question 466: An Identity

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abstract

This note presents a nontrivial identity that involve the number pi:

$$\pi = 3.1415926535 \ldots$$

keywords: number pi , integrals.

1. Identity

Let $u = \frac{3\sqrt{6}}{8}$, then

$$\pi = \frac{3\sqrt{3}}{2} + 2\sqrt{2}\int_u^\infty \left(1 - \sqrt{-\frac{1}{3} - \frac{4}{3}\cos\left(\frac{2\pi}{3} + \frac{1}{3}\cos^{-1}\left(\frac{27}{16x^2} - 1\right)\right)}\right) dx + 2\sqrt{2}\int_u^1 \left(\sqrt{-\frac{1}{3} - \frac{4}{3}\cos\left(\frac{4\pi}{3} + \frac{1}{3}\cos^{-1}\left(\frac{27}{16x^2} - 1\right)\right)}\right) dx \tag{1}$$

related integrals :

$$\pi = \frac{3\sqrt{3}}{2} + \frac{3}{2}\sqrt{\frac{3}{2}}\int_{-1}^1 \left(1 - \sqrt{-\frac{1}{3} - \frac{4}{3}\cos\left(\frac{2\pi}{3} + \frac{1}{3}\cos^{-1}x\right)}\right) \frac{1}{(1+x)^{3/2}} dx + \frac{3}{2}\sqrt{\frac{3}{2}}\int_{11/16}^1 \left(\sqrt{-\frac{1}{3} - \frac{4}{3}\cos\left(\frac{4\pi}{3} + \frac{1}{3}\cos^{-1}x\right)}\right) \frac{1}{(1+x)^{3/2}} dx \tag{2}$$

Let $v = \frac{4\pi}{3} + \frac{1}{3}\cos^{-1}\left(\frac{11}{16}\right)$, then

$$\pi = \frac{3\sqrt{3}}{2} + \frac{9}{2\sqrt{2}}\int_{2\pi/3}^{\pi} \frac{\left(\sqrt{3} - \sqrt{-1 - 4\cos x}\right)\sin(3x)}{(1+\cos(3x))^{3/2}} dx + \frac{9}{2\sqrt{2}}\int_{4\pi/3}^{v} \frac{\sqrt{-1 - 4\cos x}\,\sin(3x)}{(1+\cos(3x))^{3/2}} dx \tag{3}$$

Let $w = \frac{\pi}{6} - \frac{1}{3}\cos^{-1}\left(\frac{11}{16}\right)$, then

$$\pi = \frac{3\sqrt{3}}{2} + \frac{9}{2\sqrt{2}}\int_0^{\pi/3} \frac{\left(\sqrt{3} - \sqrt{4\cos x - 1}\right)\sin(3x)}{(1-\cos(3x))^{3/2}} dx + \frac{9}{2\sqrt{2}}\int_w^{\pi/6} \frac{\sqrt{4\cos x - 1}\,\cos(3x)}{(1+\sin(3x))^{3/2}} dx \tag{4}$$

$$\pi = \frac{3\sqrt{3}}{2} + \frac{9}{2\sqrt{2}}\int_{1/2}^{1} \frac{\left(\sqrt{3} - \sqrt{4x-1}\right)(4x^2-1)}{(1+3x-4x^3)^{3/2}}\,dx + \frac{9}{2\sqrt{2}}\int_{1/4}^{1/2} \frac{\sqrt{4x-1}\,(1-4x^2)}{(1+3x-4x^3)^{3/2}}\,dx \tag{5}$$

Let $s = \cos^{-1}\left(\frac{11}{16}\right)$, then

$$\pi = \frac{3\sqrt{3}}{2} + \frac{3}{2}\int_0^{\pi}\left(\sqrt{\frac{3}{2}} - \sqrt{\cos\left(\frac{x}{3}\right) + \sqrt{3}\sin\left(\frac{x}{3}\right) - \frac{1}{2}}\right)\frac{\sin x}{(1+\cos x)^{3/2}}\,dx + \frac{3}{2}\int_0^{s}\left(\sqrt{\cos\left(\frac{x}{3}\right) - \sqrt{3}\sin\left(\frac{x}{3}\right) - \frac{1}{2}}\right)\frac{\sin x}{(1+\cos x)^{3/2}}\,dx \tag{6}$$

$$\pi = \frac{3\sqrt{3}}{2} + \frac{3}{2\sqrt{2}}\int_0^{\pi}\left(\sqrt{\frac{3}{2}} - \sqrt{\cos\left(\frac{x}{3}\right) + \sqrt{3}\sin\left(\frac{x}{3}\right) - \frac{1}{2}}\right)\tan\left(\frac{x}{2}\right)\sec\left(\frac{x}{2}\right)dx + \frac{3}{2\sqrt{2}}\int_0^{s}\left(\sqrt{\cos\left(\frac{x}{3}\right) - \sqrt{3}\sin\left(\frac{x}{3}\right) - \frac{1}{2}}\right)\tan\left(\frac{x}{2}\right)\sec\left(\frac{x}{2}\right)dx \tag{7}$$

References

A. Boros, G. and Moll, V.H.: Irresistible Integrals, Cambridge University Press, 2004.

B. Gradshteyn, I.S. and Ryzhik, I.M.: Table of Integrals,Series and Products. 7th ed., ed. Alan Jeffrey and Daniel Zwillinger, Academic Press,2007.