Evaluating $f(x) = C$ for Infinite Set Domains

A Method for Sizing Infinite Sets

I. Abstract

In a previous paper, The Function $f(x) = C$ and the Continuum Hypothesis, posted on viXra.org (viXra:1806.0030), I demonstrated that the set of natural numbers can be put into a one to one correspondence with the set of real numbers, $f: \boldsymbol{N} \rightarrow \boldsymbol{R}$. In that paper I used the function $f(x) = C$ to create an indexed array of the function's real number domain d, the constant range, C, and the index value of each iteration of the function's evaluation, i, for each member of the domain d_i.

The purpose of the exercise was to provide constructive proof of Cantor's continuum hypothesis which has been shown to be independent of the ZFC axioms of set theory. Because the domain of $f(x) = C$ contains all real numbers, evaluating and indexing the function over the entire domain leads naturally to the bijective function $f: \boldsymbol{N} \rightarrow \boldsymbol{R}$.

In this paper I'll demonstrate how the set of natural numbers $\boldsymbol{N}$ can be put into a one to one correspondence the power set of natural numbers, $P(\boldsymbol{N})$. From this I will derive the bijective function $f: \boldsymbol{N} \rightarrow P(\boldsymbol{N})$. Lastly, I'll propose a conjecture asserting that $f(x) = C$ can be employed to construct a one to one correspondence between the natural numbers and any infinite set that can be cast as the domain of the function.

II. Introduction

In a previous paper, The Function $f(x) = C$ and the Continuum Hypothesis, posted on viXra.org (viXra:1806.0030), I demonstrated that the set of natural numbers can be put into a one to one correspondence with the set of real numbers, $f: \boldsymbol{N} \rightarrow \boldsymbol{R}$. In that paper I used the function $f(x) = C$ to create an indexed array of the function's real number domain d, the constant range, C, and the index value of each iteration of the function's evaluation, i, for each member of the domain d_i.

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Functions are normally thought of as being composed of relationships involving variables that are replaced by numbers when evaluating the function. Here, the concept of the function is expanded to include non-numeric domain sets and the range is limited to a single value, the constant C.

I'll demonstrate how the set of natural numbers $\boldsymbol{N}$ can be put into a one to one correspondence the power set of natural numbers, $P(\boldsymbol{N})$. From this I will derive the bijective function $f: \boldsymbol{N} \rightarrow P(\boldsymbol{N})$. Lastly, I'll propose a conjecture asserting that $f(x) = C$ can be employed to construct a one to one correspondence between the natural numbers and any infinite set that can be cast as the domain of the function.

III. Preliminaries

Definition 1: A **function** is an equation for which any x from the domain of the function that can be plugged into the equation will yield exactly one y out of the equation.

Definition 2: A **function index** is a count of the number of times the function has been evaluated for values of x, the index value is denoted by n.

Definition 3: Let C be called a constant.

Definition 4: The set of x values is called the **function domain**.

Definition 5: The set of y values is called the **function range**.

Definition 6: The set of n values is called the **function index**.

Notation 1: Let y be written *f*(x) to denote that the value of each member of the range of the function is dependent on the value of each member of the domain of the function.

Notation 2: Let the set of x values be denoted by {x} and called values of the domain.

Notation 3: Let the set of *f*(x) values by denoted by {*f*(x)} and be called values of the range.

Notation 4: Let the set of n values be denoted by {n} and be called the values of the index.

Notation 5: Let the domain of a function be denoted by d.

Notation 6: Let the range of the function be denoted by r.

Notation 7: Let the index of the function be denoted by i.

Equivalency 1: d = {x}.

Equivalency 2: r = {*f*(x)}

Equivalency 3: i = {n}

Operator: The process of determining individual values of the range of the function based on the values of the function's domain is called evaluating the function and is denoted by **E.**

Notation 9: Let the following notation,

$$\mathbf{E}_{i=(1,\,n)}\, f(x)\ d_i \mid i = (1, 2 \ldots)$$

be taken to mean "evaluate the function *f*(x) over the domain d where i is an index of the number of iterations of **E** and i = (1, 2 …)." For a domain composed of an infinite set n = ∞.

Examples:

1. *f*(x) = C | d = {1, 2, 3}, the domain of the function is the set {1, 2, 3}
2. *f*(x) = C | d = ***N***, {n ∈ ***N*** | 1 ≤ n}, the domain of the function is the set of natural numbers
3. *f*(x) = C | d = ***S***, {r ∈ ***S*** | a < r < b} (a, b), the domain of the function is the set of real numbers on the open interval (a, b).

Note: The function that will be central to this discussion is written as,

$$f(x) = C$$

The demonstration below proceeds along the same lines as the demonstration in The Function $f(x) = C$ and the Continuum Hypothesis.

IV. Given

1. The set of natural numbers $\boldsymbol{N}$. Let n stand for an element of $\boldsymbol{N}$ so that

$$\boldsymbol{N}, \{n \in \boldsymbol{N} \mid 1 \leq n\}$$

2. The power set P of $\boldsymbol{N}$ written P($\boldsymbol{N}$). Let {p} stand for an element of P($\boldsymbol{N}$) so that

$$P(\boldsymbol{N}), \{\{p\} \in P(\boldsymbol{N})\}$$

3. The continuous function

$$f(x) = C$$

where C is constant.

4. The evaluate function operator

$$\mathbf{E}_{i = (1, \infty)} f(x) = C\, d_i \mid i = (1, 2 \ldots)$$

V. Definitions

1. The domain d of $f(x) = C$ is the set

$$P(\boldsymbol{N}), \{\{p\} \in P(\boldsymbol{N})\}$$

that is d = P($\boldsymbol{N}$).

2. The index i is the set

$$\boldsymbol{N}, \{n \in \boldsymbol{N} \mid 1 \leq n\}$$

that is i = $\boldsymbol{N}$.

The index is constructed from repeated iterations of $\mathbf{E}$ and provides a unique identifier for each evaluation of the function on a member of the domain.

VI. Theorems 1 & 2

Theorem 1: The domain d of the continuous function $f(x) = C$ is countable where d = P($\boldsymbol{N}$), {{p} ϵ P($\boldsymbol{N}$)}.

Proof by Construction: Invoking the evaluate function operator on $f(x) = C$ we have

$$\mathbf{E}_{i\,=\,(1,\,\infty)} f(x) = C \mid d_i \text{ where } i = (1, 2 \ldots)$$

The resultant data points can can be formatted as an array showing the values of i, x and $f(x)$ respectively, depicted below.

i	1	2	3	…	n	…
	↕	↕	↕	…	↕	…
x	$\{p_1\}$	$\{p_2\}$	$\{p_3\}$	…	$\{p_n\}$	…
$f(x)$	C	C	C	…	C	…

The process of building the array of data points can go on indefinitely. There are an infinite number of elements in the domain and an infinite number of index of values of i to match with elements of the domain. There is exactly one i for every {p} and exactly one {p} for every i in the array. This shows that there exists a one to one correspondence between i and d and this completes the proof. The correspondence can be expressed as the bijective function,

$$f : i \rightarrow d$$

Having shown that $f : i \rightarrow d$ exists we can assert that the cardinal numbers of the sets comprising i and d are equal and that the sets are the same size.

Theorem 2: The set of $\boldsymbol{N}$ of natural numbers

$$\boldsymbol{N}, \{n \in \boldsymbol{N} \mid 1 \leq n\}$$

and the power set P($\boldsymbol{N}$) of $\boldsymbol{N}$

$$P(\boldsymbol{N}), \{\{p\} \in P(\boldsymbol{N})\}$$

have the same cardinality and, as a result. are the same size.

Proof by Substitution: Theorem 1 proves that $f : i \rightarrow d$ exits and definitions 1 and 2 establish that i = $\boldsymbol{N}$ and d = P($\boldsymbol{N}$). Substituting $\boldsymbol{N}$ for i and P($\boldsymbol{N}$) for d in the bijective function

$$f : i \rightarrow d$$

we have

$$f : \boldsymbol{N} \rightarrow P(\boldsymbol{N})$$

which completes the proof.

That f: $\boldsymbol{N} \rightarrow$ P($\boldsymbol{N}$) exists is not surprising in view of the fact that I demonstrated that f: $\boldsymbol{N} \rightarrow \boldsymbol{R}$ exists in The Function f(x) = C and the Continuum Hypothesis and it has long been established that f: $\boldsymbol{R} \rightarrow$ P($\boldsymbol{N}$) also exists.

VII. Conjecture

Any infinite set that can be cast as the domain of the function f(x) = C can be put into a one to one correspondence with the set of natural numbers via the application of the evaluate function operator over the entire domain.

So that for an infinite set $\boldsymbol{S}$ of members $\{m_1, m_2, m_3 \ldots\}$, f: $\boldsymbol{N} \rightarrow \boldsymbol{S}$ is true if the domain d of f(x) = C consists of the members of the set $\boldsymbol{S}$, that is d = $\{m_1, m_2, m_3 \ldots\}$.