\rangle + x_1\hat{T}_{x_1}|x\rangle = (x + x_1)\hat{T}_{x_1}|x\rangle$$

That is

$$\hat{x}\hat{T}_{x_1}|x\rangle = (x + x_1)\hat{T}_{x_1}|x\rangle$$

Thus, the state $\hat{T}_{x_1}|x\rangle$ is a position eigenstate with eigenvalue $x + x_1$.

Therefore

$$\hat{T}_{x_1}|x\rangle = A|x + x_1\rangle,$$

where A is a complex constant.

In the previous exercise, we showed that the operator $\hat{T}_{x_1}$ is unitary, and thus it preserves the norms of the states on which it acts.

Therefore, although the position eigenstates are not bound, we can assume that

$$|A| = 1$$

Omitting the physically unimportant constant phase of A, we end up to

$$\hat{T}_{x_1}|x\rangle = |x + x_1\rangle \tag{4}$$

iii) For $\hat{A} = \frac{i\hat{p}x_1}{\hbar}$ and $\hat{B} = f(\hat{x})$, the Baker-Campbell-Hausdorff formula is written as

$$\exp\left(\frac{i\hat{p}x_1}{\hbar}\right)f(\hat{x})\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right) = f(\hat{x}) + \left[\frac{i\hat{p}x_1}{\hbar}, f(\hat{x})\right] + \frac{1}{2}\left[\frac{i\hat{p}x_1}{\hbar},\left[\frac{i\hat{p}x_1}{\hbar}, f(\hat{x})\right]\right] +$$
$$+\frac{1}{3!}\left[\frac{i\hat{p}x_1}{\hbar},\left[\frac{i\hat{p}x_1}{\hbar},\left[\frac{i\hat{p}x_1}{\hbar}, f(\hat{x})\right]\right]\right] + \ldots + \frac{1}{n!}\left[\frac{i\hat{p}x_1}{\hbar},\left[\frac{i\hat{p}x_1}{\hbar},\ldots\left[\frac{i\hat{p}x_1}{\hbar}, f(\hat{x})\right]\ldots\right]\right] \tag{5}$$

We need to calculate the commutator $\left[\hat{p}, f(\hat{x})\right]$. Since the function f is unknown, it is suitable to do the calculation in the position representation, where the position operator is a scalar variable, and thus the operator function $f(\hat{x})$ becomes a scalar function.

We remind that in the position representation, $\hat{x} = x$ and $\hat{p} = -i\hbar \frac{d}{dx}$.

Choosing an arbitrary wave function $\phi(x)$ (in the position representation), the action of $\left[\hat{p}, f(\hat{x})\right]$ on $\phi(x)$ yields

$$\left[-i\hbar\frac{d}{dx}, f(x)\right]\phi(x) = \left(-i\hbar\frac{d}{dx}f(x) - f(x)\left(-i\hbar\frac{d}{dx}\right)\right)\phi(x) =$$
$$= -i\hbar\left(\frac{d}{dx}\left(f(x)\phi(x)\right) - f(x)\frac{d\phi(x)}{dx}\right) = -i\hbar\left(f'(x)\phi(x) + f(x)\phi'(x) - f(x)\phi'(x)\right) =$$
$$= -i\hbar f'(x)\phi(x)$$

Here, the prime denotes differentiation with respect to x.

That is

$$\left[-i\hbar\frac{d}{dx}, f(x)\right]\phi(x) = -i\hbar f'(x)\phi(x)$$

Since the wave function $\phi(x)$ is arbitrary,

$$\left[-i\hbar\frac{d}{dx}, f(x)\right] = -i\hbar f'(x)$$

or, in representation-free form,

$$\left[\hat{p}, f(\hat{x})\right] = -i\hbar f'(\hat{x}) \qquad (6)$$

Using (6), we have

$$\left[\frac{i\hat{p}x_1}{\hbar}, f(\hat{x})\right] = \frac{ix_1}{\hbar}\left[\hat{p}, f(\hat{x})\right] = \frac{ix_1}{\hbar}\left(-i\hbar f'(\hat{x})\right) = x_1 f'(\hat{x})$$

$$\left[\frac{i\hat{p}x_1}{\hbar}, \left[\frac{i\hat{p}x_1}{\hbar}, f(\hat{x})\right]\right] = \left[\frac{i\hat{p}x_1}{\hbar}, x_1 f'(\hat{x})\right] = \frac{i}{\hbar}x_1^2\left[\hat{p}, f'(\hat{x})\right] = \frac{i}{\hbar}x_1^2\left(-i\hbar f''(\hat{x})\right) = x_1^2 f''(\hat{x})$$

Assuming that

$$\underbrace{\left[\frac{i\hat{p}x_1}{\hbar},\left[\frac{i\hat{p}x_1}{\hbar},\ldots\left[\frac{i\hat{p}x_1}{\hbar},f(\hat{x})\right]\ldots\right]\right]}_{k \text{ commutators}}=x_1{}^k f^{(k)}(\hat{x})$$

then

$$\underbrace{\left[\frac{i\hat{p}x_1}{\hbar},\left[\frac{i\hat{p}x_1}{\hbar},\ldots\left[\frac{i\hat{p}x_1}{\hbar},f(\hat{x})\right]\ldots\right]\right]}_{(k+1) \text{ commutators}}=\left[\frac{i\hat{p}x_1}{\hbar},\underbrace{\left[\frac{i\hat{p}x_1}{\hbar},\ldots\left[\frac{i\hat{p}x_1}{\hbar},f(\hat{x})\right]\ldots\right]}_{k \text{ commutators}}\right]=\left[\frac{i\hat{p}x_1}{\hbar},x_1{}^k f^{(k)}(\hat{x})\right]=$$
$$=\frac{i}{\hbar}x_1{}^{k+1}\left[\hat{p},f^{(k)}(\hat{x})\right]=\frac{i}{\hbar}x_1{}^{k+1}\left(-i\hbar f^{(k+1)}(\hat{x})\right)=x_1{}^{k+1}f^{(k+1)}(\hat{x})$$

Thus

$$\underbrace{\left[\frac{i\hat{p}x_1}{\hbar},\left[\frac{i\hat{p}x_1}{\hbar},\ldots\left[\frac{i\hat{p}x_1}{\hbar},f(\hat{x})\right]\ldots\right]\right]}_{n \text{ commutators}}=x_1{}^n f^{(n)}(\hat{x}),$$

for every $n=1,2,\ldots$

Thus, the right-hand side of (5) is written as

$$f(\hat{x})+x_1 f'(\hat{x})+\frac{1}{2}x_1{}^2 f''(\hat{x})+\frac{1}{3!}x_1{}^3 f^{(3)}(\hat{x})+\ldots+\frac{1}{n!}x_1{}^n f^{(n)}(\hat{x})=$$
$$=f(\hat{x})+f'(\hat{x})x_1+\frac{f''(\hat{x})}{2}x_1{}^2+\frac{f^{(3)}(\hat{x})}{3!}x_1{}^3+\ldots+\frac{f^{(n)}(\hat{x})}{n!}x_1{}^n=\sum_{n=0}^{\infty}\frac{f^{(n)}(\hat{x})}{n!}x_1{}^n$$

The series $\sum_{n=0}^{\infty}\frac{f^{(n)}(\hat{x})}{n!}x_1{}^n$ is the Taylor expansion of $f(\hat{x}+x_1)$ about $\hat{x}$, i.e.

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(\hat{x})}{n!}x_1{}^n=f(\hat{x}+x_1)$$

Thus, the right-hand side of (5) is equal to $f(\hat{x}+x_1)$.

Besides, in the previous exercise, we showed that

$$\exp\left(\frac{i\hat{p}x_1}{\hbar}\right)=\left(\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)\right)^\dagger$$

Thus, the left-hand side of (5) is written as

$$\left(\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)\right)^\dagger f(\hat{x})\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)=\hat{T}_{x_1}{}^\dagger f(\hat{x})\hat{T}_{x_1}$$

Therefore, (5) is written as

$$\hat{T}_{x_1}^{\dagger} f(\hat{x}) \hat{T}_{x_1} = f(\hat{x} + x_1) \tag{7}$$

iv) Denoting by $\langle \hat{O} \rangle_{|\psi\rangle}$ and $\langle \hat{O} \rangle_{\hat{T}_{x_1}|\psi\rangle}$, respectively, the expectation values of an operator $\hat{O}$ in the untranslated state $|\psi\rangle$ and in the translated state $\hat{T}_{x_1}|\psi\rangle$, we have

$$\langle x \rangle_{\hat{T}_{x_1}|\psi\rangle} = \left(\hat{T}_{x_1}|\psi\rangle, \hat{x}\hat{T}_{x_1}|\psi\rangle\right) = \left(|\psi\rangle, \hat{T}_{x_1}^{\dagger}\hat{x}\hat{T}_{x_1}|\psi\rangle\right),$$

where, for more clarity, we use the general notation for inner products, since the spatial translation operators are non-Hermitian.

Using (7), we have

$$\hat{T}_{x_1}^{\dagger}\hat{x}\hat{T}_{x_1} = \hat{x} + x_1,$$

since, in this case, $f(\hat{x}) = \hat{x}$.

Thus, the position expectation value in the translated state is written as

$$\langle x \rangle_{\hat{T}_{x_1}|\psi\rangle} = \left(|\psi\rangle, (\hat{x} + x_1)|\psi\rangle\right) = \underbrace{\left(|\psi\rangle, \hat{x}|\psi\rangle\right)}_{\langle x \rangle_{|\psi\rangle}} + \left(|\psi\rangle, x_1|\psi\rangle\right) =$$

$$= \langle x \rangle_{|\psi\rangle} + x_1 \underbrace{\left(|\psi\rangle, |\psi\rangle\right)}_{1} = \langle x \rangle_{|\psi\rangle} + x_1$$

We assume that the initial, untranslated state $|\psi\rangle$ is normalized.

That is

$$\langle x \rangle_{\hat{T}_{x_1}|\psi\rangle} = \langle x \rangle_{|\psi\rangle} + x_1 \tag{8}$$

To calculate the position expectation value (8), we didn't use that the state $|\psi\rangle$ is a QHO state, we only used that it is bound, so that its norm is 1.

Thus, the relation (8) holds for every bound, and normalized, state of any one-dimensional quantum system.

The momentum expectation value in the translated state is written as

$$\langle p \rangle_{\hat{T}_{x_1}|\psi\rangle} = \left(\hat{T}_{x_1}|\psi\rangle, \hat{p}\hat{T}_{x_1}|\psi\rangle\right) = \left(|\psi\rangle, \hat{T}_{x_1}^{\dagger}\hat{p}\hat{T}_{x_1}|\psi\rangle\right)$$

Since the spatial translation operators depend only on the momentum operator, they commute with the momentum operator, and with every function of the momentum operator, i.e. if $g(\hat{p})$ is a function of the momentum operator, then

$$\left[g(\hat{p}),\hat{T}_{x_1}\right]=0 \qquad (9)$$

Thus

$$\hat{T}_{x_1}{}^{\dagger}\hat{p}\hat{T}_{x_1}=\underbrace{\hat{T}_{x_1}{}^{\dagger}\hat{T}_{x_1}}_{1}\hat{p}=\hat{p}$$

Then, the momentum expectation value in the translated state is written as

$$\langle p\rangle_{\hat{T}_{x_1}|\psi\rangle}=\left(|\psi\rangle,\hat{p}|\psi\rangle\right)=\langle p\rangle_{|\psi\rangle}$$

That is

$$\langle p\rangle_{\hat{T}_{x_1}|\psi\rangle}=\langle p\rangle_{|\psi\rangle} \qquad (10)$$

As in the case of the relation (8), the relation (10) also holds for every bound, and normalized, state of any one-dimensional quantum system.
The energy expectation value in the translated state is written as

$$\langle E\rangle_{\hat{T}_{x_1}|\psi\rangle}=\left(\hat{T}_{x_1}|\psi\rangle,\hat{H}\hat{T}_{x_1}|\psi\rangle\right)=\left(|\psi\rangle,\hat{T}_{x_1}{}^{\dagger}\hat{H}\hat{T}_{x_1}|\psi\rangle\right)$$

The Hamiltonian of the QHO is

$$\hat{H}=\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2$$

Thus, we have

$$\hat{T}_{x_1}{}^{\dagger}\hat{H}\hat{T}_{x_1}=\hat{T}_{x_1}{}^{\dagger}\left(\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2\right)\hat{T}_{x_1}=\frac{1}{2m}\hat{T}_{x_1}{}^{\dagger}\hat{p}^2\hat{T}_{x_1}+\frac{1}{2}m\omega^2\hat{T}_{x_1}{}^{\dagger}\hat{x}^2\hat{T}_{x_1}$$

Using (7) and that the momentum operator squared commutes with $\hat{T}_{x_1}$, we obtain

$$\hat{T}_{x_1}{}^{\dagger}\hat{H}\hat{T}_{x_1} = \frac{1}{2m}\underbrace{\hat{T}_{x_1}{}^{\dagger}\hat{T}_{x_1}}_{1}\hat{p}^2 + \frac{1}{2}m\omega^2\underbrace{\hat{T}_{x_1}{}^{\dagger}\hat{x}^2\hat{T}_{x_1}}_{(\hat{x}+x_1)^2} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\left(\hat{x}+x_1\right)^2 =$$

$$= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\left(\hat{x}^2 + 2x_1\hat{x} + x_1{}^2\right)^2 = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + m\omega^2 x_1\hat{x} + \frac{1}{2}m\omega^2 x_1{}^2 =$$

$$= \hat{H} + m\omega^2 x_1\hat{x} + \frac{1}{2}m\omega^2 x_1{}^2$$

That is

$$\hat{T}_{x_1}{}^{\dagger}\hat{H}\hat{T}_{x_1} = \hat{H} + m\omega^2 x_1\hat{x} + \frac{1}{2}m\omega^2 x_1{}^2$$

Then, the energy expectation value in the translated state is written as

$$\langle E\rangle_{\hat{T}_{x_1}|\psi\rangle} = \left(|\psi\rangle, \left(\hat{H} + m\omega^2 x_1\hat{x} + \frac{1}{2}m\omega^2 x_1{}^2\right)|\psi\rangle\right) = \underbrace{\left(|\psi\rangle, \hat{H}|\psi\rangle\right)}_{\langle E\rangle_{|\psi\rangle}} + m\omega^2 x_1\underbrace{\left(|\psi\rangle, \hat{x}|\psi\rangle\right)}_{\langle x\rangle_{|\psi\rangle}} +$$

$$+ \frac{1}{2}m\omega^2 x_1{}^2\underbrace{\left(|\psi\rangle, |\psi\rangle\right)}_{1} = \langle E\rangle_{|\psi\rangle} + m\omega^2 x_1\langle x\rangle_{|\psi\rangle} + \frac{1}{2}m\omega^2 x_1{}^2$$

That is

$$\langle E\rangle_{\hat{T}_{x_1}|\psi\rangle} = \langle E\rangle_{|\psi\rangle} + m\omega^2 x_1\langle x\rangle_{|\psi\rangle} + \frac{1}{2}m\omega^2 x_1{}^2 \tag{11}$$

The relation (11) holds for every state of the QHO.

If the untranslated state $|\psi\rangle$ is an energy eigenstate $|n\rangle$, then $\langle x\rangle_{|\psi\rangle} = 0$ and $\langle E\rangle_{|\psi\rangle} = E_n = \left(n + \frac{1}{2}\right)\hbar\omega$. Then, (11) becomes

$$\langle E\rangle_{\hat{T}_{x_1}|\psi\rangle} = \left(n + \frac{1}{2}\right)\hbar\omega + \frac{1}{2}m\omega^2 x_1{}^2 \tag{12}$$

For $n = 0$ (ground state), (12) gives (14) or (15) of the previous exercise, as it should do.

The position uncertainty in the translated state is

$$(\Delta x)_{\hat{T}_{x_1}|\psi\rangle} = \sqrt{\langle x^2\rangle_{\hat{T}_{x_1}|\psi\rangle} - \left(\langle x\rangle_{\hat{T}_{x_1}|\psi\rangle}\right)^2} \tag{13}$$

The expectation value of the position operator squared in the translated state is

$$\left\langle x^2\right\rangle_{\hat{T}_{x_1}|\psi\rangle} = \left(\hat{T}_{x_1}|\psi\rangle, \hat{x}^2\hat{T}_{x_1}|\psi\rangle\right) = \left(|\psi\rangle, \hat{T}_{x_1}{}^{\dagger}\hat{x}^2\hat{T}_{x_1}|\psi\rangle\right)$$

Using (7), we obtain

$$\hat{T}_{x_1}{}^{\dagger}\hat{x}^2\hat{T}_{x_1} = \left(\hat{x}+x_1\right)^2 = \hat{x}^2 + 2x_1\hat{x} + x_1{}^2$$

Thus

$$\left\langle x^2\right\rangle_{\hat{T}_{x_1}|\psi\rangle} = \left(|\psi\rangle, \left(\hat{x}^2 + 2x_1\hat{x} + x_1{}^2\right)|\psi\rangle\right) = \left(|\psi\rangle, \hat{x}^2|\psi\rangle\right) + 2x_1\left(|\psi\rangle, \hat{x}|\psi\rangle\right) + x_1{}^2\left(|\psi\rangle, |\psi\rangle\right) =$$
$$= \left\langle x^2\right\rangle_{|\psi\rangle} + 2x_1\left\langle x\right\rangle_{|\psi\rangle} + x_1{}^2$$

That is

$$\left\langle x^2\right\rangle_{\hat{T}_{x_1}|\psi\rangle} = \left\langle x^2\right\rangle_{|\psi\rangle} + 2x_1\left\langle x\right\rangle_{|\psi\rangle} + x_1{}^2 \tag{14}$$

Substituting (14) and (8) into (13), we obtain

$$\left(\Delta x\right)_{\hat{T}_{x_1}|\psi\rangle} = \sqrt{\left\langle x^2\right\rangle_{|\psi\rangle} + 2x_1\left\langle x\right\rangle_{|\psi\rangle} + x_1{}^2 - \left(\left\langle x\right\rangle_{|\psi\rangle} + x_1\right)^2} =$$
$$= \sqrt{\left\langle x^2\right\rangle_{|\psi\rangle} + 2x_1\left\langle x\right\rangle_{|\psi\rangle} + x_1{}^2 - \left(\left(\left\langle x\right\rangle_{|\psi\rangle}\right)^2 + 2x_1\left\langle x\right\rangle_{|\psi\rangle} + x_1{}^2\right)} =$$
$$= \sqrt{\left\langle x^2\right\rangle_{|\psi\rangle} - \left(\left\langle x\right\rangle_{|\psi\rangle}\right)^2} = \left(\Delta x\right)_{|\psi\rangle}$$

That is

$$\left(\Delta x\right)_{\hat{T}_{x_1}|\psi\rangle} = \left(\Delta x\right)_{|\psi\rangle} \tag{15}$$

Thus, the position uncertainty does not change, and this holds for every bound, and normalized, state of any one-dimensional quantum system.

Similarly, the momentum uncertainty in the translated state is

$$\left(\Delta p\right)_{\hat{T}_{x_1}|\psi\rangle} = \sqrt{\left\langle p^2\right\rangle_{\hat{T}_{x_1}|\psi\rangle} - \left(\left\langle p\right\rangle_{\hat{T}_{x_1}|\psi\rangle}\right)^2} \tag{16}$$

The expectation value of the momentum operator squared in the translated state is

$$\left\langle p^2\right\rangle_{\hat{T}_{x_1}|\psi\rangle} = \left(\hat{T}_{x_1}|\psi\rangle, \hat{p}^2\hat{T}_{x_1}|\psi\rangle\right) = \left(|\psi\rangle, \hat{T}_{x_1}{}^{\dagger}\hat{p}^2\hat{T}_{x_1}|\psi\rangle\right)$$

Since the momentum operator squared commutes with $\hat{T}_{x_1}$,

$$\hat{T}_{x_1}^{\dagger}\hat{p}^2\hat{T}_{x_1} = \underbrace{\hat{T}_{x_1}^{\dagger}\hat{T}_{x_1}}_{1}\hat{p}^2 = \hat{p}^2$$

Thus

$$\left\langle p^2 \right\rangle_{\hat{T}_{x_1}|\psi\rangle} = \left(|\psi\rangle, \hat{p}^2|\psi\rangle\right) = \left\langle p^2 \right\rangle_{|\psi\rangle}$$

That is

$$\left\langle p^2 \right\rangle_{\hat{T}_{x_1}|\psi\rangle} = \left\langle p^2 \right\rangle_{|\psi\rangle} \qquad (17)$$

Substituting (17) and (10) into (16), we obtain

$$(\Delta p)_{\hat{T}_{x_1}|\psi\rangle} = (\Delta p)_{|\psi\rangle} \qquad (18)$$

Thus, the momentum uncertainty does not change, and this also holds for every bound, and normalized, state of any one-dimensional quantum system.
From (15) and (18), we derive that the position-momentum uncertainty product is the same in the translated and untranslated state, and this also holds for every bound, and normalized, state of any one-dimensional quantum system.

Momentum translation operators

3) Similar to the case of spatial translation operators, a momentum translation operator $\hat{T}_{p_1}$ is defined as $\hat{T}_{p_1} \equiv \exp\left(\frac{ip_1\hat{x}}{\hbar}\right)$, where $\hat{x}$ is the position operator and p_1 is an eigenvalue of the momentum operator.

i) Show that the operator $\hat{T}_{p_1}$ is unitary.

ii) Show that $\left[\hat{p}, \hat{T}_{p_1}\right] = p_1\hat{T}_{p_1}$.

iii) Show that the action of $\hat{T}_{p_1}$ on a momentum eigenstate yields a momentum eigenstate with eigenvalue translated by p_1, i.e. $\hat{T}_{p_1}|p\rangle = |p + p_1\rangle$.

iv) Using the Baker-Campbell-Hausdorff formula show that, for an infinitely many times differentiable function f, $\hat{T}_{p_1}^{\dagger} f(\hat{p})\hat{T}_{p_1} = f(\hat{p} + p_1)$.

v) A QHO is in the state $\hat{T}_{p_1}|\psi\rangle$, with $|\psi\rangle$ being an arbitrary state of the QHO. Using the previous property of $\hat{T}_{p_1}$, express the expectation values of the *position, momentum, and energy, as well as the position and momentum uncertainties, and the position-momentum uncertainty product, in the translated state $\hat{T}_{p_1}|\psi\rangle$ in terms of the respective quantities in the untranslated state $|\psi\rangle$.*

Solution

i) Using the Taylor expansion of $\exp\left(\frac{ip_1\hat{x}}{\hbar}\right)$, the Hermitian conjugate of $\hat{T}_{p_1}$ is written as

$$\hat{T}_{p_1}^{\dagger}=\left(\exp\left(\frac{ip_1\hat{x}}{\hbar}\right)\right)^{\dagger}=\left(\sum_{n=0}^{\infty}\frac{\left(\frac{ip_1\hat{x}}{\hbar}\right)^{n}}{n!}\right)^{\dagger}=\sum_{n=0}^{\infty}\left(\frac{\left(\frac{ip_1\hat{x}}{\hbar}\right)^{n}}{n!}\right)^{\dagger}=\sum_{n=0}^{\infty}\frac{\left(\left(\frac{ip_1\hat{x}}{\hbar}\right)^{n}\right)^{\dagger}}{n!}$$

In the same way we proved the relation (4) of the exercise 1, we prove that

$$\left(\left(\frac{ip_1\hat{x}}{\hbar}\right)^{n}\right)^{\dagger}=\left(\left(\frac{ip_1\hat{x}}{\hbar}\right)^{\dagger}\right)^{n}$$

Then, using also that the position operator is Hermitian, we obtain

$$\hat{T}_{p_1}^{\dagger}=\sum_{n=0}^{\infty}\frac{\left(\left(\frac{ip_1\hat{x}}{\hbar}\right)^{\dagger}\right)^{n}}{n!}=\sum_{n=0}^{\infty}\frac{\left(-\frac{ip_1\hat{x}}{\hbar}\right)^{n}}{n!}=\exp\left(-\frac{ip_1\hat{x}}{\hbar}\right)$$

That is

$$\hat{T}_{p_1}^{\dagger}=\exp\left(-\frac{ip_1\hat{x}}{\hbar}\right) \qquad (1)$$

Then, the product $\hat{T}_{p_1}^{\dagger}\hat{T}_{p_1}$ is written as

$$\hat{T}_{p_1}^{\dagger}\hat{T}_{p_1}=\exp\left(-\frac{ip_1\hat{x}}{\hbar}\right)\exp\left(\frac{ip_1\hat{x}}{\hbar}\right)$$

Since the commutator of $-\frac{ip_1\hat{x}}{\hbar}$ and $\frac{ip_1\hat{x}}{\hbar}$ is zero,

$$\exp\left(-\frac{ip_1\hat{x}}{\hbar}\right)\exp\left(\frac{ip_1\hat{x}}{\hbar}\right)=\exp\left(-\frac{ip_1\hat{x}}{\hbar}+\frac{ip_1\hat{x}}{\hbar}\right)=\exp 0=1$$

That is

$$\hat{T}_{p_1}{}^{\dagger}\hat{T}_{p_1}=1$$

In the same way, we have

$$\hat{T}_{p_1}\hat{T}_{p_1}{}^{\dagger}=\exp\left(\frac{ip_1\hat{x}}{\hbar}\right)\exp\left(-\frac{ip_1\hat{x}}{\hbar}\right)=\exp\left(\frac{ip_1\hat{x}}{\hbar}-\frac{ip_1\hat{x}}{\hbar}\right)=1$$

Thus

$$\hat{T}_{p_1}{}^{\dagger}\hat{T}_{p_1}=\hat{T}_{p_1}\hat{T}_{p_1}{}^{\dagger}=1$$

Therefore, $\hat{T}_{p_1}$ is unitary.

ii) Using again the Taylor expansion of $\exp\left(\frac{ip_1\hat{x}}{\hbar}\right)$, the commutator $\left[\hat{p},\hat{T}_{p_1}\right]$ is written as

$$\left[\hat{p},\hat{T}_{p_1}\right]=\left[\hat{p},\sum_{n=0}^{\infty}\frac{\left(\frac{ip_1\hat{x}}{\hbar}\right)^n}{n!}\right]=\sum_{n=0}^{\infty}\left[\hat{p},\frac{\left(\frac{ip_1\hat{x}}{\hbar}\right)^n}{n!}\right]=\sum_{n=0}^{\infty}\frac{1}{n!}\left[\hat{p},\left(\frac{ip_1\hat{x}}{\hbar}\right)^n\right]$$

Using that the position operator commutes with the number $\frac{ip_1}{\hbar}$ (remember that p_1 is a number), we obtain

$$\left[\hat{p},\hat{T}_{p_1}\right]=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{ip_1}{\hbar}\right)^n\left[\hat{p},\hat{x}^n\right]$$

In the previous exercise, we showed that the commutator of the momentum operator with a differentiable function f of the position operator is

$$\left[\hat{p},f\left(\hat{x}\right)\right]=-i\hbar f'\left(\hat{x}\right)$$

Thus

$$\left[\hat{p},\hat{x}^n\right]=-i\hbar n\hat{x}^{n-1}$$

with $n\geq 1$. If $n=0$, the commutator is zero.

Then, the commutator $\left[\hat{p},\hat{T}_{p_1}\right]$ is written as

$$\left[\hat{p},\hat{T}_{p_1}\right]=\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{ip_1}{\hbar}\right)^{n}\left(-i\hbar n\hat{x}^{n-1}\right)=\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\left(\frac{ip_1}{\hbar}\right)^{n-1}\left(\frac{ip_1}{\hbar}\right)(-i\hbar)\,\hat{x}^{n-1}=$$

$$=\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\left(\frac{ip_1}{\hbar}\right)^{n-1}p_1\hat{x}^{n-1}=p_1\sum_{n=1}^{\infty}\frac{\left(\frac{ip_1\hat{x}}{\hbar}\right)^{n-1}}{(n-1)!}\underbrace{=}_{n'=n-1}p_1\sum_{n'=0}^{\infty}\frac{\left(\frac{ip_1\hat{x}}{\hbar}\right)^{n'}}{n'!}=$$

$$=p_1\exp\left(\frac{ip_1\hat{x}}{\hbar}\right)=p_1\hat{T}_{p_1}$$

That is

$$\left[\hat{p},\hat{T}_{p_1}\right]=p_1\hat{T}_{p_1} \qquad (2)$$

iii) Using the previous commutator, we obtain

$$\hat{p}\hat{T}_{p_1}-\hat{T}_{p_1}\hat{p}=p_1\hat{T}_{p_1}\Rightarrow\hat{p}\hat{T}_{p_1}=\hat{T}_{p_1}\hat{p}+p_1\hat{T}_{p_1}$$

Thus, the action of $\hat{p}\hat{T}_{p_1}$ on a momentum eigenstate $|p\rangle$ yields

$$\hat{p}\hat{T}_{p_1}|p\rangle=\left(\hat{T}_{p_1}\hat{p}+p_1\hat{T}_{p_1}\right)|p\rangle=\hat{T}_{p_1}\hat{p}|p\rangle+p_1\hat{T}_{p_1}|p\rangle=\hat{T}_{p_1}p|p\rangle+p_1\hat{T}_{p_1}|p\rangle$$

where p is the eigenvalue of the momentum eigenstate $|p\rangle$, i.e. it is a number, and thus it commutes with $\hat{T}_{p_1}$.

Then

$$\hat{p}\hat{T}_{p_1}|p\rangle=p\hat{T}_{p_1}|p\rangle+p_1\hat{T}_{p_1}|p\rangle=(p+p_1)\hat{T}_{p_1}|p\rangle$$

That is

$$\hat{p}\hat{T}_{p_1}|p\rangle=(p+p_1)\hat{T}_{p_1}|p\rangle$$

The state $\hat{T}_{p_1}|p\rangle$ is thus a momentum eigenstate with eigenvalue $p+p_1$.

Therefore

$$\hat{T}_{p_1}|p\rangle=B|p+p_1\rangle,$$

where B is a complex constant.

Since $\hat{T}_{p_1}$ is a unitary operator, it preserves the norms of the states on which it acts. Therefore, although the momentum eigenstates are not bound, we can assume that

$$|B|=1$$

Then, omitting the physically unimportant constant phase of B, we obtain

$$\hat{T}_{p_1}|p\rangle = |p+p_1\rangle \tag{3}$$

iv) For $\hat{A}=-\frac{ip_1\hat{x}}{\hbar}$ and $\hat{B}=f(\hat{p})$, the Baker-Campbell-Hausdorff formula,

$$\exp(\hat{A})\hat{B}\exp(-\hat{A})=\hat{B}+[\hat{A},\hat{B}]+\frac{1}{2}[\hat{A},[\hat{A},\hat{B}]]+\frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]]+...+\frac{1}{n!}\underbrace{[\hat{A},[\hat{A},...[\hat{A},\hat{B}]...]]}_{n\ \hat{A}'s\ \text{or}\ n\ \text{commutators}},$$

is written as

$$\exp\left(-\frac{ip_1\hat{x}}{\hbar}\right)f(\hat{p})\exp\left(\frac{ip_1\hat{x}}{\hbar}\right)=f(\hat{p})+\left[-\frac{ip_1\hat{x}}{\hbar},f(\hat{p})\right]+\frac{1}{2}\left[-\frac{ip_1\hat{x}}{\hbar},\left[-\frac{ip_1\hat{x}}{\hbar},f(\hat{p})\right]\right]+$$
$$+\frac{1}{3!}\left[-\frac{ip_1\hat{x}}{\hbar},\left[-\frac{ip_1\hat{x}}{\hbar},\left[-\frac{ip_1\hat{x}}{\hbar},f(\hat{p})\right]\right]\right]+...+\frac{1}{n!}\left[-\frac{ip_1\hat{x}}{\hbar},\left[-\frac{ip_1\hat{x}}{\hbar},...\left[-\frac{ip_1\hat{x}}{\hbar},f(\hat{p})\right]...\right]\right] \tag{4}$$

We'll use the momentum representation to calculate the commutator $[\hat{x},f(\hat{p})]$, since in the momentum representation, $\hat{x}=i\hbar\frac{d}{dp}$ and $\hat{p}=p$, and thus the operator function $f(\hat{p})$ becomes a usual, and easy-to-handle, scalar function.

Thus, choosing an arbitrary wave function $\phi(p)$ (in the momentum representation), we have

$$\left[i\hbar\frac{d}{dp},f(p)\right]\phi(p)=\left(i\hbar\frac{d}{dp}f(p)-f(p)i\hbar\frac{d}{dp}\right)\phi(p)=i\hbar\left(\frac{d}{dp}\big(f(p)\phi(p)\big)-f(p)\frac{d\phi(p)}{dp}\right)=$$
$$=i\hbar\big(f'(p)\phi(p)+f(p)\phi'(p)-f(p)\phi'(p)\big)=i\hbar f'(p)\phi(p)$$

Here, the prime denotes differentiation with respect to p.

That is

$$\left[i\hbar\frac{d}{dp},f(p)\right]\phi(p)=i\hbar f'(p)\phi(p)$$

Since the wave function $\phi(p)$ is arbitrary,

$$\left[i\hbar\frac{d}{dp},f(p)\right]=i\hbar f'(p)$$

or, in representation-free form,

$$\left[\hat{x},f(\hat{p})\right]=i\hbar f'(\hat{p}) \tag{5}$$

Using (5), we have

$$\left[-\frac{ip_1\hat{x}}{\hbar},f(\hat{p})\right]=-\frac{ip_1}{\hbar}\left[\hat{x},f(\hat{p})\right]=-\frac{ip_1}{\hbar}i\hbar f'(\hat{p})=p_1f'(\hat{p})$$

$$\left[-\frac{ip_1\hat{x}}{\hbar},\left[-\frac{ip_1\hat{x}}{\hbar},f(\hat{p})\right]\right]=\left[-\frac{ip_1\hat{x}}{\hbar},p_1f'(\hat{p})\right]=-\frac{ip_1^{\,2}}{\hbar}\left[\hat{x},f'(\hat{p})\right]=$$
$$=-\frac{ip_1^{\,2}}{\hbar}i\hbar f''(\hat{p})=p_1^{\,2}f''(\hat{p})$$

Assuming that

$$\underbrace{\left[-\frac{ip_1\hat{x}}{\hbar},\left[-\frac{ip_1\hat{x}}{\hbar},\ldots\left[-\frac{ip_1\hat{x}}{\hbar},f(\hat{p})\right]\ldots\right]\right]}_{k\text{ commutators}}=p_1^{\,k}f^{(k)}(\hat{p})$$

then

$$\underbrace{\left[-\frac{ip_1\hat{x}}{\hbar},\left[-\frac{ip_1\hat{x}}{\hbar},\ldots\left[-\frac{ip_1\hat{x}}{\hbar},f(\hat{p})\right]\ldots\right]\right]}_{(k+1)\text{ commutators}}=\left[-\frac{ip_1\hat{x}}{\hbar},\underbrace{\left[-\frac{ip_1\hat{x}}{\hbar},\ldots\left[-\frac{ip_1\hat{x}}{\hbar},f(\hat{p})\right]\ldots\right]}_{k\text{ commutators}}\right]=$$
$$=\left[-\frac{ip_1\hat{x}}{\hbar},p_1^{\,k}f^{(k)}(\hat{p})\right]=-\frac{ip_1^{\,k+1}}{\hbar}\left[\hat{x},f^{(k)}(\hat{p})\right]=-\frac{ip_1^{\,k+1}}{\hbar}i\hbar f^{(k+1)}(\hat{p})=p_1^{\,k+1}f^{(k+1)}(\hat{p})$$

Thus

$$\underbrace{\left[-\frac{ip_1\hat{x}}{\hbar},\left[-\frac{ip_1\hat{x}}{\hbar},\ldots\left[-\frac{ip_1\hat{x}}{\hbar},f(\hat{p})\right]\ldots\right]\right]}_{n\text{ commutators}}=p_1^{\,n}f^{(n)}(\hat{p}),$$

for every $n=1,2,\ldots$

Then, the right-hand side of (4) becomes

$$f(\hat{p}) + p_1 f'(\hat{p}) + \frac{1}{2} p_1^{\,2} f''(\hat{p}) + \frac{1}{3!} p_1^{\,3} f^{(3)}(\hat{p}) + \ldots + \frac{1}{n!} p_1^{\,n} f^{(n)}(\hat{p}) =$$

$$= f(\hat{p}) + f'(\hat{p}) p_1 + \frac{f''(\hat{p})}{2} p_1^{\,2} + \frac{f^{(3)}(\hat{p})}{3!} p_1^{\,3} + \ldots + \frac{f^{(n)}(\hat{p})}{n!} p_1^{\,n} = \sum_{n=0}^{\infty} \frac{f^{(n)}(\hat{p})}{n!} p_1^{\,n}$$

The series $\sum_{n=0}^{\infty} \frac{f^{(n)}(\hat{p})}{n!} p_1^{\,n}$ is the Taylor expansion of $f(\hat{p} + p_1)$ about $\hat{p}$, i.e.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\hat{p})}{n!} p_1^{\,n} = f(\hat{p} + p_1)$$

Thus, the right-hand side of (4) is equal to $f(\hat{p} + p_1)$.

Using (1), the left-hand side of (4), i.e. the term $\exp\left(-\frac{ip_1\hat{x}}{\hbar}\right) f(\hat{p}) \exp\left(\frac{ip_1\hat{x}}{\hbar}\right)$, is written as $\hat{T}_{p_1}^{\,\dagger} f(\hat{p}) \hat{T}_{p_1}$.

Therefore, (4) is written as

$$\hat{T}_{p_1}^{\,\dagger} f(\hat{p}) \hat{T}_{p_1} = f(\hat{p} + p_1) \tag{6}$$

v) Using again, for more clarity, the general notation for inner products, the position expectation value in the translated state $\hat{T}_{p_1} |\psi\rangle$ is written as

$$\langle x \rangle_{\hat{T}_{p_1}|\psi\rangle} = \left(\hat{T}_{p_1} |\psi\rangle, \hat{x} \hat{T}_{p_1} |\psi\rangle\right) = \left(|\psi\rangle, \hat{T}_{p_1}^{\,\dagger} \hat{x} \hat{T}_{p_1} |\psi\rangle\right)$$

The momentum translation operators depend only on the position operator, and thus they commute with the position operator, and with every function of the position operator, i.e. if $g(\hat{x})$ is a function of the position operator, then

$$\left[g(\hat{x}), \hat{T}_{p_1}\right] = 0 \tag{7}$$

Thus

$$\hat{T}_{p_1}^{\,\dagger} \hat{x} \hat{T}_{p_1} = \underbrace{\hat{T}_{p_1}^{\,\dagger} \hat{T}_{p_1}}_{1} \hat{x} = \hat{x}$$

Then, the position expectation value in the translated state is written as

$$\langle x \rangle_{\hat{T}_{p_1}|\psi\rangle} = \left(|\psi\rangle, \hat{x} |\psi\rangle\right) = \langle x \rangle_{|\psi\rangle}$$

That is

$$\langle x \rangle_{\hat{T}_{p_1}|\psi\rangle} = \langle x \rangle_{|\psi\rangle} \tag{8}$$

Since we didn't make use of the fact that the state $|\psi\rangle$ is a QHO state, the relation (8) holds for every bound, and normalized, state of any one-dimensional quantum system.
The momentum expectation value in the translated state is written as

$$\langle p \rangle_{\hat{T}_{p_1}|\psi\rangle} = \left(\hat{T}_{p_1}|\psi\rangle, \hat{p}\hat{T}_{p_1}|\psi\rangle\right) = \left(|\psi\rangle, \hat{T}_{p_1}{}^{\dagger}\hat{p}\hat{T}_{p_1}|\psi\rangle\right)$$

Using (6), we obtain

$$\hat{T}_{p_1}{}^{\dagger}\hat{p}\hat{T}_{p_1} = \hat{p} + p_1$$

Thus

$$\langle p \rangle_{\hat{T}_{p_1}|\psi\rangle} = \left(|\psi\rangle, (\hat{p} + p_1)|\psi\rangle\right) = \left(|\psi\rangle, \hat{p}|\psi\rangle\right) + p_1\left(|\psi\rangle, |\psi\rangle\right) = \langle p \rangle_{|\psi\rangle} + p_1$$

That is

$$\langle p \rangle_{\hat{T}_{p_1}|\psi\rangle} = \langle p \rangle_{|\psi\rangle} + p_1 \tag{9}$$

The relation (9) also holds for every bound, and normalized, state of any one-dimensional quantum system.
As we did in the case of the spatial translation – see the previous exercise – we can directly calculate the energy expectation value in the translated state. Alternatively, we can calculate first the expectation values of the position squared and momentum squared, since we'll need them to calculate the respective uncertainties, and from them we'll derive the energy expectation value.
The expectation value of the position squared in the translated state is written as

$$\langle x^2 \rangle_{\hat{T}_{p_1}|\psi\rangle} = \left(\hat{T}_{p_1}|\psi\rangle, \hat{x}^2\hat{T}_{p_1}|\psi\rangle\right) = \left(|\psi\rangle, \hat{T}_{p_1}{}^{\dagger}\hat{x}^2\hat{T}_{p_1}|\psi\rangle\right)$$

Using (7), we obtain

$$\hat{T}_{p_1}{}^{\dagger}\hat{x}^2\hat{T}_{p_1} = \underbrace{\hat{T}_{p_1}{}^{\dagger}\hat{T}_{p_1}}_{1}\hat{x}^2 = \hat{x}^2$$

Thus

$$\left\langle x^2 \right\rangle_{\hat{T}_{p_1}|\psi\rangle} = \left(|\psi\rangle, \hat{x}^2 |\psi\rangle\right) = \left\langle x^2 \right\rangle_{|\psi\rangle}$$

That is

$$\left\langle x^2 \right\rangle_{\hat{T}_{p_1}|\psi\rangle} = \left\langle x^2 \right\rangle_{|\psi\rangle} \tag{10}$$

The relation (10) also holds for every bound, and normalized, state of any one-dimensional quantum system.

The expectation value of the momentum squared in the translated state is written as

$$\left\langle p^2 \right\rangle_{\hat{T}_{p_1}|\psi\rangle} = \left(\hat{T}_{p_1}|\psi\rangle, \hat{p}^2 \hat{T}_{p_1}|\psi\rangle\right) = \left(|\psi\rangle, \hat{T}_{p_1}{}^{\dagger} \hat{p}^2 \hat{T}_{p_1}|\psi\rangle\right)$$

Using (6), we obtain

$$\hat{T}_{p_1}{}^{\dagger} \hat{p}^2 \hat{T}_{p_1} = \left(\hat{p} + p_1\right)^2 = \hat{p}^2 + 2p_1\hat{p} + p_1{}^2$$

Thus

$$\left\langle p^2 \right\rangle_{\hat{T}_{p_1}|\psi\rangle} = \left(|\psi\rangle, \left(\hat{p}^2 + 2p_1\hat{p} + p_1{}^2\right)|\psi\rangle\right) = \left(|\psi\rangle, \hat{p}^2|\psi\rangle\right) + 2p_1\left(|\psi\rangle, \hat{p}|\psi\rangle\right) + p_1{}^2 \underbrace{\left(|\psi\rangle, |\psi\rangle\right)}_{1} =$$

$$= \left\langle p^2 \right\rangle_{|\psi\rangle} + 2p_1 \left\langle p \right\rangle_{|\psi\rangle} + p_1{}^2$$

We remind that we've assumed that the bound, and thus normalizable, state $|\psi\rangle$ is normalized.

That is

$$\left\langle p^2 \right\rangle_{\hat{T}_{p_1}|\psi\rangle} = \left\langle p^2 \right\rangle_{|\psi\rangle} + 2p_1 \left\langle p \right\rangle_{|\psi\rangle} + p_1{}^2 \tag{11}$$

The relation (11) also holds for every bound, and normalized, state of any one-dimensional quantum system.

Using the Hamiltonian of the QHO, the energy expectation value of the QHO in the translated state is written as

$$\left\langle E \right\rangle_{\hat{T}_{p_1}|\psi\rangle} = \left\langle \hat{H} \right\rangle_{\hat{T}_{p_1}|\psi\rangle} = \left\langle \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \right\rangle_{\hat{T}_{p_1}|\psi\rangle} = \frac{\left\langle p^2 \right\rangle_{\hat{T}_{p_1}|\psi\rangle}}{2m} + \frac{1}{2}m\omega^2 \left\langle x^2 \right\rangle_{\hat{T}_{p_1}|\psi\rangle}$$

Substituting (10) and (11) yields

$$\langle E \rangle_{\hat{T}_{p1}|\psi\rangle} = \frac{\langle p^2 \rangle_{|\psi\rangle} + 2p_1 \langle p \rangle_{|\psi\rangle} + p_1^{\ 2}}{2m} + \frac{1}{2} m\omega^2 \langle x^2 \rangle_{|\psi\rangle} = \underbrace{\frac{\langle p^2 \rangle_{|\psi\rangle}}{2m} + \frac{1}{2} m\omega^2 \langle x^2 \rangle_{|\psi\rangle}}_{\langle E \rangle_{|\psi\rangle}} + \frac{p_1}{m} \langle p \rangle_{|\psi\rangle} +$$

$$+ \frac{p_1^{\ 2}}{2m} = \langle E \rangle_{|\psi\rangle} + \frac{p_1}{m} \langle p \rangle_{|\psi\rangle} + \frac{p_1^{\ 2}}{2m}$$

That is

$$\langle E \rangle_{\hat{T}_{p1}|\psi\rangle} = \langle E \rangle_{|\psi\rangle} + \frac{p_1}{m} \langle p \rangle_{|\psi\rangle} + \frac{p_1^{\ 2}}{2m} \tag{12}$$

Obviously, the relation (12) holds only for the states of the QHO.

If the untranslated state $|\psi\rangle$ is an energy eigenstate $|n\rangle$, then $\langle p \rangle_{|\psi\rangle} = 0$ and $\langle E \rangle_{|\psi\rangle} = E_n = \left(n + \frac{1}{2}\right)\hbar\omega$. Then, (12) becomes

$$\langle E \rangle_{\hat{T}_{p1}|n\rangle} = \left(n + \frac{1}{2}\right)\hbar\omega + \frac{p_1^{\ 2}}{2m} \tag{13}$$

Using (8) and (10), we derive that the position uncertainty in the translated state is the same as in the untranslated state, i.e.

$$(\Delta x)_{\hat{T}_{p1}|\psi\rangle} = (\Delta x)_{|\psi\rangle} \tag{14}$$

The relation (14) also holds for every bound, and normalized, state of any one-dimensional quantum system.

Using (9) and (11), the momentum uncertainty in the translated state is written as

$$(\Delta p)_{\hat{T}_{p1}|\psi\rangle} = \sqrt{\langle p^2 \rangle_{\hat{T}_{p1}|\psi\rangle} - \left(\langle p \rangle_{\hat{T}_{p1}|\psi\rangle}\right)^2} = \sqrt{\langle p^2 \rangle_{|\psi\rangle} + 2p_1 \langle p \rangle_{|\psi\rangle} + p_1^{\ 2} - \left(\langle p \rangle_{|\psi\rangle} + p_1\right)^2} =$$

$$= \sqrt{\langle p^2 \rangle_{|\psi\rangle} + 2p_1 \langle p \rangle_{|\psi\rangle} + p_1^{\ 2} - \left(\left(\langle p \rangle_{|\psi\rangle}\right)^2 + 2p_1 \langle p \rangle_{|\psi\rangle} + p_1^{\ 2}\right)} =$$

$$= \sqrt{\langle p^2 \rangle_{|\psi\rangle} - \left(\langle p \rangle_{|\psi\rangle}\right)^2} = (\Delta p)_{|\psi\rangle}$$

That is

$$(\Delta p)_{\hat{T}_{p1}|\psi\rangle} = (\Delta p)_{|\psi\rangle} \tag{15}$$

The momentum uncertainty in the translated state is the same as the momentum uncertainty in the untranslated state.

The relation (15) also holds for every bound, and normalized, state of any one-dimensional quantum system.

From (14) and (15), we derive that the position-momentum uncertainty product in the translated state is the same as the position-momentum uncertainty product in the untranslated state, and this also holds for every bound, and normalized, state of any one-dimensional quantum system.

The combined action of a spatial and a momentum translation operator

4) Let $\hat{T}_{x_1}$ be a spatial translation operator, i.e. $\hat{T}_{x_1} \equiv \exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)$, and $\hat{T}_{p_1}$ be a momentum translation operator, i.e. $\hat{T}_{p_1} \equiv \exp\left(\frac{ip_1\hat{x}}{\hbar}\right)$.

We also define the operator $\hat{T}_{p_1,x_1} \equiv \exp\left(\frac{i\left(p_1\hat{x}-\hat{p}x_1\right)}{\hbar}\right)$.

Since $\exp\left(\frac{i\left(p_1\hat{x}-\hat{p}x_1\right)}{\hbar}\right) = \exp\left(\frac{i\left(-\hat{p}x_1+p_1\hat{x}\right)}{\hbar}\right)$, the order of the two operators doesn't matter, and $\hat{T}_{p_1,x_1} = \hat{T}_{x_1,p_1}$.

We'll show that the three operators $\hat{T}_{x_1}\hat{T}_{p_1}$, $\hat{T}_{p_1}\hat{T}_{x_1}$, and $\hat{T}_{p_1,x_1}$ differ only by a constant phase, and thus they are physically equivalent.

This means that their action on a physical state – not necessarily a QHO state – yields states that differ only by a constant phase, and thus they are physically equivalent, i.e. they are the same state.

Solution

The physical equivalence of the three operators follows from the fact that the commutator $\left[\frac{ip_1\hat{x}}{\hbar}, -\frac{i\hat{p}x_1}{\hbar}\right]$ is a constant.

Indeed

$$\left[\frac{ip_1\hat{x}}{\hbar}, -\frac{i\hat{p}x_1}{\hbar}\right] = \left(\frac{ip_1}{\hbar}\right)\left(-\frac{ix_1}{\hbar}\right)\left[\hat{x},\hat{p}\right] = \frac{p_1x_1}{\hbar^2}i\hbar = \frac{ix_1p_1}{\hbar}$$

That is

$$\left[\frac{ip_1\hat{x}}{\hbar},-\frac{i\hat{p}x_1}{\hbar}\right]=\frac{ix_1p_1}{\hbar} \tag{1}$$

Since the previous commutator is a constant, we can use the identity

$$\exp\left(\hat{A}+\hat{B}\right)=\exp\left(\hat{A}\right)\exp\left(\hat{B}\right)\exp\left(-\frac{1}{2}\left[\hat{A},\hat{B}\right]\right)$$

which holds if the operators $\hat{A}$ and $\hat{B}$ commute with their commutator $\left[\hat{A},\hat{B}\right]$.

In our case, the commutator $\left[\frac{ip_1\hat{x}}{\hbar},-\frac{i\hat{p}x_1}{\hbar}\right]$ is a constant, and thus it commutes with both $\frac{ip_1\hat{x}}{\hbar}$ and $-\frac{i\hat{p}x_1}{\hbar}$.

Then, using the previous identity and the commutator (1), we obtain

$$\exp\left(\frac{i\left(p_1\hat{x}-\hat{p}x_1\right)}{\hbar}\right)=\exp\left(\frac{ip_1\hat{x}}{\hbar}-\frac{i\hat{p}x_1}{\hbar}\right)=\exp\left(\frac{ip_1\hat{x}}{\hbar}\right)\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)\exp\left(-\frac{1}{2}\frac{ix_1p_1}{\hbar}\right)=$$

$$=\exp\left(\frac{ip_1\hat{x}}{\hbar}\right)\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)\exp\left(-\frac{ix_1p_1}{2\hbar}\right)$$

That is

$$\hat{T}_{p_1,x_1}=\hat{T}_{p_1}\hat{T}_{x_1}\exp\left(-\frac{ix_1p_1}{2\hbar}\right)$$

The exponential $\exp\left(-\frac{ix_1p_1}{2\hbar}\right)$ is a constant complex number, and thus it commutes with both $\hat{T}_{x_1}$ and $\hat{T}_{p_1}$.

Thus

$$\hat{T}_{p_1,x_1}=\exp\left(-\frac{ix_1p_1}{2\hbar}\right)\hat{T}_{p_1}\hat{T}_{x_1} \tag{2}$$

Therefore, the operators $\hat{T}_{p_1}\hat{T}_{x_1}$ and $\hat{T}_{p_1,x_1}$ differ only by a constant phase.

Besides, from (1) we obtain

$$\left[-\frac{i\hat{p}x_1}{\hbar},\frac{ip_1\hat{x}}{\hbar}\right]=-\left[\frac{ip_1\hat{x}}{\hbar},-\frac{i\hat{p}x_1}{\hbar}\right]=-\frac{ix_1p_1}{\hbar},$$

Thus, using again the previous identity, we obtain

$$\exp\left(-\frac{i\hat{p}x_1}{\hbar}+\frac{ip_1\hat{x}}{\hbar}\right)=\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)\exp\left(\frac{ip_1\hat{x}}{\hbar}\right)\exp\left(-\frac{1}{2}\left(-\frac{ix_1p_1}{\hbar}\right)\right)=$$
$$=\exp\left(-\frac{i\hat{p}x_1}{\hbar}\right)\exp\left(\frac{ip_1\hat{x}}{\hbar}\right)\exp\left(\frac{ix_1p_1}{2\hbar}\right)$$

That is

$$\exp\left(-\frac{i\hat{p}x_1}{\hbar}+\frac{ip_1\hat{x}}{\hbar}\right)=\hat{T}_{x_1}\hat{T}_{p_1}\exp\left(\frac{ix_1p_1}{2\hbar}\right)$$

The exponential $\exp\left(\frac{ix_1p_1}{2\hbar}\right)$ is a constant complex number, and thus it commutes with both $\hat{T}_{x_1}$ and $\hat{T}_{p_1}$.

Thus

$$\exp\left(-\frac{i\hat{p}x_1}{\hbar}+\frac{ip_1\hat{x}}{\hbar}\right)=\exp\left(\frac{ix_1p_1}{2\hbar}\right)\hat{T}_{x_1}\hat{T}_{p_1} \quad (3)$$

Also

$$\exp\left(-\frac{i\hat{p}x_1}{\hbar}+\frac{ip_1\hat{x}}{\hbar}\right)=\exp\left(\frac{ip_1\hat{x}}{\hbar}-\frac{i\hat{p}x_1}{\hbar}\right)=\exp\left(\frac{i\left(p_1\hat{x}-\hat{p}x_1\right)}{\hbar}\right)=\hat{T}_{p_1,x_1}$$

Thus, (3) is written as

$$\hat{T}_{p_1,x_1}=\exp\left(\frac{ix_1p_1}{2\hbar}\right)\hat{T}_{x_1}\hat{T}_{p_1} \quad (4)$$

Therefore, the operators $\hat{T}_{x_1}\hat{T}_{p_1}$ and $\hat{T}_{p_1,x_1}$ differ only by a constant phase.

Besides, comparing (2) and (4) yields

$$\exp\left(-\frac{ix_1p_1}{2\hbar}\right)\hat{T}_{p_1}\hat{T}_{x_1}=\exp\left(\frac{ix_1p_1}{2\hbar}\right)\hat{T}_{x_1}\hat{T}_{p_1}\Rightarrow$$
$$\Rightarrow\exp\left(\frac{ix_1p_1}{2\hbar}\right)\exp\left(-\frac{ix_1p_1}{2\hbar}\right)\hat{T}_{p_1}\hat{T}_{x_1}=\exp\left(\frac{ix_1p_1}{2\hbar}\right)\exp\left(\frac{ix_1p_1}{2\hbar}\right)\hat{T}_{x_1}\hat{T}_{p_1}\Rightarrow$$
$$\Rightarrow\hat{T}_{p_1}\hat{T}_{x_1}=\exp\left(\frac{ix_1p_1}{\hbar}\right)\hat{T}_{x_1}\hat{T}_{p_1}$$

That is

$$\hat{T}_{p_1}\hat{T}_{x_1}=\exp\left(\frac{ix_1p_1}{\hbar}\right)\hat{T}_{x_1}\hat{T}_{p_1} \quad (5)$$

Therefore, the operators $\hat{T}_{x_1}\hat{T}_{p_1}$ and $\hat{T}_{p_1}\hat{T}_{x_1}$ differ only by a constant phase.

This means that the spatial translations commute with the momentum translations, or, in other words, they are independent.

Returning back to the QHO – The displacement operator

5) For the QHO, express the operator $\hat{T}_{p_1,x_1} \equiv \exp\left(\dfrac{i\left(p_1\hat{x} - \hat{p}x_1\right)}{\hbar}\right)$ *in terms of the ladder operators and show that it is written as* $\exp\left(\lambda\hat{a}^\dagger - \lambda^*\hat{a}\right)$*, with* $\lambda = \dfrac{1}{\sqrt{2}}\left(\dfrac{x_1}{x_0} + i\dfrac{p_1}{p_0}\right)$*, where* $x_0 = \sqrt{\dfrac{\hbar}{m\omega}}$ *is the length scale and* $p_0 = \sqrt{m\hbar\omega}$ *is the momentum scale of the QHO.*

Solution

Solving the definition relations of the ladder operators for the position and momentum operators, we obtain

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right)$$

$$\hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}\left(\hat{a}^\dagger - \hat{a}\right)$$

Introducing into the previous relations the length and momentum scales of the QHO, we obtain

$$\hat{x} = \frac{x_0}{\sqrt{2}}\left(\hat{a} + \hat{a}^\dagger\right) \qquad (1)$$

$$\hat{p} = i\frac{p_0}{\sqrt{2}}\left(\hat{a}^\dagger - \hat{a}\right) \qquad (2)$$

By means of (1) and (2), the operator $\dfrac{i\left(p_1\hat{x} - \hat{p}x_1\right)}{\hbar}$ is written as

$$\frac{i\left(p_1\hat{x}-\hat{p}x_1\right)}{\hbar}=\frac{i\left(p_1\frac{x_0}{\sqrt{2}}\left(\hat{a}+\hat{a}^\dagger\right)-i\frac{p_0}{\sqrt{2}}\left(\hat{a}^\dagger-\hat{a}\right)x_1\right)}{\hbar}=\frac{1}{\sqrt{2}\hbar}\left(ix_0p_1\left(\hat{a}+\hat{a}^\dagger\right)+p_0x_1\left(\hat{a}^\dagger-\hat{a}\right)\right)=$$

$$=\frac{1}{\sqrt{2}\hbar}\left(ix_0p_1\hat{a}+ix_0p_1\hat{a}^\dagger+p_0x_1\hat{a}^\dagger-p_0x_1\hat{a}\right)=\frac{1}{\sqrt{2}\hbar}\left(\left(p_0x_1+ix_0p_1\right)\hat{a}^\dagger-\left(p_0x_1-ix_0p_1\right)\hat{a}\right)$$

That is

$$\frac{i\left(p_1\hat{x}-\hat{p}x_1\right)}{\hbar}=\frac{1}{\sqrt{2}\hbar}\left(\left(p_0x_1+ix_0p_1\right)\hat{a}^\dagger-\left(p_0x_1-ix_0p_1\right)\hat{a}\right)$$

Using that $x_0p_0=\hbar$, the previous equation is written as

$$\frac{i\left(p_1\hat{x}-\hat{p}x_1\right)}{\hbar}=\frac{1}{\sqrt{2}x_0p_0}\left(\left(p_0x_1+ix_0p_1\right)\hat{a}^\dagger-\left(p_0x_1-ix_0p_1\right)\hat{a}\right)=$$

$$=\frac{1}{\sqrt{2}}\left(\left(\frac{x_1}{x_0}+i\frac{p_1}{p_0}\right)\hat{a}^\dagger-\left(\frac{x_1}{x_0}-i\frac{p_1}{p_0}\right)\hat{a}\right)=\left(\frac{1}{\sqrt{2}}\left(\frac{x_1}{x_0}+i\frac{p_1}{p_0}\right)\hat{a}^\dagger-\frac{1}{\sqrt{2}}\left(\frac{x_1}{x_0}-i\frac{p_1}{p_0}\right)\hat{a}\right)=$$

$$=\left(\frac{1}{\sqrt{2}}\left(\frac{x_1}{x_0}+i\frac{p_1}{p_0}\right)\hat{a}^\dagger-\left(\frac{1}{\sqrt{2}}\left(\frac{x_1}{x_0}+i\frac{p_1}{p_0}\right)\right)^*\hat{a}\right)$$

That is

$$\frac{i\left(p_1\hat{x}-\hat{p}x_1\right)}{\hbar}=\left(\frac{1}{\sqrt{2}}\left(\frac{x_1}{x_0}+i\frac{p_1}{p_0}\right)\hat{a}^\dagger-\left(\frac{1}{\sqrt{2}}\left(\frac{x_1}{x_0}+i\frac{p_1}{p_0}\right)\right)^*\hat{a}\right)$$

Setting

$$\lambda=\frac{1}{\sqrt{2}}\left(\frac{x_1}{x_0}+i\frac{p_1}{p_0}\right) \qquad (3)$$

we end up to

$$\frac{i\left(p_1\hat{x}-\hat{p}x_1\right)}{\hbar}=\left(\lambda\hat{a}^\dagger-\lambda^*\hat{a}\right) \qquad (4)$$

Thus, the operator $\hat{T}_{p_1,x_1}$ is written as

$$\hat{T}_{p_1,x_1}=\exp\left(\lambda\hat{a}^\dagger-\lambda^*\hat{a}\right) \qquad (5)$$

In (5), the operator $\hat{T}_{p_1,x_1}$ is written in terms of one, but complex, parameter, the parameter λ, instead of the two real parameters x_1 and p_1 (the spatial and momentum translations).

Since the parameters x_1 and p_1 can be any real number, from (3) we see that the parameter λ can be any complex number.

The operator $\exp\left(\lambda\hat{a}^\dagger - \lambda^*\hat{a}\right)$ is called the displacement operator and it is usually denoted by $\hat{D}(\lambda)$, i.e.

$$\hat{D}(\lambda) = \exp\left(\lambda\hat{a}^\dagger - \lambda^*\hat{a}\right) \tag{6}$$

The parameter λ is called the displacement parameter.

As we'll see below, for each value of the displacement parameter λ, the displacement operator, acting on the ground state of the QHO, yields an eigenstate of the annihilation operator, a so-called coherent state.

From (5) and (6), we have

$$\hat{T}_{p_1,x_1} = \hat{D}(\lambda) \tag{7}$$

where $\hat{T}_{p_1,x_1} \equiv \exp\left(\frac{i\left(p_1\hat{x} - \hat{p}x_1\right)}{\hbar}\right)$.

That is, the operator $\hat{T}_{p_1,x_1}$ is the displacement operator.

Besides, in the previous exercise, we showed that the operator $\hat{T}_{p_1,x_1}$ differs from the operators $\hat{T}_{x_1}\hat{T}_{p_1}$ and $\hat{T}_{p_1}\hat{T}_{x_1}$ only by a, physically unimportant, constant phase. Thus, the action of the displacement operator is physically the same as the combined action of a spatial and a momentum translation operator, or a momentum and a spatial translation operator. In other words, the displacement operator, acting on an arbitrary state of the QHO, yields a spatial and a momentum translation, or a momentum and a spatial translation.

6) At $t=0$, a QHO is in the state $\hat{T}_{p_1,x_1}\left|0\right\rangle$, where $\hat{T}_{p_1,x_1} \equiv \exp\left(\frac{i\left(p_1\hat{x} - \hat{p}x_1\right)}{\hbar}\right)$.

i) Show that the state $\hat{T}_{p_1,x_1}\left|0\right\rangle$ is normalized.

ii) Expand the state $\hat{T}_{p_1,x_1}|0\rangle$ *in the basis of the energy eigenstates of the QHO.*

iii) Write the time evolution of the state $\hat{T}_{p_1,x_1}|0\rangle$ *for* $t>0$.

iv) Show that the probability that the QHO is found in an energy eigenstate at time $t\geq 0$ *is given by a Poisson distribution. What is the parameter of the distribution?*

Solution

i) We showed in the exercise 4 that the operator $\hat{T}_{p_1,x_1}$ is written as

$$\hat{T}_{p_1,x_1}=\exp\left(-\frac{ix_1p_1}{2\hbar}\right)\hat{T}_{p_1}\hat{T}_{x_1} \tag{1}$$

The Hermitian conjugate of $\hat{T}_{p_1,x_1}$ is then

$$\hat{T}_{p_1,x_1}{}^{\dagger}=\left(\exp\left(-\frac{ix_1p_1}{2\hbar}\right)\hat{T}_{p_1}\hat{T}_{x_1}\right)^{\dagger}=\exp\left(\frac{ix_1p_1}{2\hbar}\right)\left(\hat{T}_{p_1}\hat{T}_{x_1}\right)^{\dagger}=\exp\left(\frac{ix_1p_1}{2\hbar}\right)\hat{T}_{x_1}{}^{\dagger}\hat{T}_{p_1}{}^{\dagger}$$

That is

$$\hat{T}_{p_1,x_1}{}^{\dagger}=\exp\left(\frac{ix_1p_1}{2\hbar}\right)\hat{T}_{x_1}{}^{\dagger}\hat{T}_{p_1}{}^{\dagger} \tag{2}$$

Using (1) and (2), and that the operators $\hat{T}_{x_1}$ and $\hat{T}_{p_1}$ are unitary, we have

$$\hat{T}_{p_1,x_1}{}^{\dagger}\hat{T}_{p_1,x_1}=\exp\left(\frac{ix_1p_1}{2\hbar}\right)\hat{T}_{x_1}{}^{\dagger}\hat{T}_{p_1}{}^{\dagger}\exp\left(-\frac{ix_1p_1}{2\hbar}\right)\hat{T}_{p_1}\hat{T}_{x_1}=$$
$$=\exp\left(\frac{ix_1p_1}{2\hbar}\right)\exp\left(-\frac{ix_1p_1}{2\hbar}\right)\hat{T}_{x_1}{}^{\dagger}\underbrace{\hat{T}_{p_1}{}^{\dagger}\hat{T}_{p_1}}_{1}\hat{T}_{x_1}=\underbrace{\exp\left(\frac{ix_1p_1}{2\hbar}-\frac{ix_1p_1}{2\hbar}\right)}_{\exp 0=1}\underbrace{\hat{T}_{x_1}{}^{\dagger}\hat{T}_{x_1}}_{1}=1$$

Also

$$\hat{T}_{p_1,x_1}\hat{T}_{p_1,x_1}{}^{\dagger}=\exp\left(-\frac{ix_1p_1}{2\hbar}\right)\hat{T}_{p_1}\hat{T}_{x_1}\exp\left(\frac{ix_1p_1}{2\hbar}\right)\hat{T}_{x_1}{}^{\dagger}\hat{T}_{p_1}{}^{\dagger}=$$
$$=\exp\left(-\frac{ix_1p_1}{2\hbar}\right)\exp\left(\frac{ix_1p_1}{2\hbar}\right)\hat{T}_{p_1}\underbrace{\hat{T}_{x_1}\hat{T}_{x_1}{}^{\dagger}}_{1}\hat{T}_{p_1}{}^{\dagger}=\underbrace{\exp\left(-\frac{ix_1p_1}{2\hbar}+\frac{ix_1p_1}{2\hbar}\right)}_{\exp 0=1}\underbrace{\hat{T}_{p_1}\hat{T}_{p_1}{}^{\dagger}}_{1}=1$$

That is

$$\hat{T}_{p_1,x_1}{}^{\dagger}\hat{T}_{p_1,x_1}=\hat{T}_{p_1,x_1}\hat{T}_{p_1,x_1}{}^{\dagger}=1$$

Therefore, the operator $\hat{T}_{p_1,x_1}$ is unitary.

We remind that the terms $\exp\left(-\frac{ix_1p_1}{2\hbar}\right)$ and $\exp\left(\frac{ix_1p_1}{2\hbar}\right)$, as constant complex numbers, commute with any of the above operators.

Since the operator $\hat{T}_{p_1,x_1}$ is unitary, it preserves the norms of the states on which it acts, and thus

$$\left\|\hat{T}_{p_1,x_1}\left|0\right\rangle\right\| = \left\|\left|0\right\rangle\right\| = 1$$

Therefore, the state $\hat{T}_{p_1,x_1}\left|0\right\rangle$ is normalized.

ii) Using the completeness relation of the energy eigenstates, i.e. $\sum_{n=0}^{\infty}\left|n\right\rangle\left\langle n\right| = 1$, the state $\hat{T}_{p_1,x_1}\left|0\right\rangle$ is written as

$$\hat{T}_{p_1,x_1}\left|0\right\rangle = \left(\sum_{n=0}^{\infty}\left|n\right\rangle\left\langle n\right|\right)\hat{T}_{p_1,x_1}\left|0\right\rangle = \sum_{n=0}^{\infty}\left|n\right\rangle\left\langle n\right|\hat{T}_{p_1,x_1}\left|0\right\rangle = \sum_{n=0}^{\infty}\left\langle n\right|\hat{T}_{p_1,x_1}\left|0\right\rangle\left|n\right\rangle$$

That is

$$\hat{T}_{p_1,x_1}\left|0\right\rangle = \sum_{n=0}^{\infty}\left\langle n\right|\hat{T}_{p_1,x_1}\left|0\right\rangle\left|n\right\rangle \tag{3}$$

This is the expansion of the state $\hat{T}_{p_1,x_1}\left|0\right\rangle$ in the energy basis of the QHO, but we have to calculate the amplitude $\left\langle n\right|\hat{T}_{p_1,x_1}\left|0\right\rangle$.

To do that, we can use that – see the previous exercise –

$$\hat{T}_{p_1,x_1} = \exp\left(\lambda\hat{a}^\dagger - \lambda^*\hat{a}\right)$$

where $\lambda = \frac{1}{\sqrt{2}}\left(\frac{x_1}{x_0} + i\frac{p_1}{p_0}\right)$.

We observe that

$$\left[\lambda\hat{a}^\dagger, -\lambda^*\hat{a}\right] = \lambda\left(-\lambda^*\right)\left[\hat{a}^\dagger, \hat{a}\right] = -\left|\lambda\right|^2\left(-\underbrace{\left[\hat{a}, \hat{a}^\dagger\right]}_{1}\right) = \left|\lambda\right|^2$$

That is

$$\left[\lambda\hat{a}^{\dagger},-\lambda^{*}\hat{a}\right]=|\lambda|^{2} \tag{4}$$

Since the previous commutator is a constant, using the identity

$$\exp\left(\hat{A}+\hat{B}\right)=\exp\left(\hat{A}\right)\exp\left(\hat{B}\right)\exp\left(-\frac{1}{2}\left[\hat{A},\hat{B}\right]\right),$$

for $\hat{A}=\lambda\hat{a}^{\dagger}$ and $\hat{B}=-\lambda^{*}\hat{a}$, we obtain

$$\exp\left(\lambda\hat{a}^{\dagger}-\lambda^{*}\hat{a}\right)=\exp\left(\lambda\hat{a}^{\dagger}\right)\exp\left(-\lambda^{*}\hat{a}\right)\exp\left(-\frac{1}{2}|\lambda|^{2}\right)$$

The term $\exp\left(-\frac{1}{2}|\lambda|^{2}\right)$ is a constant, and thus we can move it to the left and write

$$\exp\left(\lambda\hat{a}^{\dagger}-\lambda^{*}\hat{a}\right)=\exp\left(-\frac{1}{2}|\lambda|^{2}\right)\exp\left(\lambda\hat{a}^{\dagger}\right)\exp\left(-\lambda^{*}\hat{a}\right)$$

Thus, the operator $\hat{T}_{p_1,x_1}$ is written as

$$\hat{T}_{p_1,x_1}=\exp\left(-\frac{1}{2}|\lambda|^{2}\right)\exp\left(\lambda\hat{a}^{\dagger}\right)\exp\left(-\lambda^{*}\hat{a}\right) \tag{5}$$

Then, to calculate the action of $\hat{T}_{p_1,x_1}$ on the ground state, we must first calculate the action of $\exp\left(-\lambda^{*}\hat{a}\right)$ on the ground state.

Using the Taylor expansion of $\exp\left(-\lambda^{*}\hat{a}\right)$, its action on the ground state is written as

$$\exp\left(-\lambda^{*}\hat{a}\right)|0\rangle=\left(\sum_{m=0}^{\infty}\frac{\left(-\lambda^{*}\hat{a}\right)^{m}}{m!}\right)|0\rangle=\sum_{m=0}^{\infty}\frac{\left(-\lambda^{*}\hat{a}\right)^{m}}{m!}|0\rangle=\sum_{m=0}^{\infty}\frac{\left(-\lambda^{*}\right)^{m}}{m!}\hat{a}^{m}|0\rangle$$

In the last equality, we used that $\hat{a}$ commutes with the constant number $-\lambda^{*}$.

That is

$$\exp\left(-\lambda^{*}\hat{a}\right)|0\rangle=\sum_{m=0}^{\infty}\frac{\left(-\lambda^{*}\right)^{m}}{m!}\hat{a}^{m}|0\rangle$$

But, since $\hat{a}$ kills the ground state,

$\hat{a}^m |0\rangle = 0$ if $m = 1, 2, ...$

Then, in the series $\sum_{m=0}^{\infty} \frac{(-\lambda^*)^m}{m!} \hat{a}^m |0\rangle$ only the first term, with $m = 0$, survives.

Thus

$$\exp\left(-\lambda^* \hat{a}\right)|0\rangle = |0\rangle \quad (6)$$

Using (5) and (6), the action of $\hat{T}_{p_1,x_1}$ on the ground state is written as

$$\hat{T}_{p_1,x_1} |0\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right) \exp\left(\lambda \hat{a}^\dagger\right)|0\rangle$$

Using the Taylor expansion of $\exp\left(\lambda \hat{a}^\dagger\right)$, we obtain

$$\hat{T}_{p_1,x_1} |0\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\left(\sum_{m=0}^{\infty} \frac{\left(\lambda \hat{a}^\dagger\right)^m}{m!}\right)|0\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\left(\sum_{m=0}^{\infty} \frac{\left(\lambda \hat{a}^\dagger\right)^m}{m!}|0\rangle\right) =$$
$$= \exp\left(-\frac{1}{2}|\lambda|^2\right)\left(\sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \hat{a}^{\dagger m} |0\rangle\right)$$

In the last equality, we used that $\hat{a}^\dagger$ commutes with the constant number λ.

That is

$$\hat{T}_{p_1,x_1} |0\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\left(\sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \hat{a}^{\dagger m} |0\rangle\right) \quad (7)$$

Using that $\hat{a}^\dagger |n\rangle = \sqrt{n+1}|n+1\rangle$, we have

$$\hat{a}^{\dagger m} |0\rangle = \hat{a}^{\dagger m-1} |1\rangle = \sqrt{2}\hat{a}^{\dagger m-2} |2\rangle = \sqrt{2!}\hat{a}^{\dagger m-2} |2\rangle = \sqrt{2!}\sqrt{3}\hat{a}^{\dagger m-3} |3\rangle =$$
$$= \sqrt{3!}\hat{a}^{\dagger m-3} |3\rangle = ... = \sqrt{m!}\underbrace{\hat{a}^{\dagger m-m}}_{\hat{a}^{\dagger 0}=1} |m\rangle = \sqrt{m!}|m\rangle$$

That is

$$\hat{a}^{\dagger m} |0\rangle = \sqrt{m!}|m\rangle \quad (8)$$

By means of (8), (7) becomes

$$\hat{T}_{p_1,x_1}|0\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\left(\sum_{m=0}^{\infty}\frac{\lambda^m}{\sqrt{m!}}|m\rangle\right) \tag{9}$$

Using (9), the amplitude $\langle n|\hat{T}_{p_1,x_1}|0\rangle$ is written as

$$\langle n|\hat{T}_{p_1,x_1}|0\rangle = \langle n|\left(\exp\left(-\frac{1}{2}|\lambda|^2\right)\left(\sum_{m=0}^{\infty}\frac{\lambda^m}{\sqrt{m!}}|m\rangle\right)\right) =$$

$$= \exp\left(-\frac{1}{2}|\lambda|^2\right)\langle n|\left(\sum_{m=0}^{\infty}\frac{\lambda^m}{\sqrt{m!}}|m\rangle\right) = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{m=0}^{\infty}\frac{\lambda^m}{\sqrt{m!}}\langle n|m\rangle$$

That is

$$\langle n|\hat{T}_{p_1,x_1}|0\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{m=0}^{\infty}\frac{\lambda^m}{\sqrt{m!}}\langle n|m\rangle \tag{10}$$

Using the orthonormality of the energy eigenstates, i.e. $\langle n|m\rangle = \delta_{nm}$, the series $\sum_{m=0}^{\infty}\frac{\lambda^m}{\sqrt{m!}}\langle n|m\rangle$ becomes

$$\sum_{m=0}^{\infty}\frac{\lambda^m}{\sqrt{m!}}\langle n|m\rangle = \sum_{m=0}^{\infty}\frac{\lambda^m}{\sqrt{m!}}\delta_{nm} = \frac{\lambda^n}{\sqrt{n!}}$$

Substituting into (10), we obtain

$$\langle n|\hat{T}_{p_1,x_1}|0\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\frac{\lambda^n}{\sqrt{n!}} \tag{11}$$

Substituting the amplitude (11) into the expansion (3), we obtain

$$\hat{T}_{p_1,x_1}|0\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle \tag{12}$$

where $\lambda = \frac{1}{\sqrt{2}}\left(\frac{x_1}{x_0} + i\frac{p_1}{p_0}\right)$.

This is the expansion of the state $\hat{T}_{p_1,x_1}|0\rangle$ in the energy basis of the QHO.

Since $\hat{T}_{p_1,x_1} = \hat{D}(\lambda)$, (12) is also written as

$$\hat{D}(\lambda)|0\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle \tag{13}$$

iii) Denoting by $|n\rangle_t$ the time evolution of an energy eigenstate $|n\rangle$, then

$$\left|n\right\rangle_t = \exp\left(-\frac{iE_n t}{\hbar}\right)\left|n\right\rangle,$$

with $t \geq 0$ and $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$.

Then, using (12), the time evolution of the state $\hat{T}_{p_1,x_1}\left|0\right\rangle$ – let us denote it by $\left|\psi(t)\right\rangle$ – is

$$\left|\psi(t)\right\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\left|n\right\rangle_t = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\exp\left(-\frac{iE_n t}{\hbar}\right)\left|n\right\rangle$$

That is

$$\left|\psi(t)\right\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\exp\left(-\frac{iE_n t}{\hbar}\right)\left|n\right\rangle \quad (14)$$

with $t \geq 0$ and $\left|\psi(0)\right\rangle \equiv \hat{T}_{p_1,x_1}\left|0\right\rangle$.

> If the initial state is taken at $t = t_0$, instead of $t = 0$, then the time evolution of an energy eigenstate $\left|n\right\rangle$ is
>
> $$\left|n\right\rangle_t = \exp\left(-\frac{iE_n(t - t_0)}{\hbar}\right)\left|n\right\rangle,$$
>
> and the time evolution of the state $\hat{T}_{p_1,x_1}\left|0\right\rangle$ is then
>
> $$\left|\psi(t)\right\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\exp\left(-\frac{iE_n(t - t_0)}{\hbar}\right)\left|n\right\rangle$$
>
> with $t \geq t_0$ and $\left|\psi(t_0)\right\rangle \equiv \hat{T}_{p_1,x_1}\left|0\right\rangle$.

iv) The state of the QHO at $t \geq 0$ is given by (14).

The energy eigenstates are orthonormal, thus the probability amplitude that the QHO is found in an energy eigenstate $\left|m\right\rangle$, at time $t \geq 0$, is $\left\langle m \middle| \psi(t)\right\rangle$, which, using (14), is written as

$$\langle m|\psi(t)\rangle = \langle m|\left(\exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\exp\left(-\frac{iE_n t}{\hbar}\right)|n\rangle\right) =$$
$$= \exp\left(-\frac{1}{2}|\lambda|^2\right)\langle m|\left(\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\exp\left(-\frac{iE_n t}{\hbar}\right)|n\rangle\right) =$$
$$= \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\exp\left(-\frac{iE_n t}{\hbar}\right)\underbrace{\langle m|n\rangle}_{\delta_{mn}} = \exp\left(-\frac{1}{2}|\lambda|^2\right)\frac{\lambda^m}{\sqrt{m!}}\exp\left(-\frac{iE_m t}{\hbar}\right)$$

That is

$$\langle m|\psi(t)\rangle = \exp\left(-\frac{iE_m t}{\hbar}\right)\exp\left(-\frac{1}{2}|\lambda|^2\right)\frac{\lambda^m}{\sqrt{m!}} \qquad (15)$$

Then, the probability that the QHO is found in an energy eigenstate $|m\rangle$, at time $t \geq 0$, is

$$P_m(t) = |\langle m|\psi(t)\rangle|^2 = \left|\exp\left(-\frac{iE_m t}{\hbar}\right)\exp\left(-\frac{1}{2}|\lambda|^2\right)\frac{\lambda^m}{\sqrt{m!}}\right|^2$$

Using that for a complex number z, $|z|^2 = zz^*$, the probability $P_m(t)$ becomes

$$P_m(t) = \left(\exp\left(-\frac{iE_m t}{\hbar}\right)\exp\left(-\frac{1}{2}|\lambda|^2\right)\frac{\lambda^m}{\sqrt{m!}}\right)\left(\exp\left(-\frac{iE_m t}{\hbar}\right)\exp\left(-\frac{1}{2}|\lambda|^2\right)\frac{\lambda^m}{\sqrt{m!}}\right)^* =$$
$$= \exp\left(-\frac{iE_m t}{\hbar}\right)\exp\left(-\frac{1}{2}|\lambda|^2\right)\frac{\lambda^m}{\sqrt{m!}}\exp\left(\frac{iE_m t}{\hbar}\right)\exp\left(-\frac{1}{2}|\lambda|^2\right)\frac{(\lambda^m)^*}{\sqrt{m!}}$$

Using that $(\lambda^m)^* = (\lambda^*)^m$, which follows from the property $(z_1 z_2 \ldots z_m)^* = z_1^* z_2^* \ldots z_m^*$ for $z_1 = z_2 = \ldots = z_m = \lambda$, which, in turn, follows easily, by induction, from the elementary property $(z_1 z_2)^* = z_1^* z_2^*$, the probability $P_m(t)$ becomes

$$P_m(t) = \underbrace{\exp\left(-\frac{iE_m t}{\hbar}\right)\exp\left(\frac{iE_m t}{\hbar}\right)}_{1}\underbrace{\exp\left(-\frac{1}{2}|\lambda|^2\right)\exp\left(-\frac{1}{2}|\lambda|^2\right)}_{\exp\left(-|\lambda|^2\right)}\frac{\lambda^m}{\sqrt{m!}}\frac{(\lambda^*)^m}{\sqrt{m!}} =$$
$$= \exp\left(-|\lambda|^2\right)\frac{(\lambda\lambda^*)^m}{m!} = \exp\left(-|\lambda|^2\right)\frac{\left(|\lambda|^2\right)^m}{m!}$$

That is

$$P_m(t) = \exp\left(-|\lambda|^2\right)\frac{\left(|\lambda|^2\right)^m}{m!} \tag{16}$$

with $m = 0,1,...$

The probability (16) is time independent and it is given by a Poisson distribution with parameter $|\lambda|^2$.

Observe that

$$\sum_{m=0}^{\infty} P_m(t) = \sum_{m=0}^{\infty} \exp\left(-|\lambda|^2\right)\frac{\left(|\lambda|^2\right)^m}{m!} = \exp\left(-|\lambda|^2\right)\underbrace{\sum_{m=0}^{\infty}\frac{\left(|\lambda|^2\right)^m}{m!}}_{\exp\left(|\lambda|^2\right)} = 1$$

That is, the probabilities (16) add up to 1, as they should.

Using that $\lambda = \frac{1}{\sqrt{2}}\left(\frac{x_1}{x_0} + i\frac{p_1}{p_0}\right)$, we obtain the parameter $|\lambda|^2$ in terms of the spatial and the momentum translation, x_1 and p_1, respectively. Then, we have

$$|\lambda|^2 = \frac{1}{2}\left(\left(\frac{x_1}{x_0}\right)^2 + \left(\frac{p_1}{p_0}\right)^2\right) \tag{17}$$

II. The coherent states of the QHO

7) The coherent states of the QHO are defined as the eigenstates of the annihilation operator (see the references [4], [5], and [6] (section 2.1)).

i) If $|\lambda\rangle$ *is an eigenstate of the annihilation operator, i.e. if* $\hat{a}|\lambda\rangle = \lambda|\lambda\rangle$*, with* $\lambda \in \mathbb{C}$*, show that* $\hat{D}(\lambda)|0\rangle = |\lambda\rangle$*, i.e. the displacement operator, acting on the ground state, generates the coherent states.*

We remind that the eigenvalues of the annihilation operator are complex numbers, because the annihilation operator is not Hermitian.

ii) Show that the coherent states are states of minimum position-momentum uncertainty AND the two individual uncertainties, i.e. the position uncertainty and the momentum uncertainty, are equally distributed, in the sense that $\Delta x = \frac{x_0}{\sqrt{2}}$ *and* $\Delta p = \frac{p_0}{\sqrt{2}}$*, or, in dimensionless form,* $\frac{\Delta x}{x_0} = \frac{\Delta p}{p_0} = \frac{1}{\sqrt{2}}$*, with* x_0, p_0 *being, respectively, the length and momentum scales of the QHO. This property, i.e. the equal distribution of the position uncertainty and the momentum uncertainty, differentiates the coherent from the squeezed states, which are also minimum position-momentum uncertainty states, but the two individual uncertainties are NOT equally distributed.*

iii) Calculate the energy expectation value and uncertainty in a coherent state.

iv) Using the expansion of the coherent state $|\lambda\rangle$ *in the energy basis of the QHO, write its time evolution and show that although it remains a coherent state, and thus a state of minimum position-momentum uncertainty, its eigenvalue changes. What is the time evolution of the eigenvalue* λ*?*

Solution

i) We'll show that the states $\hat{D}(\lambda)|0\rangle$ and $|\lambda\rangle$ have the same expansion in the energy basis of the QHO, and thus they are the same state.

In the exercise 6, we showed that the expansion of the state $\hat{D}(\lambda)|0\rangle$ in the energy basis of the QHO is

$$\hat{D}(\lambda)|0\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle \qquad (1)$$

Let us now find the expansion of the state $|\lambda\rangle$.

Using the completeness relation of the energy eigenstates, the state $|\lambda\rangle$ is written as

$$|\lambda\rangle = \left(\sum_{n=0}^{\infty}|n\rangle\langle n|\right)|\lambda\rangle = \sum_{n=0}^{\infty}|n\rangle\langle n|\lambda\rangle = \sum_{n=0}^{\infty}\langle n|\lambda\rangle|n\rangle$$

That is

$$|\lambda\rangle = \sum_{n=0}^{\infty}\langle n|\lambda\rangle|n\rangle \qquad (2)$$

Then, the action of the annihilation operator on the state $|\lambda\rangle$ is written as

$$\hat{a}|\lambda\rangle = \hat{a}\left(\sum_{n=0}^{\infty}\langle n|\lambda\rangle|n\rangle\right) = \sum_{n=0}^{\infty}\langle n|\lambda\rangle\hat{a}|n\rangle$$

Using that $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$, we obtain

$$\hat{a}|\lambda\rangle = \sum_{n=1}^{\infty}\langle n|\lambda\rangle\sqrt{n}|n-1\rangle$$

Changing the summation index to $n' = n-1$, we obtain

$$\hat{a}|\lambda\rangle = \sum_{n'=0}^{\infty}\langle n'+1|\lambda\rangle\sqrt{n'+1}|n'\rangle$$

Renaming the summation index to n, we end up to

$$\hat{a}|\lambda\rangle = \sum_{n=0}^{\infty}\langle n+1|\lambda\rangle\sqrt{n+1}|n\rangle \qquad (3)$$

Since the state $|\lambda\rangle$ is eigenstate of $\hat{a}$ with eigenvalue λ,

$$\hat{a}|\lambda\rangle = \lambda|\lambda\rangle$$

Substituting (2) and (3) into the previous equation yields

$$\sum_{n=0}^{\infty}\langle n+1|\lambda\rangle\sqrt{n+1}|n\rangle = \lambda\left(\sum_{n=0}^{\infty}\langle n|\lambda\rangle|n\rangle\right) \Rightarrow \sum_{n=0}^{\infty}\sqrt{n+1}\langle n+1|\lambda\rangle|n\rangle = \sum_{n=0}^{\infty}\lambda\langle n|\lambda\rangle|n\rangle \Rightarrow$$

$$\Rightarrow \sum_{n=0}^{\infty}\left(\sqrt{n+1}\langle n+1|\lambda\rangle - \lambda\langle n|\lambda\rangle\right)|n\rangle = 0$$

Since the energy eigenstates are linearly independent, from the last equation we obtain

$$\sqrt{n+1}\langle n+1|\lambda\rangle-\lambda\langle n|\lambda\rangle=0\Rightarrow\sqrt{n+1}\langle n+1|\lambda\rangle=\lambda\langle n|\lambda\rangle$$

Thus

$$\langle n+1|\lambda\rangle=\frac{\lambda}{\sqrt{n+1}}\langle n|\lambda\rangle \tag{4}$$

with $n=0,1,...$

Applying the recursive relation (4) repeatedly gives

$$\langle n+1|\lambda\rangle=\frac{\lambda}{\sqrt{n+1}}\langle n|\lambda\rangle=\frac{\lambda}{\sqrt{n+1}}\frac{\lambda}{\sqrt{n}}\langle n-1|\lambda\rangle=\frac{\lambda}{\sqrt{n+1}}\frac{\lambda}{\sqrt{n}}\frac{\lambda}{\sqrt{n-1}}\langle n-2|\lambda\rangle=$$
$$=...=\frac{\lambda}{\sqrt{n+1}}\frac{\lambda}{\sqrt{n}}\frac{\lambda}{\sqrt{n-1}}...\frac{\lambda}{\sqrt{n-(n-1)}}\langle n-n|\lambda\rangle=\frac{\lambda}{\sqrt{n+1}}...\frac{\lambda}{\sqrt{1}}\langle 0|\lambda\rangle=$$
$$=\frac{\lambda^{n+1}}{\sqrt{(n+1)!}}\langle 0|\lambda\rangle$$

That is

$$\langle n+1|\lambda\rangle=\frac{\lambda^{n+1}}{\sqrt{(n+1)!}}\langle 0|\lambda\rangle$$

Thus

$$\langle n|\lambda\rangle=\frac{\lambda^{n}}{\sqrt{n!}}\langle 0|\lambda\rangle \tag{5}$$

The constant $\langle 0|\lambda\rangle$ can be calculated using that the state $|\lambda\rangle$ is normalized, i.e. $\langle\lambda|\lambda\rangle=1$.

Substituting (5) into (2) yields

$$|\lambda\rangle=\sum_{n=0}^{\infty}\frac{\lambda^{n}}{\sqrt{n!}}\langle 0|\lambda\rangle|n\rangle=\langle 0|\lambda\rangle\sum_{n=0}^{\infty}\frac{\lambda^{n}}{\sqrt{n!}}|n\rangle$$

That is

$$|\lambda\rangle=\langle 0|\lambda\rangle\sum_{n=0}^{\infty}\frac{\lambda^{n}}{\sqrt{n!}}|n\rangle \tag{6}$$

Then, the bra $\langle\lambda|$ is

$$\langle\lambda| = \langle 0|\lambda\rangle^* \sum_{n=0}^{\infty} \frac{\left(\lambda^n\right)^*}{\sqrt{n!}} \langle n| = \langle 0|\lambda\rangle^* \sum_{n=0}^{\infty} \frac{\left(\lambda^*\right)^n}{\sqrt{n!}} \langle n|$$

where, in the last equality, we used that $\left(z_1 z_2 ... z_n\right)^* = z_1^* z_2^* ... z_n^*$, a complex number property that follows easily, by induction, from the basic property $\left(z_1 z_2\right)^* = z_1^* z_2^*$.

Thus, the bra $\langle\lambda|$ is

$$\langle\lambda| = \langle 0|\lambda\rangle^* \sum_{n=0}^{\infty} \frac{\left(\lambda^*\right)^n}{\sqrt{n!}} \langle n| \qquad (7)$$

Using (6) and (7), we obtain

$$\langle\lambda|\lambda\rangle = \left(\langle 0|\lambda\rangle^* \sum_{m=0}^{\infty} \frac{\left(\lambda^*\right)^m}{\sqrt{m!}} \langle m| \right) \left(\langle 0|\lambda\rangle \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} |n\rangle \right) =$$

$$= \langle 0|\lambda\rangle^* \langle 0|\lambda\rangle \sum_{m,n=0}^{\infty} \frac{\left(\lambda^*\right)^m}{\sqrt{m!}} \frac{\lambda^n}{\sqrt{n!}} \langle m|n\rangle = \left|\langle 0|\lambda\rangle\right|^2 \sum_{m,n=0}^{\infty} \frac{\left(\lambda^*\right)^m}{\sqrt{m!}} \frac{\lambda^n}{\sqrt{n!}} \langle m|n\rangle$$

Since the two sums are independent, we use different summation indices.

Using the orthonormality of the energy eigenstates, i.e. $\langle m|n\rangle = \delta_{mn}$, we obtain

$$\langle\lambda|\lambda\rangle = \left|\langle 0|\lambda\rangle\right|^2 \sum_{m,n=0}^{\infty} \frac{\left(\lambda^*\right)^m}{\sqrt{m!}} \frac{\lambda^n}{\sqrt{n!}} \delta_{mn} = \left|\langle 0|\lambda\rangle\right|^2 \sum_{n=0}^{\infty} \frac{\left(\lambda^*\right)^n}{\sqrt{n!}} \frac{\lambda^n}{\sqrt{n!}} = \left|\langle 0|\lambda\rangle\right|^2 \sum_{n=0}^{\infty} \frac{\left(\lambda\lambda^*\right)^n}{n!} =$$

$$= \left|\langle 0|\lambda\rangle\right|^2 \sum_{n=0}^{\infty} \frac{\left(|\lambda|^2\right)^n}{n!} = \left|\langle 0|\lambda\rangle\right|^2 \exp\left(|\lambda|^2\right)$$

That is

$$\langle\lambda|\lambda\rangle = \left|\langle 0|\lambda\rangle\right|^2 \exp\left(|\lambda|^2\right)$$

Since the state $|\lambda\rangle$ is normalized,

$$1 = \left|\langle 0|\lambda\rangle\right|^2 \exp\left(|\lambda|^2\right) \Rightarrow \left|\langle 0|\lambda\rangle\right|^2 = \exp\left(-|\lambda|^2\right) \Rightarrow \left|\langle 0|\lambda\rangle\right| = \exp\left(-\frac{1}{2}|\lambda|^2\right)$$

Omitting the physically unimportant phase of $\langle 0|\lambda\rangle$, we end up to

$$\langle 0|\lambda\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right) \tag{8}$$

Substituting (8) into (6) yields

$$|\lambda\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle \tag{9}$$

Comparing (1) and (9), we obtain

$$\hat{D}(\lambda)|0\rangle = |\lambda\rangle \tag{10}$$

ii) In the exercise 4, we showed that the operator $\hat{T}_{p_1,x_1}$ is written as

$$\hat{T}_{p_1,x_1} = \exp\left(-\frac{ix_1p_1}{2\hbar}\right)\hat{T}_{p_1}\hat{T}_{x_1}$$

Then, since the term $\exp\left(-\frac{ix_1p_1}{2\hbar}\right)$ is a constant phase, the action of $\hat{T}_{p_1,x_1}$ on a physical state is equivalent, i.e. it is physically the same, to the action of $\hat{T}_{p_1}\hat{T}_{x_1}$.

Since the operator $\hat{T}_{p_1,x_1}$ is the displacement operator $\hat{D}(\lambda)$, as shown in the exercise 5, the action of $\hat{D}(\lambda)$ is physically equivalent to the action of $\hat{T}_{p_1}\hat{T}_{x_1}$.

Then, omitting the physically unimportant constant phase, we can write

$$\hat{D}(\lambda)|0\rangle = \hat{T}_{p_1}\hat{T}_{x_1}|0\rangle$$

By means of (10), the previous equation becomes

$$|\lambda\rangle = \hat{T}_{p_1}\hat{T}_{x_1}|0\rangle \tag{11}$$

with $\lambda = \frac{1}{\sqrt{2}}\left(\frac{x_1}{x_0} + i\frac{p_1}{p_0}\right)$.

The equation (11) provides a nice intuitive picture of the coherent states, as it tells us that they result from the application of spatial and momentum translations to the ground state of the QHO.

We remind that, as also shown in the exercise 4, the order of the two translations doesn't matter, i.e.

$$\hat{T}_{p_1}\hat{T}_{x_1}|0\rangle = \hat{T}_{x_1}\hat{T}_{p_1}|0\rangle$$

In the ground state of the QHO, the position and momentum uncertainties are, respectively,

$$(\Delta x)_{|0\rangle} = \frac{x_0}{\sqrt{2}} \tag{12}$$

$$(\Delta p)_{|0\rangle} = \frac{p_0}{\sqrt{2}} \tag{13}$$

The position-momentum uncertainty product is then

$$(\Delta x \Delta p)_{|0\rangle} = \frac{x_0 p_0}{2}$$

Using that $x_0 p_0 = \hbar$, we obtain

$$(\Delta x \Delta p)_{|0\rangle} = \frac{\hbar}{2} \tag{14}$$

The ground state is then a state of minimum position-momentum uncertainty.

The ground state is also a coherent state, since it is eigenstate of the annihilation operator with eigenvalue 0, i.e. $\hat{a}|0\rangle = 0|0\rangle$**.**

In the exercise 2, we showed that the application of a spatial translation does not change either the position or the momentum uncertainty.
Thus

$$(\Delta x)_{\hat{T}_{x_1}|0\rangle} = (\Delta x)_{|0\rangle}$$

$$(\Delta p)_{\hat{T}_{x_1}|0\rangle} = (\Delta p)_{|0\rangle}$$

By means of (12) and (13), the previous two equations become, respectively,

$$(\Delta x)_{\hat{T}_{x_1}|0\rangle} = \frac{x_0}{\sqrt{2}} \tag{15}$$

$$\left(\Delta p\right)_{\hat{T}_{x_1}|0\rangle} = \frac{p_0}{\sqrt{2}} \tag{16}$$

Similarly, in the exercise 3, we showed that the application of a momentum translation does not change either the position or the momentum uncertainty.

Thus, taking the state $\hat{T}_{x_1}\left|0\right\rangle$ as the initial state,

$$\left(\Delta x\right)_{\hat{T}_{p_1}\hat{T}_{x_1}|0\rangle} = \left(\Delta x\right)_{\hat{T}_{x_1}|0\rangle}$$

$$\left(\Delta p\right)_{\hat{T}_{p_1}\hat{T}_{x_1}|0\rangle} = \left(\Delta p\right)_{\hat{T}_{x_1}|0\rangle}$$

By means of (15) and (16), the previous two equations become, respectively,

$$\left(\Delta x\right)_{\hat{T}_{p_1}\hat{T}_{x_1}|0\rangle} = \frac{x_0}{\sqrt{2}}$$

$$\left(\Delta p\right)_{\hat{T}_{p_1}\hat{T}_{x_1}|0\rangle} = \frac{p_0}{\sqrt{2}}$$

By means of (11), the previous two uncertainties are written as

$$\left(\Delta x\right)_{|\lambda\rangle} = \frac{x_0}{\sqrt{2}} \tag{17}$$

$$\left(\Delta p\right)_{|\lambda\rangle} = \frac{p_0}{\sqrt{2}} \tag{18}$$

By means of (17), (18), and the relation $x_0 p_0 = \hbar$, we obtain

$$\left(\Delta x \Delta p\right)_{|\lambda\rangle} = \frac{\hbar}{2} \tag{19}$$

Therefore, the coherent states are states of minimum position-momentum uncertainty and the two individual uncertainties are equally distributed.

iii) The energy expectation value of the QHO in the coherent state $\left|\lambda\right\rangle$ is

$$\left\langle E\right\rangle_{|\lambda\rangle} = \left\langle \lambda\right|\hat{H}\left|\lambda\right\rangle$$

The Hamiltonian of the QHO is written as

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

Thus

$$\langle E\rangle_{|\lambda\rangle} = \langle\lambda|\hat{H}|\lambda\rangle = \langle E\rangle_{|\lambda\rangle} = \langle\lambda|\hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|\lambda\rangle = \hbar\omega\langle\lambda|\hat{a}^\dagger\hat{a} + \frac{1}{2}|\lambda\rangle =$$

$$= \hbar\omega\left(\langle\lambda|\hat{a}^\dagger\hat{a}|\lambda\rangle + \frac{1}{2}\underbrace{\langle\lambda|\lambda\rangle}_{1}\right) = \hbar\omega\left(\lambda\langle\lambda|\hat{a}^\dagger|\lambda\rangle + \frac{1}{2}\right)$$

where, in the last equality, we used that $\hat{a}|\lambda\rangle = \lambda|\lambda\rangle$.

Thus

$$\langle E\rangle_{|\lambda\rangle} = \left(\lambda\langle\lambda|\hat{a}^\dagger|\lambda\rangle + \frac{1}{2}\right)\hbar\omega \qquad (20)$$

Turning again – for more clarity – to the general notation for inner products, the inner product $\langle\lambda|\hat{a}^\dagger|\lambda\rangle$ is written as $\left(|\lambda\rangle, \hat{a}^\dagger|\lambda\rangle\right)$, and using the definition of the Hermitian conjugate of $\hat{a}^\dagger$, we have

$$\left(|\lambda\rangle, \hat{a}^\dagger|\lambda\rangle\right) = \left(\underbrace{\left(\hat{a}^\dagger\right)^\dagger}_{\hat{a}}|\lambda\rangle, |\lambda\rangle\right) = \left(\hat{a}|\lambda\rangle, |\lambda\rangle\right) = \left(\lambda|\lambda\rangle, |\lambda\rangle\right) = \lambda^*\underbrace{\left(|\lambda\rangle, |\lambda\rangle\right)}_{1}$$

Thus

$$\langle\lambda|\hat{a}^\dagger|\lambda\rangle = \lambda^* \qquad (21)$$

Using (21) and that $\hat{a}|\lambda\rangle = \lambda|\lambda\rangle$, we have

$$\langle\lambda|\hat{a}^\dagger\hat{a}|\lambda\rangle = \lambda\langle\lambda|\hat{a}^\dagger|\lambda\rangle = \lambda\lambda^* = |\lambda|^2$$

That is

$$\langle\lambda|\hat{a}^\dagger\hat{a}|\lambda\rangle = |\lambda|^2 \qquad (22)$$

Also, since $\hat{N} \equiv \hat{a}^\dagger\hat{a}$, (22) is the expectation value of the number operator in the coherent state $|\lambda\rangle$.

By means of (21), (20) becomes

$$\langle E\rangle_{|\lambda\rangle}=\left(|\lambda|^2+\frac{1}{2}\right)\hbar\omega \tag{23}$$

This is the energy expectation value in the coherent state $|\lambda\rangle$.

Observe that the energy expectation value depends only on the magnitude of λ, not on its phase.

The energy uncertainty in the coherent state $|\lambda\rangle$ is

$$(\Delta E)_{|\lambda\rangle}=\sqrt{\langle E^2\rangle_{|\lambda\rangle}-\left(\langle E\rangle_{|\lambda\rangle}\right)^2}$$

We'll calculate the expectation value of the energy squared in the state $|\lambda\rangle$, which is

$$\langle E^2\rangle_{|\lambda\rangle}=\langle\lambda|\hat{H}^2|\lambda\rangle$$

Using the previous expression of the QHO Hamiltonian, we have

$$\hat{H}^2=\hbar\omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)\hbar\omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)=(\hbar\omega)^2\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)=$$

$$=(\hbar\omega)^2\left(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}+\underbrace{\hat{a}^\dagger\hat{a}\frac{1}{2}+\frac{1}{2}\hat{a}^\dagger\hat{a}}_{2\frac{1}{2}\hat{a}^\dagger\hat{a}}+\frac{1}{4}\right)=(\hbar\omega)^2\left(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}+\hat{a}^\dagger\hat{a}+\frac{1}{4}\right)$$

That is

$$\hat{H}^2=(\hbar\omega)^2\left(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}+\hat{a}^\dagger\hat{a}+\frac{1}{4}\right)$$

Then, the expectation value of the energy squared is written as

$$\langle E^2\rangle_{|\lambda\rangle}=\langle\lambda|(\hbar\omega)^2\left(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}+\hat{a}^\dagger\hat{a}+\frac{1}{4}\right)|\lambda\rangle=$$

$$=(\hbar\omega)^2\left(\langle\lambda|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\lambda\rangle+\langle\lambda|\hat{a}^\dagger\hat{a}|\lambda\rangle+\frac{1}{4}\langle\lambda|\lambda\rangle\right)$$

Using (22) and that the state $|\lambda\rangle$ is normalized, we obtain

$$\langle E^2\rangle_{|\lambda\rangle}=(\hbar\omega)^2\left(\langle\lambda|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\lambda\rangle+|\lambda|^2+\frac{1}{4}\right) \tag{24}$$

We'll now calculate the inner product $\langle\lambda|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\lambda\rangle$.

Using again the general notation for inner products, we have

$$\langle\lambda|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\lambda\rangle \equiv \left(|\lambda\rangle, \hat{a}^\dagger\hat{a}\hat{a}^\dagger \underbrace{\hat{a}|\lambda\rangle}_{\lambda|\lambda\rangle}\right) = \lambda\left(|\lambda\rangle, \hat{a}^\dagger\left(\hat{a}\hat{a}^\dagger|\lambda\rangle\right)\right) = \lambda\left(\underbrace{\left(\hat{a}^\dagger\right)^\dagger}_{\hat{a}}|\lambda\rangle, \hat{a}\hat{a}^\dagger|\lambda\rangle\right) =$$

$$= \lambda\left(\underbrace{\hat{a}|\lambda\rangle}_{\lambda|\lambda\rangle}, \hat{a}\hat{a}^\dagger|\lambda\rangle\right) = \lambda\lambda^*\left(|\lambda\rangle, \hat{a}\hat{a}^\dagger|\lambda\rangle\right)$$

That is

$$\langle\lambda|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\lambda\rangle = |\lambda|^2\left(|\lambda\rangle, \hat{a}\hat{a}^\dagger|\lambda\rangle\right) \tag{25}$$

To calculate the inner product $\left(|\lambda\rangle, \hat{a}\hat{a}^\dagger|\lambda\rangle\right)$, we use the commutator $\left[\hat{a}, \hat{a}^\dagger\right] = 1$ to replace $\hat{a}\hat{a}^\dagger$ with $1+\hat{a}^\dagger\hat{a}$.

Thus, we have

$$\left(|\lambda\rangle, \hat{a}\hat{a}^\dagger|\lambda\rangle\right) = \left(|\lambda\rangle, \left(1+\hat{a}^\dagger\hat{a}\right)|\lambda\rangle\right) = \left(|\lambda\rangle, |\lambda\rangle\right) + \left(|\lambda\rangle, \hat{a}^\dagger\hat{a}|\lambda\rangle\right)$$

Using (22) and that $\left(|\lambda\rangle, |\lambda\rangle\right) = 1$ (the state $|\lambda\rangle$ is normalized), we obtain

$$\left(|\lambda\rangle, \hat{a}\hat{a}^\dagger|\lambda\rangle\right) = 1 + |\lambda|^2$$

Substituting into (25) yields

$$\langle\lambda|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\lambda\rangle = |\lambda|^4 + |\lambda|^2 \tag{26}$$

By means of (26), (24) becomes

$$\left\langle E^2\right\rangle_{|\lambda\rangle} = (\hbar\omega)^2\left(|\lambda|^4 + |\lambda|^2 + |\lambda|^2 + \frac{1}{4}\right) = (\hbar\omega)^2\left(|\lambda|^2 + \underbrace{|\lambda|^4 + |\lambda|^2 + \frac{1}{4}}_{\left(|\lambda|^2+\frac{1}{2}\right)^2}\right) =$$

$$= (\hbar\omega)^2\left(|\lambda|^2 + \left(|\lambda|^2 + \frac{1}{2}\right)^2\right) = \left(|\lambda|\hbar\omega\right)^2 + \left(\left(|\lambda|^2 + \frac{1}{2}\right)\hbar\omega\right)^2 = \left(|\lambda|\hbar\omega\right)^2 + \left(\langle E\rangle_{|\lambda\rangle}\right)^2$$

where, in the last equality, we used (23).

Thus

$$\left\langle E^2\right\rangle_{|\lambda\rangle} = \left(|\lambda|\hbar\omega\right)^2 + \left(\langle E\rangle_{|\lambda\rangle}\right)^2 \tag{27}$$

Then, the energy uncertainty in the coherent state $|\lambda\rangle$ is

$$(\Delta E)_{|\lambda\rangle} = \sqrt{(|\lambda|\hbar\omega)^2 + \left(\langle E\rangle_{|\lambda\rangle}\right)^2 - \left(\langle E\rangle_{|\lambda\rangle}\right)^2} = |\lambda|\hbar\omega$$

That is

$$(\Delta E)_{|\lambda\rangle} = |\lambda|\hbar\omega \tag{28}$$

Another way of calculating the energy expectation value and uncertainty in the coherent state $|\lambda\rangle$ is by using its expansion in the energy basis of the QHO.

In i, we showed that the state $|\lambda\rangle$ is generated by the action of the displacement operator in the ground state, i.e. $|\lambda\rangle = \hat{D}(\lambda)|0\rangle$. Since the displacement operator is the operator $\hat{T}_{p_1,x_1}$, we have

$$|\lambda\rangle = \hat{T}_{p_1,x_1}|0\rangle \tag{29}$$

with $\lambda = \frac{1}{\sqrt{2}}\left(\frac{x_1}{x_0} + i\frac{p_1}{p_0}\right)$.

Also, in the exercise 6, we showed that the expansion of the state $\hat{T}_{p_1,x_1}|0\rangle$ in the energy basis of the QHO is

$$\hat{T}_{p_1,x_1}|0\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle$$

By means of (29), the previous expansion is written as

$$|\lambda\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle \tag{30}$$

This is the expansion of the coherent state $|\lambda\rangle$ in the energy basis of the QHO.

Using (30), the action of the Hamiltonian on the state $|\lambda\rangle$ yields

$$\hat{H}|\lambda\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\hat{H}|n\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}E_n|n\rangle$$

Using that $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$, we obtain

$$\hat{H}|\lambda\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\left(n+\frac{1}{2}\right)\hbar\omega|n\rangle =$$

$$= \exp\left(-\frac{1}{2}|\lambda|^2\right)\left(\sum_{n=1}^{\infty}\frac{\lambda^n}{\sqrt{(n-1)!}}\sqrt{n}|n\rangle + \frac{1}{2}\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle\right)\hbar\omega$$

Changing the summation index of the first series to $n' = n-1$, we obtain

$$\sum_{n=1}^{\infty}\frac{\lambda^n}{\sqrt{(n-1)!}}|n\rangle = \sum_{n'=0}^{\infty}\frac{\lambda^{n'+1}}{\sqrt{n'!}}\sqrt{n'+1}|n'+1\rangle$$

Changing again the summation index to n, we end up to

$$\sum_{n=1}^{\infty}\frac{\lambda^n}{\sqrt{(n-1)!}}|n\rangle = \sum_{n=0}^{\infty}\frac{\lambda^{n+1}}{\sqrt{n!}}\sqrt{n+1}|n+1\rangle$$

Thus, the action of the Hamiltonian on $|\lambda\rangle$ yields

$$\hat{H}|\lambda\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\left(\sum_{n=0}^{\infty}\frac{\lambda^{n+1}}{\sqrt{n!}}\sqrt{n+1}|n+1\rangle + \frac{1}{2}\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle\right)\hbar\omega \tag{31}$$

From (30), the bra $\langle\lambda|$ is

$$\langle\lambda| = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{m=0}^{\infty}\frac{\left(\lambda^m\right)^*}{\sqrt{m!}}\langle m| \tag{32}$$

Using (31) and (32), the energy expectation value of the QHO in the state $|\lambda\rangle$ is

$$\langle E\rangle_{|\lambda\rangle} = \langle\lambda|\hat{H}|\lambda\rangle =$$

$$= \left(\exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{m=0}^{\infty}\frac{\left(\lambda^m\right)^*}{\sqrt{m!}}\langle m|\right)\left(\exp\left(-\frac{1}{2}|\lambda|^2\right)\left(\sum_{n=0}^{\infty}\frac{\lambda^{n+1}}{\sqrt{n!}}\sqrt{n+1}|n+1\rangle + \frac{1}{2}\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle\right)\hbar\omega\right) =$$

$$= \exp\left(-|\lambda|^2\right)\left(\sum_{m,n=0}^{\infty}\frac{\left(\lambda^m\right)^*}{\sqrt{m!}}\frac{\lambda^{n+1}}{\sqrt{n!}}\sqrt{n+1}\langle m|n+1\rangle + \frac{1}{2}\sum_{m,n=0}^{\infty}\frac{\left(\lambda^m\right)^*}{\sqrt{m!}}\frac{\lambda^n}{\sqrt{n!}}\langle m|n\rangle\right)\hbar\omega$$

Using the orthonormality of the energy eigenstates, i.e.

$\langle m|n+1\rangle = \delta_{m,n+1}$ and $\langle m|n\rangle = \delta_{mn}$

we obtain

$$\langle E\rangle_{|\lambda\rangle} = \exp\left(-|\lambda|^2\right)\left(\sum_{m,n=0}^{\infty}\frac{\left(\lambda^m\right)^*}{\sqrt{m!}}\frac{\lambda^{n+1}}{\sqrt{n!}}\sqrt{n+1}\delta_{m,n+1}+\frac{1}{2}\sum_{m,n=0}^{\infty}\frac{\left(\lambda^m\right)^*}{\sqrt{m!}}\frac{\lambda^n}{\sqrt{n!}}\delta_{mn}\right)\hbar\omega =$$

$$= \exp\left(-|\lambda|^2\right)\left(\sum_{n=0}^{\infty}\frac{\left(\lambda^{n+1}\right)^*}{\sqrt{(n+1)!}}\frac{\lambda^{n+1}}{\sqrt{n!}}\sqrt{n+1}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{\left(\lambda^n\right)^*}{\sqrt{n!}}\frac{\lambda^n}{\sqrt{n!}}\right)\hbar\omega =$$

$$= \exp\left(-|\lambda|^2\right)\left(\sum_{n=0}^{\infty}\frac{\left(\lambda^{n+1}\right)^*}{\sqrt{n!}}\frac{\lambda^{n+1}}{\sqrt{n!}}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{\left(\lambda^n\right)^*}{\sqrt{n!}}\frac{\lambda^n}{\sqrt{n!}}\right)\hbar\omega =$$

$$= \exp\left(-|\lambda|^2\right)\left(\sum_{n=0}^{\infty}\frac{\left(\lambda^n\right)^*\lambda^*}{\sqrt{n!}}\frac{\lambda^n\lambda}{\sqrt{n!}}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{\left(\lambda^n\right)^*\lambda^n}{n!}\right)\hbar\omega =$$

$$= \exp\left(-|\lambda|^2\right)\left(\lambda^*\lambda\sum_{n=0}^{\infty}\frac{\left(\lambda^n\right)^*\lambda^n}{n!}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{\left(\lambda^n\right)^*\lambda^n}{n!}\right)\hbar\omega =$$

$$= \exp\left(-|\lambda|^2\right)\left(|\lambda|^2+\frac{1}{2}\right)\left(\sum_{n=0}^{\infty}\frac{\left|\lambda^n\right|^2}{n!}\right)\hbar\omega$$

But

$$\sum_{n=0}^{\infty}\frac{\left|\lambda^n\right|^2}{n!}=\sum_{n=0}^{\infty}\frac{\left(|\lambda|^n\right)^2}{n!}=\sum_{n=0}^{\infty}\frac{|\lambda|^{2n}}{n!}=\sum_{n=0}^{\infty}\frac{\left(|\lambda|^2\right)^n}{n!}=\exp\left(|\lambda|^2\right)$$

Thus

$$\langle E\rangle_{|\lambda\rangle} = \exp\left(-|\lambda|^2\right)\left(|\lambda|^2+\frac{1}{2}\right)\exp\left(|\lambda|^2\right)\hbar\omega = \left(|\lambda|^2+\frac{1}{2}\right)\hbar\omega$$

That is

$$\langle E\rangle_{|\lambda\rangle} = \left(|\lambda|^2+\frac{1}{2}\right)\hbar\omega$$

which is the relation (23).

Working in a similar way, we have

$$\hat{H}^2|\lambda\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\hat{H}^2|n\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}E_n{}^2|n\rangle =$$

$$= \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\left(\left(n+\frac{1}{2}\right)\hbar\omega\right)^2|n\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\left(\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\left(n^2+n+\frac{1}{4}\right)|n\rangle\right)(\hbar\omega)^2 =$$

$$= \exp\left(-\frac{1}{2}|\lambda|^2\right)\left(\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\left(n^2+n+\frac{1}{4}\right)|n\rangle\right)(\hbar\omega)^2$$

Using that $n^2 = n(n-1)+n$, we obtain

$$\hat{H}^2|\lambda\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\left(\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\left(n(n-1)+2n+\frac{1}{4}\right)|n\rangle\right)(\hbar\omega)^2 =$$

$$= \exp\left(-\frac{1}{2}|\lambda|^2\right)\left(\sum_{n=2}^{\infty}\frac{\lambda^n}{\sqrt{(n-2)!}}\sqrt{n(n-1)}\,|n\rangle + 2\sum_{n=1}^{\infty}\frac{\lambda^n}{\sqrt{(n-1)!}}\sqrt{n}\,|n\rangle + \frac{1}{4}\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle\right)(\hbar\omega)^2 =$$

$$= \exp\left(-\frac{1}{2}|\lambda|^2\right)\left(\sum_{n=0}^{\infty}\frac{\lambda^{n+2}}{\sqrt{n!}}\sqrt{(n+2)(n+1)}\,|n+2\rangle + 2\sum_{n=0}^{\infty}\frac{\lambda^{n+1}}{\sqrt{n!}}\sqrt{n+1}\,|n+1\rangle + \frac{1}{4}\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle\right)(\hbar\omega)^2$$

where, in the first series, we changed the summation index, first to $n' = n-2$, and then again to n, and in the second series, we changed the summation index, first to $n'' = n-1$, and then again to n.

Thus

$$\hat{H}^2|\lambda\rangle =$$
$$= \exp\left(-\frac{1}{2}|\lambda|^2\right)\left(\sum_{n=0}^{\infty}\frac{\lambda^{n+2}}{\sqrt{n!}}\sqrt{(n+2)(n+1)}\,|n+2\rangle + 2\sum_{n=0}^{\infty}\frac{\lambda^{n+1}}{\sqrt{n!}}\sqrt{n+1}\,|n+1\rangle + \frac{1}{4}\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle\right)(\hbar\omega)^2 \quad (33)$$

The expectation value of the energy squared in the state $|\lambda\rangle$ is

$$\left\langle E^2\right\rangle_{|\lambda\rangle} = \langle\lambda|\hat{H}^2|\lambda\rangle$$

By means of (32) and (33), the previous expression becomes

$$\left\langle E^2\right\rangle_{|\lambda\rangle} = \left(\exp\left(-|\lambda|^2\right)\right)$$
$$\left(\sum_{m,n=0}^{\infty}\frac{\left(\lambda^m\right)^*}{\sqrt{m!}}\frac{\lambda^{n+2}}{\sqrt{n!}}\sqrt{(n+2)(n+1)}\,\delta_{m,n+2} + 2\sum_{m,n=0}^{\infty}\frac{\left(\lambda^m\right)^*}{\sqrt{m!}}\frac{\lambda^{n+1}}{\sqrt{n!}}\sqrt{n+1}\,\delta_{m,n+1} + \frac{1}{4}\sum_{m,n=0}^{\infty}\frac{\left(\lambda^m\right)^*}{\sqrt{m!}}\frac{\lambda^n}{\sqrt{n!}}\delta_{m,n}\right)(\hbar\omega)^2$$

But

$$\sum_{m,n=0}^{\infty}\frac{\left(\lambda^m\right)^*}{\sqrt{m!}}\frac{\lambda^{n+2}}{\sqrt{n!}}\sqrt{(n+2)(n+1)}\delta_{m,n+2}+2\sum_{m,n=0}^{\infty}\frac{\left(\lambda^m\right)^*}{\sqrt{m!}}\frac{\lambda^{n+1}}{\sqrt{n!}}\sqrt{n+1}\delta_{m,n+1}+\frac{1}{4}\sum_{m,n=0}^{\infty}\frac{\left(\lambda^m\right)^*}{\sqrt{m!}}\frac{\lambda^n}{\sqrt{n!}}\delta_{m,n}=$$

$$=\sum_{n=0}^{\infty}\frac{\left(\lambda^{n+2}\right)^*}{\sqrt{(n+2)!}}\frac{\lambda^{n+2}}{\sqrt{n!}}\sqrt{(n+2)(n+1)}+2\sum_{n=0}^{\infty}\frac{\left(\lambda^{n+1}\right)^*}{\sqrt{(n+1)!}}\frac{\lambda^{n+1}}{\sqrt{n!}}\sqrt{n+1}+\frac{1}{4}\sum_{n=0}^{\infty}\frac{\left(\lambda^n\right)^*}{\sqrt{n!}}\frac{\lambda^n}{\sqrt{n!}}=$$

$$=\sum_{n=0}^{\infty}\frac{\left(\lambda^{n+2}\right)^*}{\sqrt{n!}}\frac{\lambda^{n+2}}{\sqrt{n!}}+2\sum_{n=0}^{\infty}\frac{\left(\lambda^{n+1}\right)^*}{\sqrt{n!}}\frac{\lambda^{n+1}}{\sqrt{n!}}+\frac{1}{4}\sum_{n=0}^{\infty}\frac{\left(\lambda^n\right)^*}{\sqrt{n!}}\frac{\lambda^n}{\sqrt{n!}}=$$

$$=\sum_{n=0}^{\infty}\frac{\left(\lambda^n\right)^*\left(\lambda^2\right)^*}{\sqrt{n!}}\frac{\lambda^n\lambda^2}{\sqrt{n!}}+2\sum_{n=0}^{\infty}\frac{\left(\lambda^n\right)^*\lambda^*}{\sqrt{n!}}\frac{\lambda^n\lambda}{\sqrt{n!}}+\frac{1}{4}\sum_{n=0}^{\infty}\frac{\left(\lambda^n\right)^*}{\sqrt{n!}}\frac{\lambda^n}{\sqrt{n!}}=$$

$$=\lambda^2\left(\lambda^2\right)^*\sum_{n=0}^{\infty}\frac{\left(\lambda^n\right)^*}{\sqrt{n!}}\frac{\lambda^n}{\sqrt{n!}}+2\lambda\lambda^*\sum_{n=0}^{\infty}\frac{\left(\lambda^n\right)^*}{\sqrt{n!}}\frac{\lambda^n}{\sqrt{n!}}+\frac{1}{4}\sum_{n=0}^{\infty}\frac{\left(\lambda^n\right)^*}{\sqrt{n!}}\frac{\lambda^n}{\sqrt{n!}}=$$

$$=\left(\lambda^2\left(\lambda^2\right)^*+2\lambda\lambda^*+\frac{1}{4}\right)\sum_{n=0}^{\infty}\frac{\left|\lambda^n\right|^2}{n!}=\left(\lambda^2\left(\lambda^*\right)^2+2|\lambda|^2+\frac{1}{4}\right)\sum_{n=0}^{\infty}\frac{\left(|\lambda|^n\right)^2}{n!}=$$

$$=\left(\left(\lambda\lambda^*\right)^2+2|\lambda|^2+\frac{1}{4}\right)\sum_{n=0}^{\infty}\frac{|\lambda|^{2n}}{n!}=\left(\left(|\lambda|^2\right)^2+2|\lambda|^2+\frac{1}{4}\right)\underbrace{\sum_{n=0}^{\infty}\frac{\left(|\lambda|^2\right)^n}{n!}}_{\exp\left(|\lambda|^2\right)}=$$

$$=\left(|\lambda|^4+2|\lambda|^2+\frac{1}{4}\right)\exp\left(|\lambda|^2\right)$$

Thus

$$\left\langle E^2\right\rangle_{|\lambda\rangle}=\left(\exp\left(-|\lambda|^2\right)\right)\left(|\lambda|^4+2|\lambda|^2+\frac{1}{4}\right)\exp\left(|\lambda|^2\right)(\hbar\omega)^2=\left(|\lambda|^4+2|\lambda|^2+\frac{1}{4}\right)(\hbar\omega)^2=$$

$$=\left(\left(|\lambda|^2+\frac{1}{2}\right)^2+|\lambda|^2\right)(\hbar\omega)^2=\left(\left(|\lambda|^2+\frac{1}{2}\right)\hbar\omega\right)^2+\left(|\lambda|\hbar\omega\right)^2$$

That is

$$\left\langle E^2\right\rangle_{|\lambda\rangle}=\left(|\lambda|\hbar\omega\right)^2+\left(\left(|\lambda|^2+\frac{1}{2}\right)\hbar\omega\right)^2$$

or, using (23),

$$\left\langle E^2\right\rangle_{|\lambda\rangle}=\left(|\lambda|\hbar\omega\right)^2+\left(\langle E\rangle_{|\lambda\rangle}\right)^2$$

Therefore, the energy uncertainty in the coherent state $|\lambda\rangle$ is

$$(\Delta E)_{|\lambda\rangle} = \sqrt{(|\lambda|\hbar\omega)^2 + \left(\langle E\rangle_{|\lambda\rangle}\right)^2 - \left(\langle E\rangle_{|\lambda\rangle}\right)^2} = |\lambda|\hbar\omega$$

That is

$$(\Delta E)_{|\lambda\rangle} = |\lambda|\hbar\omega$$

which is the relation (28).

iv) As we saw, the expansion of the state $|\lambda\rangle$ in the energy basis is given by (30).

Also, the time evolution of the energy eigenstate $|n\rangle$ is

$$|n\rangle_t = \exp\left(-\frac{iE_n t}{\hbar}\right)|n\rangle$$

Thus, the time evolution of the state $|\lambda\rangle$ – let us denote it by $|\lambda(t)\rangle$ – is

$$|\lambda(t)\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle_t = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\exp\left(-\frac{iE_n t}{\hbar}\right)|n\rangle$$

That is

$$|\lambda(t)\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\exp\left(-\frac{iE_n t}{\hbar}\right)|n\rangle \qquad (34)$$

Then, the action of the annihilation operator on the time evolution of the coherent state $|\lambda\rangle$ yields

$$\hat{a}|\lambda(t)\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\exp\left(-\frac{iE_n t}{\hbar}\right)\hat{a}|n\rangle$$

Using that $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$, we obtain

$$\hat{a}|\lambda(t)\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=1}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\exp\left(-\frac{iE_n t}{\hbar}\right)\sqrt{n}|n-1\rangle =$$
$$= \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=1}^{\infty}\frac{\lambda^n}{\sqrt{(n-1)!}}\exp\left(-\frac{iE_n t}{\hbar}\right)|n-1\rangle$$

Changing the summation index to $n' = n-1$, we obtain

$$\hat{a}\left|\lambda(t)\right\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n'=0}^{\infty}\frac{\lambda^{n'+1}}{\sqrt{n'!}}\exp\left(-\frac{iE_{n'+1}t}{\hbar}\right)\left|n'\right\rangle =$$
$$= \lambda\exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n'=0}^{\infty}\frac{\lambda^{n'}}{\sqrt{n'!}}\exp\left(-\frac{iE_{n'+1}t}{\hbar}\right)\left|n'\right\rangle$$

Besides, using that $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$, we derive that

$$E_{n'+1} = E_{n'} + \hbar\omega$$

Substituting into the expression of $\hat{a}\left|\lambda(t)\right\rangle$, we obtain

$$\hat{a}\left|\lambda(t)\right\rangle = \lambda\exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n'=0}^{\infty}\frac{\lambda^{n'}}{\sqrt{n'!}}\exp\left(-\frac{i\left(E_{n'} + \hbar\omega\right)t}{\hbar}\right)\left|n'\right\rangle =$$
$$= \lambda\exp(-i\omega t)\underbrace{\exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n'=0}^{\infty}\frac{\lambda^{n'}}{\sqrt{n'!}}\exp\left(-\frac{iE_{n'}t}{\hbar}\right)\left|n'\right\rangle}_{\left|\lambda(t)\right\rangle} = \lambda\exp(-i\omega t)\left|\lambda(t)\right\rangle$$

That is

$$\hat{a}\left|\lambda(t)\right\rangle = \lambda\exp(-i\omega t)\left|\lambda(t)\right\rangle \tag{35}$$

Therefore, the state $\left|\lambda(t)\right\rangle$ is an eigenstate of the annihilation operator, and thus it remains a coherent state, but its eigenvalue, $\lambda\exp(-i\omega t)$, changes periodically with time.

We observe that the magnitude of the eigenvalue of $\left|\lambda(t)\right\rangle$ is

$$\left|\lambda\exp(-i\omega t)\right| = |\lambda|\underbrace{\left|\exp(-i\omega t)\right|}_{1} = |\lambda|$$

That is, the magnitude of the eigenvalue of $\left|\lambda(t)\right\rangle$ is constant.

To summarize, the time evolution of a coherent state is a coherent state with eigenvalue having constant magnitude but time-dependent phase.

8) Express the energy expectation value in a coherent state $\left|\lambda\right\rangle$ of the QHO in terms of the expectation values of the position and momentum.

Solution

In the previous exercise, we showed that the energy expectation value in a coherent state $|\lambda\rangle$ is

$$\langle E\rangle_{|\lambda\rangle} = \left(|\lambda|^2 + \frac{1}{2}\right)\hbar\omega \qquad (1)$$

Since the state $|\lambda\rangle$ is an eigenstate of $\hat{a}$ with eigenvalue λ, we have

$$\hat{a}|\lambda\rangle = \lambda|\lambda\rangle$$

Then, the expectation value of $\hat{a}$ in the state $|\lambda\rangle$ is

$$\langle \hat{a}\rangle_{|\lambda\rangle} = \langle\lambda|\hat{a}|\lambda\rangle = \langle\lambda|\lambda|\lambda\rangle = \lambda\underbrace{\langle\lambda|\lambda\rangle}_{1}$$

That is

$$\langle \hat{a}\rangle_{|\lambda\rangle} = \lambda \qquad (2)$$

In terms of the length and momentum scales, the annihilation operator is written as

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{x_0} + i\frac{\hat{p}}{p_0}\right)$$

Thus

$$\langle \hat{a}\rangle_{|\lambda\rangle} = \frac{1}{\sqrt{2}}\left(\frac{\langle x\rangle_{|\lambda\rangle}}{x_0} + i\frac{\langle p\rangle_{|\lambda\rangle}}{p_0}\right)$$

Comparing the previous equation with (2) yields

$$\frac{1}{\sqrt{2}}\left(\frac{\langle x\rangle_{|\lambda\rangle}}{x_0} + i\frac{\langle p\rangle_{|\lambda\rangle}}{p_0}\right) = \lambda = \operatorname{Re}\lambda + i\operatorname{Im}\lambda$$

or

$$\frac{1}{\sqrt{2}}\frac{\langle x\rangle_{|\lambda\rangle}}{x_0} + i\frac{1}{\sqrt{2}}\frac{\langle p\rangle_{|\lambda\rangle}}{p_0} = \operatorname{Re}\lambda + i\operatorname{Im}\lambda$$

Since the expectation values of the position and momentum are real, the previous equation gives

$$\operatorname{Re}\lambda = \frac{1}{\sqrt{2}}\frac{\langle x\rangle_{|\lambda\rangle}}{x_0} \tag{3}$$

$$\operatorname{Im}\lambda = \frac{1}{\sqrt{2}}\frac{\langle p\rangle_{|\lambda\rangle}}{p_0} \tag{4}$$

Using (3) and (4), the square of the magnitude of λ is

$$|\lambda|^2 = (\operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda)^2 = \frac{1}{2}\left(\left(\frac{\langle x\rangle_{|\lambda\rangle}}{x_0}\right)^2 + \left(\frac{\langle p\rangle_{|\lambda\rangle}}{p_0}\right)^2\right)$$

That is

$$|\lambda|^2 = \frac{1}{2}\left(\left(\frac{\langle x\rangle_{|\lambda\rangle}}{x_0}\right)^2 + \left(\frac{\langle p\rangle_{|\lambda\rangle}}{p_0}\right)^2\right) \tag{5}$$

By means of (5), (1) becomes

$$\langle E\rangle_{|\lambda\rangle} = \frac{1}{2}\left(\left(\frac{\langle x\rangle_{|\lambda\rangle}}{x_0}\right)^2 + \left(\frac{\langle p\rangle_{|\lambda\rangle}}{p_0}\right)^2 + 1\right)\hbar\omega$$

Substituting $x_0 = \sqrt{\frac{\hbar}{m\omega}}$ and $p_0 = \sqrt{m\hbar\omega}$ into the previous expression, we obtain

$$\langle E\rangle_{|\lambda\rangle} = \frac{1}{2}\left(\frac{\left(\langle x\rangle_{|\lambda\rangle}\right)^2}{\frac{\hbar}{m\omega}} + \frac{\left(\langle p\rangle_{|\lambda\rangle}\right)^2}{m\hbar\omega} + 1\right)\hbar\omega = \frac{1}{2}\left(m\omega^2\left(\langle x\rangle_{|\lambda\rangle}\right)^2 + \frac{\left(\langle p\rangle_{|\lambda\rangle}\right)^2}{m} + \hbar\omega\right) =$$

$$= \frac{\left(\langle p\rangle_{|\lambda\rangle}\right)^2}{2m} + \frac{1}{2}m\omega^2\left(\langle x\rangle_{|\lambda\rangle}\right)^2 + \frac{\hbar\omega}{2}$$

That is

$$\langle E\rangle_{|\lambda\rangle} = \frac{\left(\langle p\rangle_{|\lambda\rangle}\right)^2}{2m} + \frac{1}{2}m\omega^2\left(\langle x\rangle_{|\lambda\rangle}\right)^2 + \frac{\hbar\omega}{2} \tag{6}$$

Observe that $\langle E\rangle_{|\lambda\rangle} \geq \frac{\hbar\omega}{2}$, as expected, since $\frac{\hbar\omega}{2}$ is the ground-state energy of the QHO, and the equality holds if and only if the expectation values of the position and momentum are both zero in the coherent state $|\lambda\rangle$. There is only one coherent state in which the position and momentum expectation values are both zero, and that coherent state is the ground state.

In the previous exercise (question ii), we showed that

$$|\lambda\rangle = \hat{T}_{p_1}\hat{T}_{x_1}|0\rangle \tag{7}$$

Using (7), it is easily shown that $\langle x\rangle_{|\lambda\rangle} = x_1$ and $\langle p\rangle_{|\lambda\rangle} = p_1$.

Then, (6) is written as

$$\langle E\rangle_{|\lambda\rangle} = \frac{p_1^{\ 2}}{2m} + \frac{1}{2}m\omega^2 x_1^{\ 2} + \frac{\hbar\omega}{2} \tag{8}$$

We see that $\langle E\rangle_{|\lambda\rangle} = \dfrac{\hbar\omega}{2}$ if and only if $x_1 = 0$ and $p_1 = 0$, thus if and only if

$$|\lambda\rangle = \underbrace{\hat{T}_0}_{1}\underbrace{\hat{T}_0}_{1}|0\rangle = |0\rangle$$

Therefore, $\langle E\rangle_{|\lambda\rangle} = \dfrac{\hbar\omega}{2}$ if and only if the coherent state is the ground state of the QHO.

9) Calculate the wave functions describing a coherent state $|\lambda\rangle$ in the position and momentum representations, respectively. Express the results in terms of the expectation values of the position and momentum.

Solution

The wave function of the state $|\lambda\rangle$ in the position representation is

$$\psi_\lambda(x) = \langle x|\lambda\rangle$$

In the exercise 7 (question ii), we showed that

$$|\lambda\rangle = \hat{T}_{p_1}\hat{T}_{x_1}|0\rangle$$

Then, the wave function $\psi_\lambda(x)$ is written as

$$\psi_\lambda(x) = \langle x|\hat{T}_{p_1}\hat{T}_{x_1}|0\rangle = \hat{T}_{p_1}(x)\hat{T}_{x_1}(x)\underbrace{\langle x|0\rangle}_{\psi_0(x)} = \hat{T}_{p_1}(x)\hat{T}_{x_1}(x)\psi_0(x)$$

That is

$$\psi_\lambda(x)=\hat{T}_{p_1}(x)\hat{T}_{x_1}(x)\psi_0(x) \tag{1}$$

where, by $\hat{T}_{p_1}(x)$ we denote the momentum translation operator $\hat{T}_{p_1}$ in the position representation, by $\hat{T}_{x_1}(x)$ we denote the spatial translation operator $\hat{T}_{x_1}$ in the position representation, and $\psi_0(x)$ is the ground-state wave function of the QHO in the position representation.

In the position representation, $\hat{x}=x$ and $\hat{p}=-i\hbar\dfrac{d}{dx}$.

Thus

$$\hat{T}_{p_1}(x)=\exp\left(\frac{ip_1x}{\hbar}\right)$$

and

$$\hat{T}_{x_1}(x)=\exp\left(-\frac{i\left(-i\hbar\dfrac{d}{dx}\right)x_1}{\hbar}\right)=\exp\left(-x_1\frac{d}{dx}\right)$$

Substituting into (1) yields

$$\psi_\lambda(x)=\exp\left(\frac{ip_1x}{\hbar}\right)\exp\left(-x_1\frac{d}{dx}\right)\psi_0(x) \tag{2}$$

Using the Taylor expansion of $\exp\left(-x_1\dfrac{d}{dx}\right)$, i.e.

$$\exp\left(-x_1\frac{d}{dx}\right)=\sum_{n=0}^{\infty}\frac{\left(-x_1\dfrac{d}{dx}\right)^n}{n!}=\sum_{n=0}^{\infty}\frac{(-x_1)^n}{n!}\frac{d^n}{dx^n}$$

we obtain

$$\exp\left(-x_1\frac{d}{dx}\right)\psi_0(x)=\left(\sum_{n=0}^{\infty}\frac{(-x_1)^n}{n!}\frac{d^n}{dx^n}\right)\psi_0(x)=\sum_{n=0}^{\infty}\frac{(-x_1)^n}{n!}\frac{d^n}{dx^n}\psi_0(x)=$$

$$=\sum_{n=0}^{\infty}\frac{(-x_1)^n}{n!}\psi_0^{(n)}(x)=\sum_{n=0}^{\infty}\frac{\psi_0^{(n)}(x)}{n!}(-x_1)^n=\psi_0(x-x_1)$$

We remind that the Taylor series of a (proper) function $f(x)$ about x' is

$$f(x)=\sum_{m=0}^{\infty}\frac{f^{(m)}(x')}{m!}(x-x')$$

Thus

$$\exp\left(-x_1\frac{d}{dx}\right)\psi_0(x)=\psi_0(x-x_1) \tag{3}$$

Obviously, (3) also holds for an arbitrary function $\psi(x)$ that has derivatives of all orders.

By means of (3), (2) becomes

$$\psi_\lambda(x)=\exp\left(\frac{ip_1x}{\hbar}\right)\psi_0(x-x_1) \tag{4}$$

Using that $x_1=\langle x\rangle_{|\lambda\rangle}$ and $p_1=\langle p\rangle_{|\lambda\rangle}$, (4) becomes

$$\psi_\lambda(x)=\exp\left(\frac{i\langle p\rangle_{|\lambda\rangle}x}{\hbar}\right)\psi_0\left(x-\langle x\rangle_{|\lambda\rangle}\right) \tag{5}$$

This is the wave function of the coherent state $|\lambda\rangle$ in the position representation, expressed in terms of the position and momentum expectation values.

Using that $\psi_0(x)=\frac{1}{\sqrt{x_0}}\frac{1}{\pi^{\frac{1}{4}}}\exp\left(-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right)$, the wave function $\psi_\lambda(x)$ takes the form

$$\psi_\lambda(x)=\frac{1}{\sqrt{x_0}}\frac{1}{\pi^{\frac{1}{4}}}\exp\left(-\frac{1}{2}\left(\frac{x-\langle x\rangle_{|\lambda\rangle}}{x_0}\right)^2+\frac{i\langle p\rangle_{|\lambda\rangle}x}{\hbar}\right) \tag{6}$$

where $x_0=\sqrt{\frac{\hbar}{m\omega}}$ is the length scale of the QHO.

The wave function $\tilde{\psi}_\lambda(p)$, i.e. the wave function describing the coherent state $|\lambda\rangle$ in the momentum representation, can be calculated by taking the Fourier transform of the wave function $\psi_\lambda(x)$, i.e.

$$\tilde{\psi}_\lambda(p) = \left(\frac{1}{2\pi\hbar}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} dx \psi_\lambda(x) \exp\left(-\frac{ipx}{\hbar}\right)$$

Since the wave function $\psi_\lambda(x)$ is Gaussian, the previous integral can be calculated using that

$$\int_{-\infty}^{\infty} dx \exp\left(-ax^2 + bx + c\right) = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a} + c\right), \text{ with } a > 0$$

See, for instance, https://en.wikipedia.org/wiki/Gaussian_integral.
Alternatively, we may again use that

$$|\lambda\rangle = \hat{T}_{p_1} \hat{T}_{x_1} |0\rangle$$

or better, that

$$|\lambda\rangle = \hat{T}_{x_1} \hat{T}_{p_1} |0\rangle$$

> We remind that, as shown in the exercise 4, the operators $\hat{T}_{p_1}\hat{T}_{x_1}$ and $\hat{T}_{x_1}\hat{T}_{p_1}$ differ only by a constant phase, and thus their action on an arbitrary state is physically the same.

Using the previous equation, the wave function $\tilde{\psi}_\lambda(p) = \langle p|\lambda\rangle$ is written as

$$\tilde{\psi}_\lambda(p) = \langle p|\hat{T}_{x_1}\hat{T}_{p_1}|0\rangle = \hat{T}_{x_1}(p)\hat{T}_{p_1}(p)\underbrace{\langle p|0\rangle}_{\tilde{\psi}_0(p)} = \hat{T}_{x_1}(p)\hat{T}_{p_1}(p)\tilde{\psi}_0(p)$$

That is

$$\tilde{\psi}_\lambda(p) = \hat{T}_{x_1}(p)\hat{T}_{p_1}(p)\tilde{\psi}_0(p) \tag{7}$$

where, by $\hat{T}_{x_1}(p)$ we denote the spatial translation operator $\hat{T}_{x_1}$ in the momentum representation, by $\hat{T}_{p_1}(p)$ we denote the momentum translation operator $\hat{T}_{p_1}$ in the momentum representation, and $\tilde{\psi}_0(p)$ is the ground-state wave function of the QHO in the momentum representation.

In the momentum representation, $\hat{x} = i\hbar\frac{d}{dp}$ and $\hat{p} = p$.

Thus

$$\hat{T}_{x_1}(p) = \exp\left(-\frac{ipx_1}{\hbar}\right)$$

$$\hat{T}_{p_1}(p) = \exp\left(\frac{ip_1 i\hbar\frac{d}{dp}}{\hbar}\right) = \exp\left(-p_1\frac{d}{dp}\right)$$

Substituting into (7) yields

$$\tilde{\psi}_\lambda(p) = \exp\left(-\frac{ipx_1}{\hbar}\right)\exp\left(-p_1\frac{d}{dp}\right)\tilde{\psi}_0(p)$$

From (3), it is obvious that

$$\exp\left(-p_1\frac{d}{dp}\right)\tilde{\psi}_0(p) = \tilde{\psi}_0(p - p_1)$$

Then, the wave function $\tilde{\psi}_\lambda(p)$ is written as

$$\tilde{\psi}_\lambda(p) = \exp\left(-\frac{ipx_1}{\hbar}\right)\tilde{\psi}_0(p - p_1)$$

Using again that $x_1 = \langle x\rangle_{|\lambda\rangle}$ and $p_1 = \langle p\rangle_{|\lambda\rangle}$, we end up to

$$\tilde{\psi}_\lambda(p) = \exp\left(-\frac{ip\langle x\rangle_{|\lambda\rangle}}{\hbar}\right)\tilde{\psi}_0\left(p - \langle p\rangle_{|\lambda\rangle}\right) \qquad (8)$$

This is the wave function of the coherent state $|\lambda\rangle$ in the momentum representation, expressed in terms of the position and momentum expectation values.

Using that $\tilde{\psi}_0(p) = \frac{1}{\sqrt{p_0}}\frac{1}{\pi^{\frac{1}{4}}}\exp\left(-\frac{1}{2}\left(\frac{p}{p_0}\right)^2\right)$, the wave function $\tilde{\psi}_\lambda(p)$ takes the form

$$\tilde{\psi}_\lambda(p)=\frac{1}{\sqrt{p_0}}\frac{1}{\pi^{\frac{1}{4}}}\exp\left(-\frac{1}{2}\left(\frac{p-\langle p\rangle_{|\lambda\rangle}}{p_0}\right)^2-\frac{ip\langle x\rangle_{|\lambda\rangle}}{\hbar}\right) \tag{9}$$

Note

The energy eigenfunctions of the QHO in the position representation, i.e. the functions $\psi_n(x)$, and the energy eigenfunctions of the QHO in the momentum representation, i.e. the functions $\tilde{\psi}_n(p)$, are related by $\psi_n(x)\overset{\overbrace{x_0\to p_0}^{x\to p}}{\to}\tilde{\psi}_n(p)$ and $\tilde{\psi}_n(p)\overset{\overbrace{p_0\to x_0}^{p\to x}}{\to}\psi_n(x)$.

In other words, the functions $\tilde{\psi}_n(p)$ are derived from $\psi_n(x)$ by replacing position with momentum and the length scale with the momentum scale, and, likewise, the functions $\psi_n(x)$ are derived from $\tilde{\psi}_n(p)$ by replacing momentum with position and the momentum scale with the length scale.

This is a unique property of the QHO, which is due to the form of the harmonic oscillator potential.

10) Find the time evolution of the wave functions $\psi_\lambda(x)$ and $\tilde{\psi}_\lambda(p)$ of the previous exercise.

Solution

In the exercise 7, we showed that the state $|\lambda(t)\rangle$, which is the time evolution of the coherent state $|\lambda\rangle$, is also a coherent state, but with time-varying eigenvalue

$$\lambda(t)=\lambda\exp(-i\omega t) \tag{1}$$

Since the state $|\lambda(t)\rangle$ is a coherent state, it can be written as

$$|\lambda(t)\rangle=\hat{T}_{p_1(t)}\hat{T}_{x_1(t)}|0\rangle \tag{2}$$

with $\lambda(t)=\frac{1}{\sqrt{2}}\left(\frac{x_1(t)}{x_0}+i\frac{p_1(t)}{p_0}\right)$.

We remind that, as shown in the exercise 7 (question ii), the coherent state $|\lambda\rangle$ is written as $|\lambda\rangle = \hat{T}_{p_1}\hat{T}_{x_1}|0\rangle$, with $\lambda = \frac{1}{\sqrt{2}}\left(\frac{x_1}{x_0} + i\frac{p_1}{p_0}\right)$.

Then, the time evolution of the wave function $\psi_\lambda(x)$, which describes the state $|\lambda\rangle$, is the wave function

$$\psi_\lambda(x,t) = \langle x|\lambda(t)\rangle \tag{3}$$

which describes the state $|\lambda(t)\rangle$.

Using (2), (3) becomes

$$\psi_\lambda(x,t) = \langle x|\hat{T}_{p_1(t)}\hat{T}_{x_1(t)}|0\rangle = \hat{T}_{p_1(t)}(x)\hat{T}_{x_1(t)}(x)\underbrace{\langle x|0\rangle}_{\psi_0(x)} = \hat{T}_{p_1(t)}(x)\hat{T}_{x_1(t)}(x)\psi_0(x)$$

That is

$$\psi_\lambda(x,t) = \hat{T}_{p_1(t)}(x)\hat{T}_{x_1(t)}(x)\psi_0(x) \tag{4}$$

where, by $\hat{T}_{p_1(t)}(x)$ we denote the momentum translation operator $\hat{T}_{p_1(t)}$ in the position representation, by $\hat{T}_{x_1(t)}(x)$ we denote the spatial translation operator $\hat{T}_{x_1(t)}$ in the position representation, and $\psi_0(x)$ is the ground-state wave function of the QHO in the position representation, i.e.

$$\psi_0(x) = \frac{1}{\sqrt{x_0}}\frac{1}{\pi^{\frac{1}{4}}}\exp\left(-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right)$$

In the position representation, $\hat{x} = x$ and $\hat{p} = -i\hbar\frac{d}{dx}$.

Thus

$$\hat{T}_{p_1(t)}(x) = \exp\left(\frac{ip_1(t)x}{\hbar}\right)$$

$$\hat{T}_{x_1(t)}(x) = \exp\left(-\frac{i\left(-i\hbar\frac{d}{dx}\right)x_1(t)}{\hbar}\right) = \exp\left(-x_1(t)\frac{d}{dx}\right)$$

Since the function $x_1(t)$ depends only on time, it commutes with the operator $\frac{d}{dx}$.

Thus

$$\psi_\lambda(x,t) = \exp\left(\frac{ip_1(t)x}{\hbar}\right)\exp\left(-x_1(t)\frac{d}{dx}\right)\psi_0(x) \tag{5}$$

In the previous exercise, we proved that

$$\exp\left(-x_1\frac{d}{dx}\right)\psi_0(x) = \psi_0(x-x_1)$$

Then, obviously,

$$\exp\left(-x_1(t)\frac{d}{dx}\right)\psi_0(x) = \psi_0(x-x_1(t))$$

Substituting into (5) yields

$$\psi_\lambda(x,t) = \exp\left(\frac{ip_1(t)x}{\hbar}\right)\psi_0(x-x_1(t)) \tag{6}$$

In the exercise 8, we explained that for the coherent state $|\lambda\rangle = \hat{T}_{p_1}\hat{T}_{x_1}|0\rangle$, it holds that $\langle x\rangle_{|\lambda\rangle} = x_1$ and $\langle p\rangle_{|\lambda\rangle} = p_1$.

Then, for the state (2) we have $\langle x\rangle_{|\lambda(t)\rangle} = x_1(t)$ and $\langle p\rangle_{|\lambda(t)\rangle} = p_1(t)$.

Thus, (6) becomes

$$\psi_\lambda(x,t) = \exp\left(\frac{i\langle p\rangle_{|\lambda(t)\rangle}x}{\hbar}\right)\psi_0\left(x-\langle x\rangle_{|\lambda(t)\rangle}\right) \tag{7}$$

Since the state $|\lambda(t)\rangle$ is the time evolution of the state $|\lambda\rangle$, the expectation values $\langle x\rangle_{|\lambda(t)\rangle}$ and $\langle p\rangle_{|\lambda(t)\rangle}$ are the time evolution of the expectation values $\langle x\rangle_{|\lambda\rangle}$ and $\langle p\rangle_{|\lambda\rangle}$.

Besides, substituting $\lambda = \frac{1}{\sqrt{2}}\left(\frac{x_1}{x_0} + i\frac{p_1}{p_0}\right)$ and $\lambda(t) = \frac{1}{\sqrt{2}}\left(\frac{x_1(t)}{x_0} + i\frac{p_1(t)}{p_0}\right)$ into (1), we obtain

$$\frac{1}{\sqrt{2}}\left(\frac{x_1(t)}{x_0}+i\frac{p_1(t)}{p_0}\right)=\frac{1}{\sqrt{2}}\left(\frac{x_1}{x_0}+i\frac{p_1}{p_0}\right)\exp(-i\omega t)=$$

$$=\frac{1}{\sqrt{2}}\left(\frac{x_1}{x_0}+i\frac{p_1}{p_0}\right)(\cos\omega t-i\sin\omega t)=$$

$$=\frac{1}{\sqrt{2}}\left(\frac{x_1}{x_0}\cos\omega t-i\frac{x_1}{x_0}\sin\omega t+i\frac{p_1}{p_0}\cos\omega t+\frac{p_1}{p_0}\sin\omega t\right)=$$

$$=\frac{1}{\sqrt{2}}\left(\frac{x_1}{x_0}\cos\omega t+\frac{p_1}{p_0}\sin\omega t+i\left(\frac{p_1}{p_0}\cos\omega t-\frac{x_1}{x_0}\sin\omega t\right)\right)$$

That is

$$\frac{1}{\sqrt{2}}\left(\frac{x_1(t)}{x_0}+i\frac{p_1(t)}{p_0}\right)=\frac{1}{\sqrt{2}}\left(\frac{x_1}{x_0}\cos\omega t+\frac{p_1}{p_0}\sin\omega t+i\left(\frac{p_1}{p_0}\cos\omega t-\frac{x_1}{x_0}\sin\omega t\right)\right)$$

or

$$\frac{x_1(t)}{x_0}+i\frac{p_1(t)}{p_0}=\frac{x_1}{x_0}\cos\omega t+\frac{p_1}{p_0}\sin\omega t+i\left(\frac{p_1}{p_0}\cos\omega t-\frac{x_1}{x_0}\sin\omega t\right)$$

Since the translations $x_1(t)$ and $p_1(t)$ are real, the previous equation gives

$$\frac{x_1(t)}{x_0}=\frac{x_1}{x_0}\cos\omega t+\frac{p_1}{p_0}\sin\omega t \qquad (8)$$

$$\frac{p_1(t)}{p_0}=\frac{p_1}{p_0}\cos\omega t-\frac{x_1}{x_0}\sin\omega t \qquad (9)$$

From (8) we obtain

$$x_1(t)=x_1\cos\omega t+\frac{x_0}{p_0}p_1\sin\omega t$$

Substituting the length and momentum scales into the previous equation yields

$$x_1(t)=x_1\cos\omega t+\frac{p_1}{m\omega}\sin\omega t$$

Since $\langle x\rangle_{|\lambda(t)\rangle}=x_1(t)$, the previous equation gives

$$\langle x\rangle_{|\lambda(t)\rangle}=x_1\cos\omega t+\frac{p_1}{m\omega}\sin\omega t \qquad (10)$$

Similarly, from (9) we obtain

$p_1(t) = -m\omega x_1 \sin\omega t + p_1 \cos\omega t$

Since $\langle p\rangle_{|\lambda(t)\rangle} = p_1(t)$, the previous equation gives

$$\langle p\rangle_{|\lambda(t)\rangle} = -m\omega x_1 \sin\omega t + p_1 \cos\omega t \tag{11}$$

Thus, to summarize, the time evolution of the wave function $\psi_\lambda(x)$ is given by (7), where the expectation values $\langle x\rangle_{|\lambda(t)\rangle}$ and $\langle p\rangle_{|\lambda(t)\rangle}$ are given by (10) and (11), respectively.

In the same way, we find that the time evolution of the wave function $\tilde{\psi}_\lambda(p)$ is the wave function

$$\tilde{\psi}_\lambda(p,t) = \exp\left(-\frac{ip\langle x\rangle_{|\lambda(t)\rangle}}{\hbar}\right)\tilde{\psi}_0\left(p - \langle p\rangle_{|\lambda(t)\rangle}\right) \tag{12}$$

where $\tilde{\psi}_0(p) = \frac{1}{\sqrt{p_0}}\frac{1}{\pi^{\frac{1}{4}}}\exp\left(-\frac{1}{2}\left(\frac{p}{p_0}\right)^2\right)$ is the ground-state wave function of the QHO in the momentum representation.

We leave to the reader to verify (12).

11) Overlap and overcompleteness of the coherent states.

i) Calculate the overlap between two coherent states $|\lambda\rangle$ and $|\lambda'\rangle$. What do you observe?

ii) Show that the set of all coherent states satisfy a completeness relation.

Solution

i) Using the expansion of a coherent state in the energy basis of the QHO, which we proved in the exercise 7, we have

$$|\lambda\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle$$

Thus

$$\langle\lambda'| = \exp\left(-\frac{1}{2}|\lambda'|^2\right)\sum_{m=0}^{\infty}\frac{\left(\lambda'^m\right)^*}{\sqrt{m!}}\langle m|$$

Then, the inner product $\langle\lambda'|\lambda\rangle$ is written as

$$\langle\lambda'|\lambda\rangle = \left(\exp\left(-\frac{1}{2}|\lambda'|^2\right)\sum_{m=0}^{\infty}\frac{\left(\lambda'^m\right)^*}{\sqrt{m!}}\langle m|\right)\left(\exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle\right) =$$

$$= \exp\left(-\frac{1}{2}|\lambda'|^2 - \frac{1}{2}|\lambda|^2\right)\sum_{m,n=0}^{\infty}\frac{\left(\lambda'^m\right)^*}{\sqrt{m!}}\frac{\lambda^n}{\sqrt{n!}}\langle m|n\rangle$$

Using the orthonormality of the energy eigenstates, i.e. $\langle m|n\rangle = \delta_{mn}$, we obtain

$$\langle\lambda'|\lambda\rangle = \exp\left(-\frac{1}{2}|\lambda'|^2 - \frac{1}{2}|\lambda|^2\right)\sum_{m,n=0}^{\infty}\frac{\left(\lambda'^m\right)^*}{\sqrt{m!}}\frac{\lambda^n}{\sqrt{n!}}\delta_{mn} =$$

$$= \exp\left(-\frac{1}{2}|\lambda'|^2 - \frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\left(\lambda'^n\right)^*\lambda^n}{n!}$$

Now, using that $\left(\lambda'^n\right)^* = \left(\lambda'^*\right)^n$, we obtain

$$\langle\lambda'|\lambda\rangle = \exp\left(-\frac{1}{2}|\lambda'|^2 - \frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\left(\lambda'^*\right)^n\lambda^n}{n!} = \exp\left(-\frac{1}{2}|\lambda'|^2 - \frac{1}{2}|\lambda|^2\right)\underbrace{\sum_{n=0}^{\infty}\frac{\left(\lambda'^*\lambda\right)^n}{n!}}_{\exp\left(\lambda'^*\lambda\right)} =$$

$$= \exp\left(-\frac{1}{2}|\lambda'|^2 - \frac{1}{2}|\lambda|^2\right)\exp\left(\lambda'^*\lambda\right) = \exp\left(-\frac{1}{2}|\lambda'|^2 - \frac{1}{2}|\lambda|^2 + \lambda'^*\lambda\right) =$$

$$= \exp\left(-\frac{1}{2}\left(|\lambda|^2 + |\lambda'|^2 - 2\lambda\lambda'^*\right)\right)$$

That is

$$\langle\lambda'|\lambda\rangle = \exp\left(-\frac{1}{2}\left(|\lambda|^2 + |\lambda'|^2 - 2\lambda\lambda'^*\right)\right) \tag{1}$$

If $\lambda \neq \lambda'$, (1) gives $\langle\lambda'|\lambda\rangle \neq 0$, i.e. the states are not orthogonal, they overlap.

The annihilation operator is not Hermitian, thus two eigenstates with different eigenvalues, i.e. two different coherent states, are not orthogonal.

We'll now write the term $|\lambda|^2 + |\lambda'|^2 - 2\lambda\lambda'^*$ as a square plus or minus something.

We have

$$|\lambda|^2 + |\lambda'|^2 - 2\lambda\lambda'^* = \lambda\lambda^* + \lambda'\lambda'^* - \lambda\lambda'^* - \lambda\lambda'^* = \underbrace{\lambda\lambda^* + \lambda'\lambda'^* - \lambda\lambda'^* - \lambda^*\lambda'}_{(\lambda-\lambda')(\lambda^*-\lambda'^*)} + \lambda^*\lambda' - \lambda\lambda'^* =$$

$$= (\lambda - \lambda')(\lambda^* - \lambda'^*) + \underbrace{\lambda^*\lambda' - (\lambda^*\lambda')^*}_{2i\,\mathrm{Im}(\lambda^*\lambda')} = (\lambda - \lambda')(\lambda - \lambda')^* + 2i\,\mathrm{Im}(\lambda^*\lambda') =$$

$$= |\lambda - \lambda'|^2 + 2i\,\mathrm{Im}(\lambda^*\lambda')$$

That is

$$|\lambda|^2 + |\lambda'|^2 - 2\lambda\lambda'^* = |\lambda - \lambda'|^2 + 2i\,\mathrm{Im}(\lambda^*\lambda')$$

Substituting into (1) yields

$$\langle\lambda'|\lambda\rangle = \exp\left(-\frac{1}{2}\left(|\lambda - \lambda'|^2 + 2i\,\mathrm{Im}(\lambda^*\lambda')\right)\right) = \exp\left(-\frac{1}{2}|\lambda - \lambda'|^2 - i\,\mathrm{Im}(\lambda^*\lambda')\right) =$$

$$= \exp\left(-\frac{1}{2}|\lambda - \lambda'|^2\right)\exp\left(-i\,\mathrm{Im}(\lambda^*\lambda')\right)$$

That is

$$\langle\lambda'|\lambda\rangle = \exp\left(-i\,\mathrm{Im}(\lambda^*\lambda')\right)\exp\left(-\frac{1}{2}|\lambda - \lambda'|^2\right) \qquad (2)$$

The overlap between the two coherent states is the absolute value of $\langle\lambda'|\lambda\rangle$, i.e.

$$|\langle\lambda'|\lambda\rangle| = \exp\left(-\frac{1}{2}|\lambda - \lambda'|^2\right) \qquad (3)$$

Since $|\lambda - \lambda'| \geq 0$, the maximum overlap happens when $\lambda = \lambda'$, i.e. when the two eigenvalues are equal, i.e. when the two coherent states coincide, and it is equal to the norm of the state to the square, i.e. 1.
If the distance between the eigenvalues of the two coherent states is large, then the term $|\lambda - \lambda'|$ is large, and the overlap between the two states is very small.

On the contrary, if the distance between the two eigenvalues is small, i.e. if the eigenvalues are close to each other in the complex plane, the overlap tends to 1.
ii) Using again the expansion of a coherent state in the energy basis of the QHO, the integral $\int d\lambda |\lambda\rangle\langle\lambda|$, where $\lambda \in \mathbb{C}$, is written as

$$\int d\lambda |\lambda\rangle\langle\lambda| = \int d\lambda \left(\exp\left(-\frac{1}{2}|\lambda|^2\right) \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} |n\rangle \right) \left(\exp\left(-\frac{1}{2}|\lambda|^2\right) \sum_{m=0}^{\infty} \frac{\left(\lambda^m\right)^*}{\sqrt{m!}} \langle m| \right) =$$

$$= \int d\lambda \exp\left(-|\lambda|^2\right) \sum_{m,n=0}^{\infty} \frac{\left(\lambda^*\right)^m \lambda^n}{\sqrt{m!n!}} |n\rangle\langle m|$$

In polar coordinates, the complex number λ is written as $\lambda = re^{i\varphi}$. Thus, its complex conjugate λ^* is $\lambda^* = re^{-i\varphi}$. In polar coordinates, the differential $d\lambda$ is written as $d\lambda = rdrd\varphi$, where r is from 0 to ∞, while the polar angle φ is from 0 to 2π (excluded). Thus

$$\int d\lambda |\lambda\rangle\langle\lambda| = \int rdrd\varphi \exp\left(-r^2\right) \sum_{m,n=0}^{\infty} \frac{\left(re^{-i\varphi}\right)^m \left(re^{i\varphi}\right)^n}{\sqrt{m!n!}} |n\rangle\langle m| =$$

$$= \int rdrd\varphi \exp\left(-r^2\right) \sum_{m,n=0}^{\infty} \frac{r^{m+n} e^{i(n-m)\varphi}}{\sqrt{m!n!}} |n\rangle\langle m| =$$

$$= \sum_{m,n=0}^{\infty} \left(\int_0^{2\pi} d\varphi e^{i(n-m)\varphi} \right) \left(\int_0^{\infty} drr^{m+n+1} \exp\left(-r^2\right) \right) \frac{|n\rangle\langle m|}{\sqrt{m!n!}}$$

The integral $\int_0^{2\pi} d\varphi e^{i(n-m)\varphi}$ is zero when $m \neq n$ and 2π when $m = n$, i.e.

$$\int_0^{2\pi} d\varphi e^{i(n-m)\varphi} = 2\pi\delta_{mn}$$

Thus, the integral $\int d\lambda |\lambda\rangle\langle\lambda|$ becomes

$$\int d\lambda |\lambda\rangle\langle\lambda| = \sum_{m,n=0}^{\infty} 2\pi\delta_{mn} \left(\int_0^{\infty} drr^{m+n+1} \exp\left(-r^2\right) \right) \frac{|n\rangle\langle m|}{\sqrt{m!n!}} =$$

$$= 2\pi \sum_{n=0}^{\infty} \left(\int_0^{\infty} drr^{2n+1} \exp\left(-r^2\right) \right) \frac{|n\rangle\langle n|}{n!}$$

But

$$\int_0^{\infty} drr^{2n+1} \exp\left(-r^2\right) = \frac{1}{2} \int_0^{\infty} dr^2 r^{2n} \exp\left(-r^2\right) = \frac{1}{2} \int_0^{\infty} dr^2 \left(r^2\right)^n \exp\left(-r^2\right) =$$

$$\overset{r^2=s}{=} \frac{1}{2} \int_0^{\infty} dss^n \exp\left(-s\right) = \frac{1}{2} \Gamma\left(n+1\right) = \frac{1}{2} n!$$

where we made use of the property $\Gamma(n)=(n-1)!$ of the gamma function $\Gamma(t)=\int_0^\infty dx x^{t-1}\exp(-x)$, $\operatorname{Re} t>0$.

Substituting into the expression of the integral $\int d\lambda|\lambda\rangle\langle\lambda|$, we obtain

$$\int d\lambda|\lambda\rangle\langle\lambda| = 2\pi\sum_{n=0}^{\infty}\frac{1}{2}n!\frac{|n\rangle\langle n|}{n!} = \pi\underbrace{\sum_{n=0}^{\infty}|n\rangle\langle n|}_{1} = \pi$$

where we used the completeness relation of the energy eigenstates.

Thus

$$\frac{1}{\pi}\int d\lambda|\lambda\rangle\langle\lambda| = 1 \tag{4}$$

This is the completeness relation of the set of the coherent states. Moreover, since the coherent states overlap, the set is said to be overcomplete.

III. An intuitive introduction to the squeezed states of the QHO

We know that the ground-state wave function of a QHO having length scale x_0 is

$$\psi_0(x) = \frac{1}{\sqrt{x_0}} \frac{1}{\pi^{\frac{1}{4}}} \exp\left(-\frac{1}{2} \left(\frac{x}{x_0} \right)^2 \right)$$

Let us now consider the wave function

$$\psi(x;\xi) = \frac{1}{\sqrt{\xi x_0}} \frac{1}{\pi^{\frac{1}{4}}} \exp\left(-\frac{1}{2} \left(\frac{x}{\xi x_0} \right)^2 \right)$$

where ξ is a dimensionless, positive real parameter.

If $\xi \neq 1$, the wave function $\psi(x;\xi)$ is not the ground-state wave function of the QHO having length scale x_0, but it can be thought of as the ground-state wave function of another QHO, having length scale ξx_0.

Moreover, for every value of ξ, $\psi(x;\xi)$ describes the ground state of a, different each time, QHO.

Since $\psi(x;\xi)$ always describes the ground state of a QHO, the position-momentum uncertainty product will be minimum in the state described by $\psi(x;\xi)$.

If $x_0{}' = \xi x_0$ is the length scale of the new QHO, with ground-state wave function $\psi(x;\xi)$, then its momentum scale will be $p_0{}' = \frac{p_0}{\xi}$, where p_0 is the momentum scale of the first QHO.

Indeed, since the product of the two scales must be equal to $\hbar$, we have

$$x_0{}' p_0{}' = \hbar = x_0 p_0$$

Thus

$$x_0{}' p_0{}' = x_0 p_0 \Rightarrow \xi x_0 p_0{}' = x_0 p_0 \Rightarrow p_0{}' = \frac{p_0}{\xi}$$

Since the wave function $\psi(x;\xi)$ describes the ground state of a QHO with scales x_0' and p_0', the position and momentum uncertainties in the state described by $\psi(x;\xi)$ will be

$$\Delta x = \frac{x_0'}{\sqrt{2}} \text{ and } \Delta p = \frac{p_0'}{\sqrt{2}}.$$

In terms of the scales x_0 and p_0 of the first QHO, the previous two uncertainties are respectively written as

$$\Delta x = \frac{\xi x_0}{\sqrt{2}} \text{ and } \Delta p = \frac{p_0}{\sqrt{2}\xi}.$$

If $\xi \neq 1$, we have

$$\frac{\Delta x}{x_0} = \frac{\xi}{\sqrt{2}} \neq \frac{1}{\sqrt{2}\xi} = \frac{\Delta p}{p_0}$$

That is

$$\frac{\Delta x}{x_0} \neq \frac{\Delta p}{p_0}$$

Thus, with respect to the first QHO, in the state described by $\psi(x;\xi)$, which is a state of minimum position-momentum uncertainty product, the two individual uncertainties are not equally distributed.

Therefore, for the first QHO, the state described by $\psi(x;\xi)$ is always, i.e. for every value of the parameter ξ, a state of minimum position-momentum uncertainty product, i.e. $\Delta x \Delta p = \frac{\hbar}{2}$, but the uncertainties of the position and momentum are not equally distributed, taking different values each time the parameter ξ changes.

For each value of the parameter ξ, the state described by $\psi(x;\xi)$ is called a squeezed state of the first QHO, and particularly, it is a squeezed state of the ground state of the first QHO.

In the same way, from each coherent state of the first QHO, we construct squeezed states of the first QHO.

Thus, making the change $x_0 \to \xi x_0$ in each coherent state of a QHO having length scale x_0, we construct squeezed states of that QHO.

We defined the squeezed states in the position representation, as this provides a better intuitive picture of the squeezed states. However, working in the same way, we may well define the squeezed states in the momentum representation too.

> The parameter ξ determines the squeezing of the position and momentum uncertainties, and thus we may call it squeezing parameter.

The coherent states as states of minimum energy expectation value

12) Show that the energy expectation value of a squeezed state is always greater than the energy expectation value of its respective coherent state, and only when the squeezed state coincides with the respective coherent state, i.e. only when the squeezing parameter is 1, the two energy expectation values are equal.

Solution

The expectation values do not depend on the representation we may use to calculate them – they are representation free – and thus we choose to work in the position representation.

In the exercise 9, we showed that, in the position representation, the coherent state $|\lambda\rangle$ is described by the wave function

$$\psi_\lambda(x) = \frac{1}{\sqrt{x_0}} \frac{1}{\pi^{\frac{1}{4}}} \exp\left(-\frac{1}{2}\left(\frac{x - \langle x \rangle_{|\lambda\rangle}}{x_0} \right)^2 + \frac{i \langle p \rangle_{|\lambda\rangle} x}{\hbar} \right) \qquad (1)$$

To construct a squeezed state of the coherent state $|\lambda\rangle$, we make in (1) the change $x_0 \to \xi x_0$, with $\xi > 0$.

Thus, in the position representation, a squeezed state is described by the wave function

$$\psi_\lambda(x;\xi)=\frac{1}{\sqrt{\xi x_0}}\frac{1}{\pi^{\frac{1}{4}}}\exp\left(-\frac{1}{2}\left(\frac{x-\langle x\rangle_{|\lambda\rangle}}{\xi x_0}\right)^2+\frac{i\langle p\rangle_{|\lambda\rangle}x}{\hbar}\right) \tag{2}$$

As shown in the exercise 8, the energy expectation value of the QHO in the coherent state $|\lambda\rangle$ is

$$\langle E\rangle_{|\lambda\rangle}=\frac{\left(\langle p\rangle_{|\lambda\rangle}\right)^2}{2m}+\frac{1}{2}m\omega^2\left(\langle x\rangle_{|\lambda\rangle}\right)^2+\frac{\hbar\omega}{2} \tag{3}$$

Denoting by $|\lambda;\xi\rangle$ the squeezed state that is described by the wave function $\psi_\lambda(x;\xi)$, the energy expectation value of the QHO in the state $|\lambda;\xi\rangle$ is

$$\langle E\rangle_{|\lambda;\xi\rangle}=\langle\lambda;\xi|\hat{H}|\lambda;\xi\rangle \tag{4}$$

We may calculate the previous energy expectation value directly in the position representation, and we urge the reader to do the relevant calculations.
Alternatively, we may use that the state $|\lambda;\xi\rangle$ is a coherent state of a second QHO, having length and momentum scales ξx_0 and $\frac{p_0}{\xi}$, respectively, where x_0 and p_0 are, respectively, the length and momentum scales of the first QHO.
Then, as shown in the exercise 7, with respect to the second QHO, the position and momentum uncertainties in the state $|\lambda;\xi\rangle$ are equally distributed, i.e.

$$(\Delta x)_{|\lambda;\xi\rangle}=\frac{\xi x_0}{\sqrt{2}}$$

and

$$(\Delta p)_{|\lambda;\xi\rangle}=\frac{p_0}{\sqrt{2}\xi}$$

Thus

$$\sqrt{\langle x^2\rangle_{|\lambda;\xi\rangle}-\left(\langle x\rangle_{|\lambda;\xi\rangle}\right)^2}=\frac{\xi x_0}{\sqrt{2}}\Rightarrow\langle x^2\rangle_{|\lambda;\xi\rangle}=\frac{\xi^2x_0{}^2}{2}+\left(\langle x\rangle_{|\lambda;\xi\rangle}\right)^2 \tag{5}$$

and

$$\sqrt{\langle p^2\rangle_{|\lambda;\xi\rangle} - \left(\langle p\rangle_{|\lambda;\xi\rangle}\right)^2} = \frac{p_0}{\sqrt{2}\xi} \Rightarrow \langle p^2\rangle_{|\lambda;\xi\rangle} = \frac{p_0^{\ 2}}{2\xi^2} + \left(\langle p\rangle_{|\lambda;\xi\rangle}\right)^2 \tag{6}$$

We'll use the wave function $\psi_\lambda(x;\xi)$ to calculate the position and momentum expectation values. Then, from (5) and (6), we'll calculate the expectation values of the position squared and momentum squared, and then, we'll use them to calculate the energy expectation value.

In the position representation, the position expectation value in the state $|\lambda;\xi\rangle$ is written as

$$\langle x\rangle_{|\lambda;\xi\rangle} = \int_{-\infty}^{\infty} dx \psi_\lambda^{\ *}(x;\xi)\, x \psi_\lambda(x;\xi)$$

Substituting into the integral the wave function $\psi_\lambda(x;\xi)$ from (2), we obtain

$$\langle x\rangle_{|\lambda;\xi\rangle} = \frac{1}{\xi x_0}\frac{1}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty} dx x \exp\left(-\left(\frac{x-\langle x\rangle_{|\lambda\rangle}}{\xi x_0}\right)^2\right) \tag{7}$$

Changing the integration variable to $x' = x - \langle x\rangle_{|\lambda\rangle}$, the previous integral becomes

$$\int_{-\infty}^{\infty} dx x \exp\left(-\left(\frac{x-\langle x\rangle_{|\lambda\rangle}}{\xi x_0}\right)^2\right) = \int_{-\infty}^{\infty} dx' \left(x' + \langle x\rangle_{|\lambda\rangle}\right) \exp\left(-\left(\frac{x'}{\xi x_0}\right)^2\right) =$$

$$= \int_{-\infty}^{\infty} dx' x' \exp\left(-\frac{x'^2}{(\xi x_0)^2}\right) + \langle x\rangle_{|\lambda\rangle} \int_{-\infty}^{\infty} dx' \exp\left(-\frac{x'^2}{(\xi x_0)^2}\right)$$

The first integral is zero, because the function $x' \exp\left(-\frac{x'^2}{(\xi x_0)^2}\right)$ is odd, as product of the odd function x' with the even function $\exp\left(-\frac{x'^2}{(\xi x_0)^2}\right)$, and the integration interval is symmetric.

Using that $\int_{-\infty}^{\infty} dx \exp(-ax^2) = \sqrt{\frac{\pi}{a}}$, where $a > 0$, the second integral is

$$\int_{-\infty}^{\infty} dx' \exp\left(-\frac{x'^2}{(\xi x_0)^2}\right) = \sqrt{\pi(\xi x_0)^2} = \pi^{\frac{1}{2}} \xi x_0$$

Thus

$$\int_{-\infty}^{\infty} dxx\exp\left(-\left(\frac{x-\langle x\rangle_{|\lambda\rangle}}{\xi x_0}\right)^2\right)=\pi^{\frac{1}{2}}\xi x_0\langle x\rangle_{|\lambda\rangle}$$

Substituting into (7) yields

$$\langle x\rangle_{|\lambda;\xi\rangle}=\langle x\rangle_{|\lambda\rangle} \tag{8}$$

Therefore, the position expectation value in a squeezed state is equal to the position expectation value in the respective coherent state.

In other words, the squeezing of a coherent state does not change the position expectation value.

In the same way, we calculate the momentum expectation value.
In the position representation, we have

$$\langle p\rangle_{|\lambda;\xi\rangle}=\int_{-\infty}^{\infty} dx\psi_\lambda^*(x;\xi)\left(-i\hbar\frac{d}{dx}\right)\psi_\lambda(x;\xi)=-i\hbar\int_{-\infty}^{\infty} dx\psi_\lambda^*(x;\xi)\frac{d\psi_\lambda(x;\xi)}{dx}$$

Using (2), the derivative $\frac{d\psi_\lambda(x;\xi)}{dx}$ is

$$\frac{d\psi_\lambda(x;\xi)}{dx}=\left(-\frac{1}{\xi x_0}\left(\frac{x-\langle x\rangle_{|\lambda\rangle}}{\xi x_0}\right)+\frac{i\langle p\rangle_{|\lambda\rangle}}{\hbar}\right)\psi_\lambda(x;\xi)=\left(\frac{\langle x\rangle_{|\lambda\rangle}-x}{(\xi x_0)^2}+\frac{i\langle p\rangle_{|\lambda\rangle}}{\hbar}\right)\psi_\lambda(x;\xi)$$

Thus, the momentum expectation value is written as

$$\langle p\rangle_{|\lambda;\xi\rangle}=-i\hbar\int_{-\infty}^{\infty} dx\psi_\lambda^*(x;\xi)\left(\frac{\langle x\rangle_{|\lambda\rangle}-x}{(\xi x_0)^2}+\frac{i\langle p\rangle_{|\lambda\rangle}}{\hbar}\right)\psi_\lambda(x;\xi)=$$

$$=\frac{i\hbar}{(\xi x_0)^2}\int_{-\infty}^{\infty} dx\psi_\lambda^*(x;\xi)\left(x-\langle x\rangle_{|\lambda\rangle}\right)\psi_\lambda(x;\xi)+\langle p\rangle_{|\lambda\rangle}\int_{-\infty}^{\infty} dx\psi_\lambda^*(x;\xi)\psi_\lambda(x;\xi)=$$

$$=\frac{i\hbar}{(\xi x_0)^2}\left(\underbrace{\int_{-\infty}^{\infty} dx\psi_\lambda^*(x;\xi)x\psi_\lambda(x;\xi)}_{\langle x\rangle_{|\lambda;\xi\rangle}}-\langle x\rangle_{|\lambda\rangle}\underbrace{\int_{-\infty}^{\infty} dx\psi_\lambda^*(x;\xi)\psi_\lambda(x;\xi)}_{1}\right)+$$

$$+\langle p\rangle_{|\lambda\rangle}\underbrace{\int_{-\infty}^{\infty} dx\psi_\lambda^*(x;\xi)\psi_\lambda(x;\xi)}_{1}=\frac{i\hbar}{(\xi x_0)^2}\left(\langle x\rangle_{|\lambda;\xi\rangle}-\langle x\rangle_{|\lambda\rangle}\right)+\langle p\rangle_{|\lambda\rangle}$$

Using (8), we end up to

$$\langle p\rangle_{|\lambda;\xi\rangle}=\langle p\rangle_{|\lambda\rangle} \tag{9}$$

Therefore, as in the case of the position expectation value, the squeezing of a coherent state does not change the momentum expectation value.

The integral $\int_{-\infty}^{\infty} dx\psi_\lambda^*(x;\xi)\psi_\lambda(x;\xi)$ is 1, as the state $|\lambda;\xi\rangle$ is normalized. This follows from the fact that the state $|\lambda;\xi\rangle$ is a coherent state of the second QHO, which has scales ξx_0 and $\frac{p_0}{\xi}$, and thus it is generated by the action of a displacement operator, which is unitary, on the ground state of the second QHO.

Substituting (8) and (9) into (5) and (6), respectively, we obtain

$$\langle x^2\rangle_{|\lambda;\xi\rangle}=\frac{\xi^2 x_0^{\ 2}}{2}+\left(\langle x\rangle_{|\lambda\rangle}\right)^2$$

$$\langle p^2\rangle_{|\lambda;\xi\rangle}=\frac{p_0^{\ 2}}{2\xi^2}+\left(\langle p\rangle_{|\lambda\rangle}\right)^2$$

Substituting into the previous two equations the length and momentum scales, $x_0=\sqrt{\frac{\hbar}{m\omega}}$ and $p_0=\sqrt{m\hbar\omega}$, we obtain, respectively,

$$\langle x^2\rangle_{|\lambda;\xi\rangle}=\frac{\xi^2\hbar}{2m\omega}+\left(\langle x\rangle_{|\lambda\rangle}\right)^2 \tag{10}$$

$$\langle p^2\rangle_{|\lambda;\xi\rangle}=\frac{m\hbar\omega}{2\xi^2}+\left(\langle p\rangle_{|\lambda\rangle}\right)^2 \tag{11}$$

By means of (10) and (11), the energy expectation value of the QHO in the squeezed state $|\lambda;\xi\rangle$ is

$$\langle E\rangle_{|\lambda;\xi\rangle} = \frac{\langle p^2\rangle_{|\lambda;\xi\rangle}}{2m} + \frac{1}{2}m\omega^2\langle x^2\rangle_{|\lambda;\xi\rangle} = \frac{\frac{m\hbar\omega}{2\xi^2} + \left(\langle p\rangle_{|\lambda\rangle}\right)^2}{2m} + \frac{1}{2}m\omega^2\left(\frac{\xi^2\hbar}{2m\omega} + \left(\langle x\rangle_{|\lambda\rangle}\right)^2\right) =$$

$$= \left(\frac{1}{\xi^2} + \xi^2\right)\frac{\hbar\omega}{4} + \frac{\left(\langle p\rangle_{|\lambda\rangle}\right)^2}{2m} + \frac{1}{2}m\omega^2\left(\langle x\rangle_{|\lambda\rangle}\right)^2$$

That is

$$\langle E\rangle_{|\lambda;\xi\rangle} = \frac{\left(\langle p\rangle_{|\lambda\rangle}\right)^2}{2m} + \frac{1}{2}m\omega^2\left(\langle x\rangle_{|\lambda\rangle}\right)^2 + \left(\frac{1}{\xi^2} + \xi^2\right)\frac{\hbar\omega}{4} \qquad (12)$$

By means of (3), (12) is written as

$$\langle E\rangle_{|\lambda;\xi\rangle} = \langle E\rangle_{|\lambda\rangle} - \frac{\hbar\omega}{2} + \left(\frac{1}{\xi^2} + \xi^2\right)\frac{\hbar\omega}{4} = \langle E\rangle_{|\lambda\rangle} + \left(\frac{1}{\xi^2} + \xi^2 - 2\right)\frac{\hbar\omega}{4} =$$

$$= \langle E\rangle_{|\lambda\rangle} + \left(\xi - \frac{1}{\xi}\right)^2\frac{\hbar\omega}{4}$$

That is

$$\langle E\rangle_{|\lambda;\xi\rangle} = \langle E\rangle_{|\lambda\rangle} + \left(\xi - \frac{1}{\xi}\right)^2\frac{\hbar\omega}{4} \qquad (13)$$

Since $\left(\xi - \frac{1}{\xi}\right)^2\frac{\hbar\omega}{4} \geq 0$, (13) gives

$$\langle E\rangle_{|\lambda;\xi\rangle} \geq \langle E\rangle_{|\lambda\rangle}$$

and the equality holds only when

$$\xi - \frac{1}{\xi} = 0 \Rightarrow \xi^2 = 1 \underbrace{\Rightarrow}_{\xi>0} \xi = 1$$

Therefore, the energy expectation value in a squeezed state is always greater than the energy expectation value in the respective coherent state, and only when the squeezing parameter is 1, i.e. only when there is no squeezing, and thus the

squeezed state coincides with the coherent state, the two energy expectation values are equal.

It is also worth noting that, as seen from the equation (13), the energy expectation value in the states $|\lambda;\xi\rangle$ and $\left|\lambda;\frac{1}{\xi}\right\rangle$ is the same, i.e.

$$\langle E\rangle_{|\lambda;\xi\rangle}=\langle E\rangle_{\left|\lambda;\frac{1}{\xi}\right\rangle}$$

13) What is the energy expectation value in the squeezed state $|\lambda;\xi\rangle$ when $\xi\to 0^+$ and $\xi\to\infty$? Comment on the results.

Solution

We showed in the previous exercise that the energy expectation value in the squeezed state $|\lambda;\xi\rangle$ is

$$\langle E\rangle_{|\lambda;\xi\rangle}=\langle E\rangle_{|\lambda\rangle}+\left(\xi-\frac{1}{\xi}\right)^2\frac{\hbar\omega}{4}$$

where $\langle E\rangle_{|\lambda\rangle}$ is the energy expectation value in the respective coherent state, and it is

$$\langle E\rangle_{|\lambda\rangle}=\frac{\left(\langle p\rangle_{|\lambda\rangle}\right)^2}{2m}+\frac{1}{2}m\omega^2\left(\langle x\rangle_{|\lambda\rangle}\right)^2+\frac{\hbar\omega}{2}$$

We observe that

$$\lim_{\xi\to 0^+}\left(\xi-\frac{1}{\xi}\right)^2=\left(0-\underbrace{\frac{1}{0^+}}_{\infty}\right)^2=(-\infty)^2=\infty$$

and

$$\lim_{\xi\to\infty}\left(\xi-\frac{1}{\xi}\right)^2=\left(\infty-\underbrace{\frac{1}{\infty}}_{0}\right)^2=\infty$$

Thus, in both cases, the energy expectation value tends to infinity, i.e.

$$\lim_{\xi \to 0^+, \infty} \langle E \rangle_{|\lambda;\xi\rangle} \to \infty$$

In the previous exercise, we showed that the position and momentum uncertainties in the squeezed state $|\lambda;\xi\rangle$ are, respectively,

$$(\Delta x)_{|\lambda;\xi\rangle} = \frac{\xi x_0}{\sqrt{2}} \text{ and } (\Delta p)_{|\lambda;\xi\rangle} = \frac{p_0}{\sqrt{2}\xi}.$$

We see that, when $\xi \to 0^+$, the position uncertainty tends to zero and the momentum uncertainty tends to infinity. Then, the squeezed state $|\lambda;\xi\rangle$ tends to become a position eigenstate, i.e. $\lim_{\xi \to 0^+} |\lambda;\xi\rangle = \left| x - \langle x \rangle_\lambda \right\rangle$, and the position eigenstates are states of infinite energy for the QHO.

Similarly, when $\xi \to \infty$, the position uncertainty tends to infinity and the momentum uncertainty tends to zero. Then, the squeezed state $|\lambda;\xi\rangle$ tends to become a momentum eigenstate, i.e. $\lim_{\xi \to \infty} |\lambda;\xi\rangle = \left| p - \langle p \rangle_\lambda \right\rangle$, and the momentum eigenstates are also states of infinite energy for the QHO.

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