Thermodynamics and the Virial Theorem, Gravitational Collapse and the Virial Theorem: Insight from the Laws of Thermodynamics

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Application of the virial theorem, when combined with results from the kinetic theory of gases, has been linked to gravitational collapse when the mass of the resulting assembly is greater than the Jeans mass, M_J. While the arguments appear straightforward, the incorporation of temperature into these equations, using kinetic theory, results in a conflict with the laws of thermodynamics. Temperature must always be viewed as an intensive property. However, it is readily demonstrated that this condition is violated when the gravitational collapse of a free gas is considered using these approaches. The result implies star formation cannot be based on the collapse of a self-gravitating gaseous mass.

1 Introduction

While the virial theorem derives its name from the work of Calusius [1], credit for its initial formulation has also been ascribed to Lagrange [2], as the theorem can be derived from the Lagrange identity [3, 4]. The virial theorem represents one of the most powerful axioms in physics and has been used to address a wide array of problems [4–6]. Jeans utilized the theorem at length in his classic text, *The Dynamical Theory of Gases* [7], in order to derive some of the well-known gas laws. However, it was not until seven years later that the virial theorem was introduced into astrophysics by Poincaré [8]. Soon after, A. S. Eddington [9], apparently unaware of Poincaré's contribution, applied the theorem to a star cluster. This work centered on kinetic energy of motion and did not attempt to introduce temperature as a variable. Each of these developments adheres to the laws of physics.

Eventually, Eddington [10] came to use the virial theorem when addressing the general theory of star formation. In doing so, it appears that he was the first to combine gravitational potential energy with the kinetic energy for a gas, as derived from the ideal gas law, and thereby obtained an expression defining the mean temperature of a star. Jeans [11] and Chandrasekhar [12] soon followed the same steps. Today, many of these ideas relative to stellar equilibrium and temperature remain ([13], [14, see Eq. 26.7]). In this case, the use of the virial theorem appears to be in conflict with the laws of thermodynamics.

2 Theoretical considerations

The existence of intensive (*e.g.* temperature, pressure, density, molar mass, thermal conductivity, ...) and extensive (*e.g.* mass, volume, internal energy, heat capacity, ...) properties has been recognized. In fact, Landsberg [15] has argued that this concept is so vital as to constitute the 4$^{\text{th}}$ law of thermodynamics. By necessity, intensive properties must be measured in terms of extensive properties. Extensive properties must be additive and are directly related to the mass of a system. Conversely, intensive properties are independent of total mass. When two extensive properties are divided, an intensive property is obtained (*e.g.* mass/volume = density). However, not all properties can be characterized as either intensive or extensive [16]. Still, it is clear that *"if one side of an equation is extensive (or intensive), then so must be the other side"* [17]. These last two realities urge some caution when advancing new relations. The point can be made by first examining the ideal gas law and then, a result from the inappropriate application of the virial theorem.

The ideal gas law is usually expressed as $PV = nRT$, where P, V, n, R and T correspond to the pressure, the volume, the number of moles, the universal gas constant, and the absolute temperature, respectively. If one considers that $n = M/\mathcal{M}$, (where M is the total mass and $\mathcal{M}$ corresponds to the molar mass) and that the mean density, ρ_0, can be expressed as $\rho_0 = M/V$, then the ideal gas law takes the following form:

$$P = \rho_0 \frac{R}{\mathcal{M}} T \,. \tag{1}$$

Recognizing that $R/\mathcal{M}$ is also known as the specific gas constant, R_s, then the ideal gas law can simply be expressed as $P = \rho_0 R_s T$. Note that this equation does not contain any extensive properties, as both the mass of the system and its volume have been replaced by density, ρ_0, which is an intensive property. Similarly, P and T are intensive properties, while R_s is a constant for any given system. In accordance with the state postulate, this simple system is fully defined by any two intensive properties [18].

At the same time, an intensive property must remain a function of only intensive properties, or of extensive properties which in combination, result in an intensive property. This is especially important when considering temperature in light of the 0$^{\text{th}}$ law of thermodynamics. If the ideal gas law is re-expressed in terms of temperature,

$$T = \frac{P}{R_s \rho_0}, \tag{2}$$

it is observed that this property remains defined only in terms of intensive properties for this system, namely pressure and density.

When considering the kinetic theory as applied to an ideal gas (see Jeans [19]), any of the associated results are inherently linked to the conditions which gave rise to the ideal gas law. For instance, 1) a large number of rapidly moving particles must be considered, 2) these must be negligibly small relative to the total volume, 3) all collisions must be elastic, 4) no net forces must exist between the particles, 5) the walls of the enclosure must be rigid, 6) the only force or change in momentum with time, dp/dt, which is experienced to define pressure, P, must occur at the walls, and 7) the sum of forces everywhere else must be zero. In this instance, temperature becomes linked to the total kinetic energy of the enclosed system, $K.E. = \frac{3}{2} N k_B T$, where N represents the total number of particles and k_B is Boltzmann's constant. Note that this expression does not address any contribution to the total kinetic energy which this enclosed system might gain if it were in motion relative to another object. Such motion would increase the total kinetic energy of the system, but not its temperature.

When the virial theorem is applied to a self-gravitating gaseous mass, wherein the kinetic theory of gases has been used to insert temperature dependence [10–12], it is well-established ([13], [14, see Eq. 26.7]) that this combination results in the following expression for temperature:

$$T = \frac{G M m_p}{5 k_B r}, \tag{3}$$

where G, M, m_p, k_B, and r corresponds to the gravitational constant, the mass of the system, the particle mass, Boltzmann's constant, and the radius. With dimensional analysis, this expression appears valid, equating Kelvin on each side. However, this is not true, relative to analysis of intensive and extensive properties.

Observe that G, m_p, and k_B are constants for this system. Mass, M, is an extensive property. However, the radius, r, is neither extensive nor intensive [18]. In order to see that radius is not an extensive property, one simply needs to recall that for an ideal gas, volume, an extensive property, is directly related to mass, M. In fact, mass is usually divided by volume in order to lead to density, ρ_0, an intensive property. However, since $V = \frac{4}{3}\pi r^3$, it is evident that radius is not directly related to mass, M, but rather to $M^{1/3}$. As such, r cannot be an extensive property. Thus, temperature in (3) is being defined in terms of two properties, M and r, which in combination *do not* result in an intensive property. This constitutes a direct violation of the 0$^{\text{th}}$ law which seeks, first and foremost, to define temperature as an intensive property, a reality well-established in thermodynamics (*e.g.* [17]).

In arriving at (3), the kinetic energy of the gas, $K.E.$, was assumed to be equal to $\frac{3}{2} N k_B T$, as presented above. However, the temperature obtained from kinetic theory is a manifestation of the internal motion of the gas within an enclosure. That energy represents heat energy and it is not related to the kinetic energy of translational motion which should be combined in the virial theorem with gravitational potential energy, when considering a bound system.

Furthermore, this expression was obtained for a gas enclosed by a rigid wall. Such a wall is not present when considering gravitational collapse. Yet, the results relative to the ideal gas law were critically dependent on the presence of this enclosure. The relationship between pressure, volume, and temperature was extracted using real walls. This is critical as the only forces used in defining pressure in this system occur at this boundary. It is not proper to remove the wall and then assume that the kinetic energy of the gaseous system remains equal to $\frac{3}{2} N k_B T$.

A thermodynamic problem also occurs with any expression attempting to define the Jeans mass, M_J, an extensive property, in terms of temperature and mean density, both of which are intensive properties. Consider the following expression:

$$M_J = \left(\frac{5 k_B T}{G m_p}\right)^{3/2} \left(\frac{3}{4\pi\rho_0}\right)^{1/2}, \tag{4}$$

which is analogous to Eq. 12.14 in [14]. Note in (4) that all terms are raised to either the 3/2 or 1/2 power. As such, no term on the right side of this equation could have been considered to behave as an extensive property. Extensive properties must be additive, a feature which is lost when they are raised to an exponential power. In (4), the only terms which are not constants are T and ρ_0, but these are intensive, not extensive properties. As such, the concept of Jeans mass is not supported by the laws of thermodynamics as no extensive properties exist on the right side of (4).

3 Discussion

When applying the virial theorem, it is important to differentiate the kinetic energy associated with temperature from the kinetic energy of motion. For instance, when Chandrasekhar [20] applied the virial theorem to rotating fluid masses, he made a clear distinction between heat energy and kinetic energy of motion. If this is not done and the two are considered the same, as with all applications to a gaseous mass [10–14], then violations of thermodynamics ensue.

It is not solely that an intensive property, like temperature, is being defined in terms of properties which, in combination, do not yield an intensive property. While this is a violation of the 0$^{\text{th}}$ law, the 3$^{\text{rd}}$ law is also being violated, as 0 K is a temperature. One cannot, by (3), increase the radius to infinity and, therefore, define 0 K as an intensive property. These considerations illustrate that gases cannot undergo gravitational collapse.

Lane's Law [21, 22], or the self-compression of a gaseous mass, also constitutes a violation of the 1st law of thermodynamics. A system cannot do work upon itself and thereby raise its own temperature. This results in a perpertual motion machine of the first kind. Additionally, a gravitationally collapsing gaseous cloud, which obeys the ideal gas law, violates the 2nd law of thermodynamics. An ideal gas is elastic by definition. It has no means of dissipating heat into the heat sink of its surroundings. Moreover, the system lacks an "engine" whereby compression can be achieved. Work must be done on the system in order to increase its order. To argue otherwise consitutes a perpertual motion machine of the second kind.

4 Conclusion

The idea that a gaseous mass can undergo gravitational collapse ([9, 11–13], [14, see Eq. 26.7]) stands in violation of the 0th, 1st, 2nd and 3rd laws of thermodynamics. It is well-established in the laboratory that gases expand to fill the void. According to the laws of thermodynamics a system cannot do work upon itself. When dealing with an ideal gas without net translation, all of the energy should be considered as kinetic energy, exclusively. It is not appropriate to add a potential energy term, if the total energy has already been defined as kinetic energy, thereby establishing temperature.

At the same time, the question remains: *How do stars form?* They do not arise from gravitational collapse. The only feasible solution is that they are the result of condensation reactions, whereby material, as it condenses and forms a new system, emits photons into its surroundings. Insight relative to this issue can be gained by considering the work of Konig *et al* [23], wherein the condensation of silver clusters at low temperatures has been associated with the emission of photons. It is highly likely that hydrogen ion clusters [24] will be found someday to behave in the same fashion. Along with other advancements in condensed matter physics [25], such discoveries may well provide the necessary force to help astronomers recognize that the stars are comprised of condensed matter [26].

Dedication

This work is dedicated to my wife, Patricia Anne.

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