Exactly Solving Second Order Linear Ordinary Differential Equations

---

Theorem I.1 & Corollary I.4 & Theorem II.1, from previous [8]:

Theorem I.1: Any Second Order Homogeneous Linear Ordinary Differential Equation may be factored via two linear differential operators.

Corollary I.4: A Second Order Linear Ordinary Differential Equation may be factored via two linear differential operators.

$$(D+h)(D+g)y = W \Rightarrow y = y_{h_1}\int y_{h_1}^{-2} e^{-\int Pdx}\left(\int W y_{h_1} e^{\int Pdx} dx\right)dx$$

(where: $y_h'' + Py_h' + Qy_h = 0$)

Theorem II.1: Any differential expression of the form: $y'' + Py' + Qy$ may be written as a pair of linear differential operators, i.e. $\forall y,P,Q : \exists g,h$: $y'' + Py' + Qy = (D+h)(D+g)y$.

Proof:

A differential expression of the form: $y'' + Py' + Qy$, has a value $\forall y,P,Q$, say W .

Thus, $\forall y,P,Q,W : y'' + Py' + Qy = W$ represents a Second Order Linear Ordinary Differential Equation.

So, by Theorem I.1 & Corollary I.4 may be factored via two linear differential operators.

$\Rightarrow \forall y,P,Q,W : \exists g,h : W = y'' + Py' + Qy = (D+h)(D+g)y$

□

And, by previous [9]:

Corollary 1.3: If : $y'' + Py' + Qy = 0$ and:

$$u = ye^{\frac{1}{2}\int(P-R)dx} \quad ;$$

then

$$u'' + Ru' + \left\{Q - \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right] + \left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right]\right\}u = 0$$

So:

Theorem II.1: For all differentiable P,R :

$$u = \left(c_1\int e^{-\int Pdx}dx + c_2\right)e^{\frac{1}{2}\int(P-R)dx}$$

$$\Rightarrow u'' + Ru' + \left\{\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right]\right\}u = 0$$

Proof:

$y'' + Py' + Qy = 0$

By the above corollary I.3:

$$u = ye^{\frac{1}{2}\int(P-R)dx} \Rightarrow u'' + Ru' + \left\{Q - \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right] + \left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right]\right\}u = 0$$

Is satisfied whenever: $Q = 0$:

$$\Rightarrow 0 = y'' + Py' = \left(y'e^{\int Pdx}\right)'e^{-\int Pdx} \Rightarrow y = c_1\int e^{-\int Pdx}dx + c_2$$

$$\Rightarrow u = \left(c_1\int e^{-\int Pdx}dx + c_2\right)e^{\frac{1}{2}\int(P-R)dx}$$

$$\Rightarrow u'' + Ru' + \left\{\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right]\right\}u = 0$$

□

Theorem II.2: For all differentiable P,R :

$$u = \left(e^{-\int Pdx}\left(c_1\int e^{\int Pdx}dx + c_2\right)\right)e^{\frac{1}{2}\int(P-R)dx} = \left(c_1\int e^{\int Pdx}dx + c_2\right)e^{-\frac{1}{2}\int(P+R)dx}$$

$$\Rightarrow u'' + Ru' + \left\{\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] + \left[\left(\tfrac{1}{2}P\right)' - \left(\tfrac{1}{2}P\right)^2\right]\right\}u = 0$$

Proof:

$y'' + Py' + Qy = 0$

By the above corollary I.3:

$$u = ye^{\frac{1}{2}\int(P-R)dx} \Rightarrow u'' + Ru' + \left\{Q - \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right] + \left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right]\right\}u = 0$$

Is satisfied whenever: $Q = P'$:

$$\Rightarrow 0 = y'' + Py' + P'y = (y' + Py)' \Rightarrow y = e^{=\int Pdx}\left(c_1\int e^{\int Pdx}dx + c_2\right)$$

$$\Rightarrow u = \left(e^{=\int Pdx}\left(c_1\int e^{\int Pdx}dx + c_2\right)\right)e^{\frac{1}{2}\int(P-R)dx}$$

$$\Rightarrow u'' + Ru' + \left\{\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] + \left[\left(\tfrac{1}{2}P\right)' - \left(\tfrac{1}{2}P\right)^2\right]\right\}u = 0$$

□

This is more easily obtained using the Ricatti equivalent.

Theorem II.1a: Any two differentiable functions Ψ_1, Ψ_2 satisfy the Ricatti ODE:

$$w' + w^2 + (2\Psi_2)w = (\Psi_1' + \Psi_1^2) - (\Psi_2' + \Psi_2^2) \quad , \qquad (w = \Psi_1 - \Psi_2) \, .$$

Proof:

Let: $w = \Psi_1 - \Psi_2$

$$\Rightarrow \Psi_1' + \Psi_1^2 = (\Psi_2 + w)' + (\Psi_2 + w)^2$$
$$= \Psi_2' + \Psi_2^2 + w' + (2\Psi_2)w + w^2$$

$$\Rightarrow w' + w^2 + (2\Psi_2)w = (\Psi_1' + \Psi_1^2) - (\Psi_2' + \Psi_2^2)$$

□

Corollary II.1a: Any two differentiable functions Ψ_1, P satisfy the Ricatti ODE:

$$w' + w^2 + Pw = -\left[\left(\frac{1}{2}P\right)' + \left(\frac{1}{2}P\right)^2\right] + (\Psi_1' + \Psi_1^2) \quad , \qquad \left(w = -\frac{1}{2}P + \Psi_1\right) .$$

Proof:

Let: $P = 2\Psi_2 \Rightarrow \Psi_2 = \frac{1}{2}P$

$$\Rightarrow w' + w^2 + Pw = (\Psi_1' + \Psi_1^2) - \left[\left(\frac{1}{2}P\right)' + \left(\frac{1}{2}P\right)^2\right] \quad , \qquad \left(w = \Psi_1 - \frac{1}{2}P\right)$$

□

Recalling the connection transformation between the Ricatti equation and the Homogeneous Linear Second Order Ordinary Differential Equation: $u = (\log y)' = \frac{y'}{y} \Leftrightarrow y = e^{\int u dx}$

$$\Rightarrow u' = \left(\frac{y'}{y}\right)' = -\frac{1}{y^2}y'y' + \frac{y''}{y} = -\left(\frac{y'}{y}\right)^2 + \frac{y''}{y} = -u^2 + \frac{y''}{y}$$

$$\Rightarrow u' + u^2 + Pu + Q = \frac{1}{y}(y'' + Py' + Qy)$$

$$\Rightarrow u = (\log y)' \, : \, u' + u^2 + Pu = -Q \Leftrightarrow y = e^{\int u dx} \, : \, y'' + Py' + Qy = 0$$

Thus:

$$\Psi_1 - \frac{1}{2}P = w = (\log y)' \, : \, w' + w^2 + Pw = -Q \Leftrightarrow y = e^{\int \left(\Psi_1 - \frac{1}{2}P\right)dx} \, : \, y'' + Py' + Qy = 0$$

Lemma II.1b: $u' + u^2 = v' + v^2 \Rightarrow u - v \in \left\{0, \left(\log\left[\int e^{-2\int v dx} dx\right]\right)'\right\}$

Proof:

$$u' + u^2 = v' + v^2 \Rightarrow (u - v)' + u^2 - v' = (u - v)' + u^2 - 2uv + v^2 + 2uv - 2v^2 = 0$$

if: $u \neq v$:

$\Rightarrow (u - v)' + (u - v)^2 + 2v(u - v) = 0$ is Bernoulli

So, let: $w^{-1} = (u - v) \Rightarrow -w^{-2}w' + w^{-2} + 2vw^{-1} = 0$

$$\Rightarrow 1 = w' - 2vw = \left(we^{-2\int v dx}\right)' e^{2\int v dx}$$

$$\Rightarrow e^{2\int v dx} \int e^{-2\int v dx} dx = w = (u - v)^{-1}$$

$$\Rightarrow u = v + \frac{e^{-2\int v dx}}{\int e^{-2\int v dx} dx} = v + \left(\log\left[\int e^{-2\int v dx} dx\right]\right)' \qquad \text{(easily verifed)}$$

alternatively:

$$u = \left(c_1 \int e^{-\int P dx} dx + c_2\right) e^{\frac{1}{2}\int (P-R)dx}$$

$$\Rightarrow u'' + Ru' + \left\{\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right]\right\} u = 0$$

$$\text{So, } \exists \Psi_1 : u = e^{\int \left(\Psi_1 - \frac{1}{2}R\right)dx} = \left(c_1 \int e^{-\int P dx} dx + c_2\right) e^{\frac{1}{2}\int (P-R)dx} \quad , \quad \Psi_1' + \Psi_1^2 = \left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2$$

$$\Rightarrow u = e^{\int \left(\Psi_1 - \frac{1}{2}P\right)dx} = c_1 \int e^{-\int P dx} dx + c_2$$

$$\Rightarrow e^{\int \Psi_1 dx} = e^{\frac{1}{2}\int P dx}\left(c_1 \int e^{-\int P dx} dx + c_2\right)$$

$$\Rightarrow \int \Psi_1 dx = \log\left[e^{\frac{1}{2}\int P dx}\left(c_1 \int e^{-\int P dx} dx + c_2\right)\right]$$

$$\Rightarrow \Psi_1 = \left(\log\left[e^{\frac{1}{2}\int P dx}\left(c_1 \int e^{-\int P dx} dx + c_2\right)\right]\right)' = \tfrac{1}{2}P + \left(\log\left[c_1 \int e^{-\int P dx} dx + c_2\right]\right)'$$

the same, with: $c_1 = 1 \quad , \quad c_2 = 0$

□

Corollary II.1b0: $u' + u^2 - [v' + v^2] = (u - v)' + (u - v)^2 + 2v(u - v)$

Proof:

$$u' + u^2 - [v' + v^2] = (u - v)' + u^2 - v' = (u - v)' + u^2 - 2uv + v^2 + 2uv - 2v^2$$
$$= (u - v)' + (u - v)^2 + 2v(u - v)$$

□

Corollary II.1b1:

$$u = \left(c_1 \int e^{-\int P dx} dx + c_2\right) e^{\frac{1}{2}\int (P-R)dx}$$

$$\Rightarrow u'' + Ru' +$$

$$+\left\{\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(\tfrac{1}{2}P + \left(\log\left[\int e^{-\int Pdx}dx\right]\right)'\right)' + \left(\tfrac{1}{2}P + \left(\log\left[\int e^{-\int Pdx}dx\right]\right)'\right)^2\right]\right\}u = 0$$

Proof:

By corollary II.1b:

$$u = \left(c_1\int e^{-\int Pdx}dx + c_2\right)e^{\frac{1}{2}\int(P-R)dx} \Rightarrow u'' + Ru' + \left\{\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right]\right\}u = 0$$

And, by corollary II.1a:

P->R:

$$w' + w^2 + Rw = -\left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right] + (\Psi_1' + \Psi_1^2) \quad , \qquad \left(w = -\frac{1}{2}R + \Psi_1\right).$$

$$w = (\log u)' = \frac{u'}{u} \Leftrightarrow u = e^{\int wdx}$$

$$\Rightarrow w' = \left(\frac{u'}{u}\right)' = -\frac{1}{u^2}u'u' + \frac{u''}{u} = -\left(\frac{u'}{u}\right)^2 + \frac{u''}{u} = -w^2 + \frac{u''}{u}$$

$$\Rightarrow w' + w^2 + Rw + Q = \frac{1}{u}(u'' + Ru' + Qu)$$

$$\Rightarrow w = (\log u)' : w' + w^2 + Rw = -Q \Leftrightarrow u = e^{\int wdx} : u'' + Ru' + Qu = 0$$

$$\Rightarrow \Psi_1 - \frac{1}{2}R = w = (\log u)' :$$

$$\Rightarrow w' + w^2 + Rw = -Q \Leftrightarrow u = e^{\int\left(\Psi_1 - \frac{1}{2}R\right)dx} : u'' + Ru' + \left[-\left\{-\left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right] + (\Psi_1' + \Psi_1^2)\right\}\right]u = 0$$

$$\Rightarrow u = e^{\int\left(\Psi_1 - \frac{1}{2}R\right)dx} \Rightarrow u'' + Ru' + \left\{\left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right] - (\Psi_1' + \Psi_1^2)\right\}u = 0$$

$$\Rightarrow \Psi_1' + \Psi_1^2 = \left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2$$

So, by lemma II.1b: $\Psi_1 - \frac{1}{2}P \in \left\{0, \left(\log\left[\int e^{-\int Pdx}dx\right]\right)'\right\}$

$$\Rightarrow u = e^{\frac{1}{2}\int(P-R)dx} \Rightarrow u'' + Ru' + \left\{\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right]\right\}u = 0$$

$$u = e^{\int\left(\frac{1}{2}P + \left(\log\left[\int e^{-\int Pdx}dx\right]\right)' - \frac{1}{2}R\right)dx} = \left[\int e^{-\int Pdx}dx + c\right]e^{\frac{1}{2}\int(P-R)dx} = \left(c_1\int e^{-\int Pdx}dx + c_2\right)e^{\frac{1}{2}\int(P-R)dx}$$

$$\Rightarrow u'' + Ru' +$$

$$+\left\{\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(\tfrac{1}{2}P + \left(\log\left[\int e^{-\int Pdx}dx\right]\right)'\right)' + \left(\tfrac{1}{2}P + \left(\log\left[\int e^{-\int Pdx}dx\right]\right)'\right)^2\right]\right\}u = 0$$

$$\Rightarrow u = e^{\frac{1}{2}\log\left[\int e^{-\int Pdx}dx\right] + \frac{1}{2}\int(P-R)dx} = \left(c_1\int e^{-\int Pdx}dx + c_2\right)e^{\frac{1}{2}\int(P-R)dx}$$

$$\Rightarrow u'' + Ru' +$$

$$+\left\{\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(\tfrac{1}{2}P + \left(\log\left[\int e^{-\int Pdx}dx\right]\right)'\right)' + \left(\tfrac{1}{2}P + \left(\log\left[\int e^{-\int Pdx}dx\right]\right)'\right)^2\right]\right\}u = 0$$

□

Lemma II.1c: $u' + u^2 = -v' + v^2 \Rightarrow u + v \in \left\{0, \left(\log\left[\int e^{2\int vdx}dx\right]\right)'\right\}$

Proof:

$$u' + u^2 = -v' + v^2 \Rightarrow (u+v)' + u^2 - v' = (u+v)' + u^2 + 2uv + v^2 - 2uv - 2v^2 = 0$$

if: $u \neq -v$:

$\Rightarrow (u+v)' + (u+v)^2 - 2v(u+v) = 0$ is Bernoulli

So, let: $w^{-1} = (u+v) \Rightarrow -w^{-2}w' + w^{-2} - 2vw^{-1} = 0$

$$\Rightarrow 1 = w' + 2vw = \left(we^{2\int vdx}\right)'e^{-2\int vdx}$$

$$\Rightarrow e^{-2\int vdx}\int e^{2\int vdx}dx = w = (u+v)^{-1}$$

$$\Rightarrow u = -v + \frac{e^{2\int vdx}}{\int e^{2\int vdx}dx} = v + \left(\log\left[\int e^{2\int vdx}dx\right]\right)'$$

(essentially lemma II.1b under transformation: $v \Rightarrow -v$)

□

Corollary II.2: For all differentiable R : a solution to: $u'' + Ru' + \left\{\left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right] - \frac{1}{4}A^2\right\}u = 0$

is: $u = \left(c_1e^{\frac{1}{2}Ax} + c_2e^{=\frac{1}{2}Ax}\right)e^{-\frac{1}{2}\int Rdx}$

Proof:

By the above theorem II.1, a solution to: $u'' + Ru' + \left\{\left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right] - \left[\left(\frac{1}{2}P\right)' + \left(\frac{1}{2}P\right)^2\right]\right\}u = 0$

is: $u = \left(c_1\int e^{-\int Pdx}dx + c_2\right)e^{\frac{1}{2}\int(P-R)dx}$

and:

By the above theorem II.2, a solution to: $u'' + Ru' + \left\{\left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right] + \left[\left(\frac{1}{2}P\right)' - \left(\frac{1}{2}P\right)^2\right]\right\}u = 0$

is: $u = \left(c_1\int e^{\int Pdx}dx + c_2\right)e^{-\frac{1}{2}\int(P+R)dx}$

So, with: $P = A$ constant:

a solution to: $u'' + Ru' + \left\{\left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right] - \frac{1}{4}A^2\right\}u = 0$

By the above theorem II.1, is: $u = \left(-\frac{c_1}{A}e^{-Ax} + c_2\right)e^{\frac{1}{2}Ax - \frac{1}{2}\int Rdx}$

and:

By the above theorem II.2, is: $u = \left(\frac{c_1}{A} + c_2 e^{-Ax}\right) e^{\frac{1}{2}Ax - \frac{1}{2}\int R dx}$

□

Corollary 1I.3: For all differentiable P : a solution to: $u'' + Au' + \left\{\frac{1}{4}A^2 - \left[\left(\frac{1}{2}P\right)' + \left(\frac{1}{2}P\right)^2\right]\right\}u = 0$

is: $u = \left(c_1 \int e^{-\int P dx} dx + c_2\right) e^{\frac{1}{2}\int P dx - \frac{1}{2}Ax}$

Proof:

By the above theorem II.1, a solution to: $u'' + Ru' + \left\{\left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right] - \left[\left(\frac{1}{2}P\right)' + \left(\frac{1}{2}P\right)^2\right]\right\}u = 0$

is: $u = \left(c_1 \int e^{-\int P dx} dx + c_2\right) e^{\frac{1}{2}\int (P-R) dx}$

and:

By the above theorem II.2, a solution to: $u'' + Ru' + \left\{\left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right] + \left[\left(\frac{1}{2}P\right)' - \left(\frac{1}{2}P\right)^2\right]\right\}u = 0$

is: $u = \left(e^{-\int P dx}\left(c_1 \int e^{\int P dx} dx + c_2\right)\right) e^{\frac{1}{2}\int (P-R) dx} = \left(c_1 \int e^{\int P dx} dx + c_2\right) e^{-\frac{1}{2}\int (P+R) dx}$

So, with: $R = A$ constant:

a solution to: $u'' + Au' + \left[\frac{1}{4}A^2 - \left(\frac{1}{2}P\right)' - \left(\frac{1}{2}P\right)^2\right]u = 0$

By the above theorem II.1, is: $u = \left(c_1 \int e^{-\int P dx} dx + c_2\right) e^{\frac{1}{2}\int P dx - \frac{1}{2}Ax}$

and:

a solution to: $u'' + Au' + \left[\frac{1}{4}A^2 + \left(\frac{1}{2}P\right)' - \left(\frac{1}{2}P\right)^2\right]u = 0$

By the above theorem II.2, is: $u = \left(e^{=\int P dx}\left(c_1 \int e^{\int P dx} dx + c_2\right)\right) e^{\frac{1}{2}\int P dx - \frac{1}{2}Ax}$

□

Corollary 1I.4: For all differentiable R :

$$u = \begin{cases} (c_1 \log x + c_2) x^{\frac{1}{2}} e^{-\frac{1}{2}\int R dx} \Rightarrow & u'' + Ru' + \left\{\left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right] + \dfrac{\frac{1}{4}}{x^2}\right\}u = 0 \\ \left(c_1 \dfrac{x^{-m+1}}{-m+1} + c_2\right) x^{\frac{m}{2}} e^{-\frac{1}{2}\int R dx} & u'' + Ru' + \left\{\left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right] - \dfrac{\frac{m}{2}\left(\frac{m}{2} - 1\right)}{x^2}\right\}u = 0 \quad , \ (m \neq 1) \end{cases}$$

Proof:

By the above theorem II.1, a solution to: $u'' + Ru' + \left\{\left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right] - \left[\left(\frac{1}{2}P\right)' + \left(\frac{1}{2}P\right)^2\right]\right\}u = 0$

is: $u = \left(c_1 \int e^{-\int P dx} dx + c_2\right) e^{\frac{1}{2}\int (P-R) dx}$

and:

By the above theorem II.2, a solution to: $u'' + Ru' + \left\{\left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right] + \left[\left(\frac{1}{2}P\right)' - \left(\frac{1}{2}P\right)^2\right]\right\}u = 0$

is: $u = \left(c_1 \int e^{\int P dx} dx + c_2\right) e^{-\frac{1}{2}\int (P+R) dx}$

So, with: $P = \frac{m}{x}$ constant:

a solution to: $u'' + Ru' + \left\{\left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right] - \dfrac{\frac{m}{2}\left(\frac{m}{2} - 1\right)}{x^2}\right\}u = 0$

By the above theorem II.1, is: $u = \left(c_1 \int x^{-m} dx + c_2\right) x^{\frac{m}{2}} e^{-\frac{1}{2}\int R dx}$

and:

a solution to: $u'' + Ru' + \left\{\left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right] - \dfrac{\frac{m}{2}\left(\frac{m}{2} + 1\right)}{x^2}\right\}u = 0$

By the above theorem II.2, is: $u = \left(c_1 \int x^{m} dx + c_2\right) x^{-\frac{m}{2}} e^{-\frac{1}{2}\int R dx}$

So:

$$u = \begin{cases} (c_1 \log x + c_2) x^{\frac{1}{2}} e^{-\frac{1}{2}\int R dx} \Rightarrow & u'' + Ru' + \left\{\left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right] + \dfrac{\frac{1}{4}}{x^2}\right\}u = 0 \\ \left(c_1 \dfrac{x^{-m+1}}{-m+1} + c_2\right) x^{\frac{m}{2}} e^{-\frac{1}{2}\int R dx} & u'' + Ru' + \left\{\left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right] - \dfrac{\frac{m}{2}\left(\frac{m}{2} - 1\right)}{x^2}\right\}u = 0 \quad , \ (m \neq 1) \end{cases}$$

and:

$$u = \begin{cases} (c_1 \log x + c_2) x^{\frac{1}{2}} e^{-\frac{1}{2}\int R dx} \Rightarrow & u'' + Ru' + \left\{\left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right] + \dfrac{\frac{1}{4}}{x^2}\right\}u = 0 \\ \left(c_1 \dfrac{x^{m+1}}{m+1} + c_2\right) x^{-\frac{m}{2}} e^{-\frac{1}{2}\int R dx} & u'' + Ru' + \left\{\left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right] - \dfrac{\frac{m}{2}\left(\frac{m}{2} + 1\right)}{x^2}\right\}u = 0 \quad , \ (m \neq -1) \end{cases}$$

which are the same under the transformation: $m \to -m$

□

Lemma II.2a: $u' + u^2 + Pu = -Q \Rightarrow \exists\Psi_1 : Q = -\left[\left(\frac{1}{2}P\right)' + \left(\frac{1}{2}P\right)^2\right] + (\Psi_1' + \Psi_1^2)$

Proof:

$$u' + \left(\tfrac{1}{2}P\right)' - \left(\tfrac{1}{2}P\right)' + u^2 + Pu + \left(\tfrac{1}{2}P\right)^2 - \left(\tfrac{1}{2}P\right)^2 = Q$$
$$\Rightarrow \left(u + \tfrac{1}{2}P\right)' + \left(u + \tfrac{1}{2}P\right)^2 - \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right] = Q$$

□

Corollary II.2a: $y'' + Py' + Qy = 0 \Rightarrow \exists\Psi_1 : Q = -\left[\left(\frac{1}{2}P\right)' + \left(\frac{1}{2}P\right)^2\right] + (\Psi_1' + \Psi_1^2)$

Proof:

lemma II.2a under transformation: $y = e^{\int u dx} \Leftrightarrow u = (\log y)'$

□

Lemma III.1: $u' + u^2 - [v' + v^2] = -Q = (u-v)' + (u-v)^2 + 2v(u-v)$

Proof:

$$u' + u^2 - [v' + v^2] = -Q$$
$$\Rightarrow (u-v)' + u^2 - 2uv + v^2 + 2uv - 2v^2 = -Q$$
$$\Rightarrow (u-v)' + (u-v)^2 + 2v(u-v) = -Q$$

□

Corollary III.1: $\left(u - \frac{1}{x}\right)' + \left(u - \frac{1}{x}\right)^2 + \frac{2}{x}\left(u - \frac{1}{x}\right) = -Q = u' + u^2$

Proof:

$$v = \frac{1}{x} \Rightarrow v' = -\frac{1}{x^2} \quad \& \quad v^2 = \frac{1}{x^2} \Rightarrow v' + v^2 = 0$$
$$\Rightarrow (u-v)' + (u-v)^2 + 2v(u-v) = -Q = u' + u^2 - [v' + v^2] = u' + u^2$$
$$\Rightarrow \left(u - \frac{1}{x}\right)' + \left(u - \frac{1}{x}\right)^2 + \frac{2}{x}\left(u - \frac{1}{x}\right) = -Q = u' + u^2 - [v' + v^2] = u' + u^2$$

□

Corollary III.1a: $(u - \frac{m}{x})' + (u - \frac{m}{x})^2 + \frac{2m}{x}(u - \frac{m}{x}) = -Q = u' + u^2 - \frac{m(m-1)}{x^2}$

Proof:

$$v = \frac{m}{x} \Rightarrow v' = -\frac{m}{x^2} \quad \& \quad v^2 = \frac{m^2}{x^2} \Rightarrow v' + v^2 = \frac{m(m-1)}{x^2}$$
$$\Rightarrow (u-v)' + (u-v)^2 + 2v(u-v) = -Q = u' + u^2 - [v' + v^2] = u' + u^2 - \frac{m(m-1)}{x^2}$$
$$\Rightarrow (u - \tfrac{m}{x})' + (u - \tfrac{m}{x})^2 + \frac{2m}{x}(u - \tfrac{m}{x}) = -Q = u' + u^2 - [v' + v^2] = u' + u^2 - \frac{m(m-1)}{x^2}$$

□

Lemma III.2: $y = e^{\int (u-v)dx} \Leftrightarrow u - v = (\log y)' \Rightarrow y'' + 2vy' - \left[(u-v)' + (u-v)^2 + 2v(u-v)\right]y = 0$

Proof:

$$y = e^{\int (u-v)dx} \Leftrightarrow u - v = (\log y)'$$
$$y' = (u-v)y$$
$$y'' = (u-v)y' + (u-v)'y = \left[(u-v)^2 + (u-v)'\right]y$$
$$\Rightarrow y'' + 2vy' = \left[(u-v)^2 + (u-v)' + 2v(u-v)\right]y$$

□

Theorem III.1: For all differentiable P, R :

$$u'' + Ru' + \left\{\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right]\right\}u = 0$$
$$\Rightarrow \begin{cases} \left[D + \frac{1}{2}(R+P)\right]\left[D + \frac{1}{2}(R-P)\right]u = 0 \\ u = \left(c_1 \int e^{-\int P dx} dx + c_2\right) e^{\frac{1}{2}\int (P-R)dx} = c_2 e^{\frac{1}{2}\int (P-R)dx} + c_1 e^{\frac{1}{2}\int (P-R)dx} \int e^{-\int P dx} dx \end{cases}$$

Proof:

By the above theorem II.1, a solution to: $u'' + Ru' + \left\{\left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right] - \left[\left(\frac{1}{2}P\right)' + \left(\frac{1}{2}P\right)^2\right]\right\}u = 0$

is: $u = \left(c_1 \int e^{-\int P dx} dx + c_2\right) e^{\frac{1}{2}\int (P-R)dx} = c_3 e^{\frac{1}{2}\int (P-R)dx} + c_4 e^{\frac{1}{2}\int (P-R)dx} \int e^{-\int P dx} dx$

But, as shown previously, any 2nd order linear the total solution of a 2nd Order HLODE:

$y'' + Ry' + Ty = 0$ may be written:

$$y = c_1 e^{-\int g dx} + c_2 e^{-\int g dx} \int e^{\int (2g-R)dx} dx$$
$$R = g + h \quad , \quad T = g' + gh$$

Let: $y = u$ and matching the respective parts:

$$-g = \tfrac{1}{2}(P-R) \quad \& \quad 2g - R = -P = 2\left[-\tfrac{1}{2}(P-R)\right] - R = -P + R - R \quad \checkmark$$
$$\Rightarrow h = R - g = R + \tfrac{1}{2}(P-R) = \tfrac{1}{2}(R+P)$$
$$\Rightarrow T = \left(\tfrac{1}{2}(R-P)\right)' + \left(\tfrac{1}{2}(R-P)\right)\left(\tfrac{1}{2}(R+P)\right)$$
$$= \left(\tfrac{1}{2}(R-P)\right)' + \left(\tfrac{1}{2}(R-P)\right)\left(\tfrac{1}{2}(R-P) + P\right)$$
$$= \left(\tfrac{1}{2}(R-P)\right)' + \left(\tfrac{1}{2}(R-P)\right)^2 + \left(\tfrac{1}{2}(R-P)\right)P$$
$$= \left(\tfrac{1}{2}R - \tfrac{1}{2}P\right)' + \left(\left(\tfrac{1}{2}R - \tfrac{1}{2}P\right)\right)^2 + 2\left(\tfrac{1}{2}P\right)\left(\left(\tfrac{1}{2}R - \tfrac{1}{2}P\right)\right)$$

So, by corollary II.1b0 or lemma III.1:

$$T = \left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2 - \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right]$$

So:

$$u'' + Ru' + \left\{\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right]\right\}u = 0$$
$$\Rightarrow (D+h)(D+g)u = \left[D + \tfrac{1}{2}(R+P)\right]\left[D + \tfrac{1}{2}(R-P)\right]u = 0$$

□

Corollary III.1c: For all differentiable P :

$$u'' + \left\{-\left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right]\right\}u = 0$$
$$\Rightarrow \begin{cases} \left(D + \tfrac{1}{2}P\right)\left(D - \tfrac{1}{2}P\right)u = 0 \\ u = \left(c_1 \int e^{-\int Pdx}dx + c_2\right)e^{\frac{1}{2}\int Pdx} = c_2 e^{\frac{1}{2}\int Pdx} + c_1 e^{\frac{1}{2}\int Pdx}\int e^{-\int Pdx}dx \end{cases}$$

Proof:

By theorem III.1, a solution to: $u'' + Ru' + \left\{\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right]\right\}u = 0$

is: $u = \left(c_1 \int e^{-\int Pdx}dx + c_2\right)e^{\frac{1}{2}\int (P-R)dx} = c_2 e^{\frac{1}{2}\int (P-R)dx} + c_1 e^{\frac{1}{2}\int (P-R)dx}\int e^{-\int Pdx}dx$

$$R = 0 \Rightarrow \begin{cases} \left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right] = -\left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right] \\ \left[D + \tfrac{1}{2}(R+P)\right]\left[D + \tfrac{1}{2}(R-P)\right]u = u'' - \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right]u \\ \left(c_1 \int e^{-\int Pdx}dx + c_2\right)e^{\frac{1}{2}\int (P-R)dx} = \left(c_1 \int e^{-\int Pdx}dx + c_2\right)e^{\frac{1}{2}\int Pdx} \end{cases}$$

□

Corollary III.1c.1: For all differentiable P :

$$u'' + \left\{-\left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right]\right\}u = 0$$
$$\Rightarrow \begin{cases} \left(D + \tfrac{1}{2}P\right)\left(D - \tfrac{1}{2}P\right)u = 0 \\ u = \left(c_1 \int e^{-\int Pdx}dx + c_2\right)e^{\frac{1}{2}\int Pdx} = e^{\int\left[\frac{1}{2}P + \left(\log\left(\int e^{-\int Pdx}dx\right)\right)\right]dx} \end{cases}$$

Proof:

By corollary III.1c, a solution to: $u'' + \left\{-\left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right]\right\}u = 0$

is: $u = \left(c_1 \int e^{-\int Pdx}dx + c_2\right)e^{\frac{1}{2}\int Pdx} = c_2 e^{\frac{1}{2}\int Pdx} + c_1 e^{\frac{1}{2}\int Pdx}\int e^{-\int Pdx}dx$

So, as usual, under the transformation: $w = (\log u)' = \frac{u'}{u} \Leftrightarrow u = e^{\int wdx}$

$$\Rightarrow w' = \left(\frac{u'}{u}\right)' = -\frac{1}{u^2}u'u' + \frac{u''}{u} = -\left(\frac{u'}{u}\right)^2 + \frac{u''}{u} = -w^2 + \frac{u''}{u}$$
$$\Rightarrow w' + w^2 + Q = \frac{1}{u}(u'' + Qu) \Rightarrow w' + w^2 + \left\{-\left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right]\right\} = \frac{1}{u}\left(u'' + \left\{-\left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right]\right\}u\right)$$
$$\Rightarrow u'' + \left\{-\left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right]\right\}u = 0 \Rightarrow w' + w^2 = \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right]$$
$$\Rightarrow w' + w^2 = \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right] \Rightarrow w = \left(\log\left[\left(c_1 \int e^{-\int Pdx}dx + c_2\right)e^{\frac{1}{2}\int Pdx}\right]\right)'$$
$$= (\log[c_2])' + \left(\log\left[e^{\frac{1}{2}\int Pdx}\right]\right)' + (\log[c_1])' + \left(\log\left(\int e^{-\int Pdx}dx\right)\right)'$$
$$= \tfrac{1}{2}P + \left(\log\left(\int e^{-\int Pdx}dx\right)\right) \quad \text{matches lemma II.1b}$$

□

Corollary III.1d: For all differentiable P, R, T :

$$u'' + Ru' + Tu = 0$$
$$\Rightarrow \begin{cases} \left[D + \tfrac{1}{2}(R+P)\right]\left[D + \tfrac{1}{2}(R-P)\right]u = 0 \\ u = c_2 e^{\int\left(-\frac{1}{2}R + \frac{1}{2}P\right)dx} + c_1 e^{\int\left(-\frac{1}{2}R - \frac{1}{2}P\right)dx}\left(\dfrac{\int e^{-\int Pdx}dx}{e^{-\int Pdx}}\right) \\ \text{where: } \left(\tfrac{1}{2}P\right)^2 + \left(\tfrac{1}{2}P\right)' = \tfrac{1}{4}[R^2 - 4T + 2R'] \end{cases}$$

Proof:

By the above theorem III.1:

$$u'' + Ru' + \left\{\left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right]\right\}u = 0$$
$$\Rightarrow \begin{cases} \left[D + \tfrac{1}{2}(R+P)\right]\left[D + \tfrac{1}{2}(R-P)\right]u = 0 \\ u = \left(c_1 \int e^{-\int Pdx}dx + c_2\right)e^{\frac{1}{2}\int (P-R)dx} = c_2 e^{\frac{1}{2}\int (P-R)dx} + c_1 e^{\frac{1}{2}\int (P-R)dx}\int e^{-\int Pdx}dx \end{cases}$$
$$\Rightarrow u = c_2 e^{\int\left(-\frac{1}{2}R + \frac{1}{2}P\right)dx} + c_1 e^{\int\left(-\frac{1}{2}R + \frac{1}{2}P\right)dx}\int e^{-\int Pdx}dx$$
$$= c_2 e^{\int\left(-\frac{1}{2}R + \frac{1}{2}P\right)dx} + c_1 e^{\int\left(-\frac{1}{2}R - \frac{1}{2}P\right)dx} e^{\int Pdx}\int e^{-\int Pdx}dx$$
$$= c_2 e^{\int\left(-\frac{1}{2}R + \frac{1}{2}P\right)dx} + c_1 e^{\int\left(-\frac{1}{2}R - \frac{1}{2}P\right)dx}\left(\dfrac{\int e^{-\int Pdx}dx}{e^{-\int Pdx}}\right)$$

where: $T \equiv \left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right]$

$$\Rightarrow \left(\tfrac{1}{2}P\right)^2 + \left(\tfrac{1}{2}P\right)' = \tfrac{1}{4}[R^2 - 4T + 2R']$$

□

This form leads to a generalization of Euler's 2nd order constant coefficients solution formula; and more.

Corollary III.1d1: For differentiable R, T, Φ, Ψ and constant A :

$$u'' + Ru' + Tu = 0$$

$$\Rightarrow \begin{cases} \left[D + \tfrac{1}{2}(R + A\Phi)\right]\left[D + \tfrac{1}{2}(R - A\Phi)\right]u = 0 \\ u = c_2 e^{\int\left(-\frac{1}{2}R + \frac{1}{2}A\Phi\right)dx} + c_1 e^{\int\left(-\frac{1}{2}R - \frac{1}{2}A\Phi\right)dx}\left(\dfrac{\int e^{-\int A\Phi dx}dx}{e^{-\int A\Phi dx}}\right) \end{cases}$$

$$\text{where: } \exists A\Phi = \left(\pm\sqrt{R^2 - 4T + 2R' + \left(\frac{\Phi'}{\Phi}\right)^2 - 2\left[\Psi' - \left(\frac{\Phi'}{\Phi}\right)\Psi\right]} - \left[\frac{\Phi'}{\Phi} + \Psi\right]\right)$$

Proof:

By corollary III.1d:

$$u'' + Ru' + Tu = 0$$

$$\Rightarrow \begin{cases} \left[D + \tfrac{1}{2}(R + P)\right]\left[D + \tfrac{1}{2}(R - P)\right]u = 0 \\ u = c_2 e^{\int\left(-\frac{1}{2}R + \frac{1}{2}P\right)dx} + c_1 e^{\int\left(-\frac{1}{2}R - \frac{1}{2}P\right)dx}\left(\dfrac{\int e^{-\int P dx}dx}{e^{-\int P dx}}\right) \end{cases}$$

$$\text{where: } \left(\tfrac{1}{2}P\right)^2 + \left(\tfrac{1}{2}P\right)' = \tfrac{1}{4}[R^2 - 4T + 2R']$$

For: $P = A\Phi + \Psi$, A constant:

$$\begin{aligned} \text{where: } \tfrac{1}{4}[R^2 - 4T + 2R'] &= \left(\tfrac{1}{2}P\right)^2 + \left(\tfrac{1}{2}P\right)' \\ &= \tfrac{1}{4}\left(A\Phi + \Psi\right)^2 + \tfrac{1}{2}\left(A\Phi + \Psi\right)' \\ &= \tfrac{1}{4}\left(A^2\Phi^2 + 2A\Phi\Psi + \Psi^2\right) + \tfrac{1}{2}\left(A\Phi' + \Psi'\right) \end{aligned}$$

$$\Rightarrow \tfrac{1}{4}\Phi^2 A^2 + \tfrac{1}{2}A\Phi\Psi + \tfrac{1}{4}\Psi^2 + \tfrac{1}{2}A\Phi' + \tfrac{1}{2}\Psi' - \tfrac{1}{4}[R^2 - 4T + 2R'] = 0$$

$$\Rightarrow \Phi^2 A^2 + 2A\Phi\Psi + \Psi^2 + 2A\Phi' + 2\Psi' - [R^2 - 4T + 2R'] = 0$$

$$\Rightarrow \Phi^2 A^2 + 2(\Phi\Psi + \Phi')A + \Psi^2 + 2\Psi' - [R^2 - 4T + 2R'] = 0$$

$$\Rightarrow A^2 + 2\left(\frac{\Psi}{\Phi} + \frac{\Phi'}{\Phi^2}\right)A + \left(\frac{\Psi^2}{\Phi^2} + 2\frac{\Psi'}{\Phi^2}\right) - \frac{1}{\Phi^2}[R^2 - 4T + 2R'] = 0$$

$$\begin{aligned} \Rightarrow A &= \tfrac{1}{2}\left[-2\left(\frac{\Psi}{\Phi} + \frac{\Phi'}{\Phi^2}\right) \pm \sqrt{4\left(\frac{\Psi}{\Phi} + \frac{\Phi'}{\Phi^2}\right)^2 - 4\left(\left[\frac{\Psi^2}{\Phi^2} + 2\frac{\Psi'}{\Phi^2}\right] - \frac{1}{\Phi^2}[R^2 - 4T + 2R']\right)}\right] \\ &= -\left(\frac{\Psi}{\Phi} + \frac{\Phi'}{\Phi^2}\right) \pm \sqrt{\frac{1}{\Phi^2}[R^2 - 4T + 2R'] - \left[\frac{\Psi^2}{\Phi^2} + 2\frac{\Psi'}{\Phi^2}\right] + \left(\frac{\Psi}{\Phi} + \frac{\Phi'}{\Phi^2}\right)^2} \\ &= \pm\sqrt{\frac{1}{\Phi^2}[R^2 - 4T + 2R'] - \frac{\Psi^2}{\Phi^2} - 2\frac{\Psi'}{\Phi^2} + \frac{\Psi^2}{\Phi^2} + 2\frac{\Psi}{\Phi}\frac{\Phi'}{\Phi^2} + \left(\frac{\Phi'}{\Phi^2}\right)^2} - \left(\frac{\Psi}{\Phi} + \frac{\Phi'}{\Phi^2}\right) \\ &= \pm\sqrt{\frac{1}{\Phi^2}[R^2 - 4T + 2R'] - 2\frac{\Psi'}{\Phi^2} + 2\frac{\Psi}{\Phi}\frac{\Phi'}{\Phi^2} + \left(\frac{\Phi'}{\Phi^2}\right)^2} - \left(\frac{\Psi}{\Phi} + \frac{\Phi'}{\Phi^2}\right) \\ &= \frac{1}{\Phi}\left(\pm\sqrt{R^2 - 4T + 2R' + \left(\frac{\Phi'}{\Phi}\right)^2 - 2\left[\Psi' - \left(\frac{\Phi'}{\Phi}\right)\Psi\right]} - \left[\frac{\Phi'}{\Phi} + \Psi\right]\right) \end{aligned}$$

$$\Rightarrow A\Phi = \left(\pm\sqrt{R^2 - 4T + 2R' + \left(\frac{\Phi'}{\Phi}\right)^2 - 2\left[\Psi' - \left(\frac{\Phi'}{\Phi}\right)\Psi\right]} - \left[\frac{\Phi'}{\Phi} + \Psi\right]\right)$$

$$\Rightarrow \begin{cases} \left[D + \tfrac{1}{2}(R + A\Phi)\right]\left[D + \tfrac{1}{2}(R - A\Phi)\right]u = 0 \\ u = c_2 e^{\int\left(-\frac{1}{2}R + \frac{1}{2}A\Phi\right)dx} + c_1 e^{\int\left(-\frac{1}{2}R - \frac{1}{2}A\Phi\right)dx}\left(\dfrac{\int e^{-\int A\Phi dx}dx}{e^{-\int A\Phi dx}}\right) \end{cases}$$

□

Obviously, this is satisfied $\exists A$ for the constant coefficients and Cauchy-Euler HLODEs with:
$(\Phi, \Psi) = (1, 0)$; $(\Phi, \Psi) = \left(\frac{1}{x}, 0\right)$, respectively.

Corollary III.1d2: For all differentiable R, T and constant A :

$$u'' + Ru' + Tu = 0$$

$$\Rightarrow \begin{cases} \left[D + \tfrac{1}{2}(R + A)\right]\left[D + \tfrac{1}{2}(R - A)\right]u = 0 \\ u = c_2 e^{\int\left(-\frac{1}{2}R + \frac{1}{2}A\right)dx} + c_1 e^{\int\left(-\frac{1}{2}R - \frac{1}{2}A\right)dx}\left(\dfrac{\int e^{-\int A dx}dx}{e^{-\int A dx}}\right) \end{cases} \quad , \quad A \neq 0$$

$$\Rightarrow \begin{cases} \left[D + \tfrac{1}{2}R\right]\left[D + \tfrac{1}{2}R\right]u = 0 \\ u = (c_2 + c_1 x)e^{-\frac{1}{2}\int R dx} \end{cases} \quad , \quad A = 0$$

$$\text{where: } A = \pm\sqrt{R^2 - 4T + 2R'}$$

Proof:

By corollary III.1d1:

$$u'' + Ru' + Tu = 0$$

$$\Rightarrow \begin{cases} \left[D+\tfrac{1}{2}(R+A\Phi)\right]\left[D+\tfrac{1}{2}(R-A\Phi)\right]u = 0 \\ u = c_2 e^{\int\left(-\frac{1}{2}R+\frac{1}{2}A\Phi\right)dx} + c_1 e^{\int\left(-\frac{1}{2}R-\frac{1}{2}A\Phi\right)dx}\left(\dfrac{\int e^{-\int A\Phi dx}dx}{e^{-\int A\Phi dx}}\right) \end{cases}$$

$$\text{where: } \exists A\Phi = \left(\pm\sqrt{R^2-4T+2R'+\left(\frac{\Phi'}{\Phi}\right)^2-2\left[\Psi'-\left(\frac{\Phi'}{\Phi}\right)\Psi\right]}-\left[\frac{\Phi'}{\Phi}+\Psi\right]\right)$$

For: $(\Phi,\Psi) = (1,0)$:

$$u''+Ru'+Tu = 0$$

$$\Rightarrow \begin{cases} \left[D+\tfrac{1}{2}(R+A)\right]\left[D+\tfrac{1}{2}(R-A)\right]u = 0 \\ u = c_2 e^{\int\left(-\frac{1}{2}R+\frac{1}{2}A\right)dx} + c_1 e^{\int\left(-\frac{1}{2}R-\frac{1}{2}A\right)dx}\left(\dfrac{\int e^{-Ax}dx}{e^{-Ax}}\right) \end{cases}$$

$$\text{where: } A = \tfrac{1}{2}\left[\pm\sqrt{R^2-4T+2R'}\right]$$

and:

$$\frac{\int e^{-\int Adx}dx}{e^{-\int Adx}} = \frac{\int e^{-Ax}dx}{e^{-Ax}} = \begin{cases} = \dfrac{1}{A} & , \quad A \neq 0 \\ = x & , \quad A = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \left[D+\tfrac{1}{2}(R+A)\right]\left[D+\tfrac{1}{2}(R-A)\right]u = 0 \\ u = c_2 e^{\int\left(-\frac{1}{2}R+\frac{1}{2}A\right)dx} + c_1 e^{\int\left(-\frac{1}{2}R-\frac{1}{2}A\right)dx}\left(\dfrac{\int e^{-\int Adx}dx}{e^{-\int Adx}}\right) \end{cases} \quad , \quad A \neq 0$$

$$\Rightarrow \begin{cases} \left[D+\tfrac{1}{2}R\right]\left[D+\tfrac{1}{2}R\right]u = 0 \\ u = (c_2+c_1x)e^{-\frac{1}{2}\int Rdx} \end{cases} \quad , \quad A = 0$$

□

Corollary III.1d3: For all differentiable R,T,u,g :

$$u''+Ru'+Tu = 0$$

$$\Rightarrow \begin{cases} \left[D+\tfrac{1}{2}\left(R+2\frac{g'}{g}\right)\right]\left[D+\tfrac{1}{2}\left(R-2\frac{g'}{g}\right)\right]u = 0 \\ u = ge^{-\frac{1}{2}\int Rdx}\left(c_2+c_1\int\dfrac{dx}{g^2}\right) \end{cases}$$

$$\text{where: } g''-\tfrac{1}{4}[R^2-4T+2R']g = 0$$

Proof:

By corollary III.1d:

$$u''+Ru'+Tu = 0$$

$$\Rightarrow \begin{cases} \left[D+\tfrac{1}{2}(R+P)\right]\left[D+\tfrac{1}{2}(R-P)\right]u = 0 \\ u = c_2 e^{\int\left(-\frac{1}{2}R+\frac{1}{2}P\right)dx} + c_1 e^{\int\left(-\frac{1}{2}R-\frac{1}{2}P\right)dx}\left(\dfrac{\int e^{-\int Pdx}dx}{e^{-\int Pdx}}\right) \end{cases}$$

$$\text{where: } \left(\tfrac{1}{2}P\right)^2+\left(\tfrac{1}{2}P\right)' = \tfrac{1}{4}[R^2-4T+2R']$$

Let: $\frac{g'}{g} = \frac{1}{2}P \Rightarrow \frac{1}{4}[R^2-4T+2R'] = \left(\frac{1}{2}P\right)^2+\left(\frac{1}{2}P\right)' = \left(\frac{g'}{g}\right)^2+\left(\frac{g'}{g}\right)'$

$$= \left(\frac{g'}{g}\right)^2+\left(\frac{gg''-g'g'}{g^2}\right) = \left(\frac{g'}{g}\right)^2+\left(\frac{g''}{g}\right)-\left(\frac{g'}{g}\right)^2$$

$$= \frac{g''}{g} \Rightarrow g''-\tfrac{1}{4}[R^2-4T+2R']g = 0$$

$$\Rightarrow (\log g)' = \frac{g'}{g} = \tfrac{1}{2}P \Rightarrow g = e^{\frac{1}{2}\int Pdx} \Rightarrow \left(e^{\frac{1}{2}\int Pdx}\right)''-\tfrac{1}{4}[R^2-4T+2R']\left(e^{\frac{1}{2}\int Pdx}\right) = 0$$

$$\Rightarrow u = c_2 e^{\int\left(-\frac{1}{2}R+\frac{g'}{g}\right)dx} + c_1 e^{\int\left(-\frac{1}{2}R-\frac{g'}{g}\right)dx}\left(\frac{\int e^{-2\int\frac{g'}{g}dx}dx}{e^{-2\int\frac{g'}{g}dx}}\right)$$

$$\Rightarrow u = c_2 e^{-\frac{1}{2}\int Rdx}e^{\int\frac{dg}{g}} + c_1 e^{-\frac{1}{2}\int Rdx}e^{-\int\frac{dg}{g}}\left(\frac{\int e^{-2\int\frac{dg}{g}}dx}{e^{-2\int\frac{dg}{g}}}\right)$$

$$\Rightarrow u = c_2 e^{-\frac{1}{2}\int Rdx}e^{\log g} + c_1 e^{-\frac{1}{2}\int Rdx}e^{-\log g}\left(\frac{\int e^{-2\log g}dx}{e^{-2\log g}}\right)$$

$$\Rightarrow u = c_2 e^{-\frac{1}{2}\int Rdx}g + c_1 e^{-\frac{1}{2}\int Rdx}\frac{1}{g}\left(\frac{\int g^{-2}dx}{g^{-2}}\right)$$

$$\Rightarrow u = c_2 e^{-\frac{1}{2}\int Rdx}g + c_1 e^{-\frac{1}{2}\int Rdx}g\left(\int\frac{dx}{g^2}\right)$$

$$\Rightarrow$$

$$\Rightarrow u = ge^{-\frac{1}{2}\int Rdx}\left(c_2+c_1\int\frac{dx}{g^2}\right)$$

$$\Rightarrow \begin{cases} \left[D+\tfrac{1}{2}\left(R+2\frac{g'}{g}\right)\right]\left[D+\tfrac{1}{2}\left(R-2\frac{g'}{g}\right)\right]u = 0 \\ u = ge^{-\frac{1}{2}\int Rdx}\left(c_2+c_1\int \frac{dx}{g^2}\right) \\ \text{where: } g''-\tfrac{1}{4}[R^2-4T+2R']g = 0 \end{cases}$$

□

Now, recall from previous [9]-theorem #1:

If $y_1''+P_1y_1'+Q_1y_1 = 0$ and $y_2''+P_2y_2'+Q_2y_2 = 0$ and:

$$u = \frac{y_2}{y_1}$$

then

$$0 = u''+\left(2\frac{y_1'}{y_1}+P_2\right)u'+\left[(P_2-P_1)\frac{y_1'}{y_1}+Q_2-Q_1\right]u$$

Leads to:

Theorem IV: If $y_1 = \left(c_{11}\int e^{-\int R_1dx}dx+c_{21}\right)e^{\frac{1}{2}\int(R_1-P_1)dx}$

and $y_2 = \left(c_{12}\int e^{-\int R_2dx}dx+c_{22}\right)e^{\frac{1}{2}\int(R_2-P_2)dx}$

and: $u = \dfrac{y_2}{y_1}$

then

$$0 = u''+\left[(P_2-P_1)+R_1+2\left(\frac{e^{-\int R_1dx}}{\int e^{-\int R_1dx}dx+c}\right)\right]u'+$$
$$+\left[(P_2-P_1)\left\{\tfrac{1}{2}(R_1-P_1)+\left(\frac{e^{-\int R_1dx}}{\int e^{-\int R_1dx}dx+\frac{c_{21}}{c_{11}}}\right)\right\}+\right.$$
$$+\left\{\left[\left(\tfrac{1}{2}P_2\right)'+\left(\tfrac{1}{2}P_2\right)^2\right]-\left[\left(\tfrac{1}{2}R_2\right)'+\left(\tfrac{1}{2}R_2\right)^2\right]\right\}+$$
$$\left.+\left\{\left[\left(\tfrac{1}{2}R_1\right)'+\left(\tfrac{1}{2}R_1\right)^2\right]-\left[\left(\tfrac{1}{2}P_1\right)'+\left(\tfrac{1}{2}P_1\right)^2\right]\right\}\right]u$$

Proof:

By theorem II.1:

For differentiable P_1, R_1 :

$$y_1 = \left(c_{11}\int e^{-\int R_1dx}dx+c_{21}\right)e^{\frac{1}{2}\int(R_1-P_1)dx}$$
$$\Rightarrow y_1''+P_1y_1'+\left\{\left[\left(\tfrac{1}{2}P_1\right)'+\left(\tfrac{1}{2}P_1\right)^2\right]-\left[\left(\tfrac{1}{2}R_1\right)'+\left(\tfrac{1}{2}R_1\right)^2\right]\right\}y_1 = 0$$

and:

For differentiable P_2, R_2 :

$$y_2 = \left(c_{12}\int e^{-\int R_2dx}dx+c_{22}\right)e^{\frac{1}{2}\int(R_2-P_2)dx}$$
$$\Rightarrow y_2''+P_2y_2'+\left\{\left[\left(\tfrac{1}{2}P_2\right)'+\left(\tfrac{1}{2}P_2\right)^2\right]-\left[\left(\tfrac{1}{2}R_2\right)'+\left(\tfrac{1}{2}R_2\right)^2\right]\right\}y_2 = 0$$

But, by [10]-theorem #1:

If $y_1''+P_1y_1'+Q_1y_1 = 0$ and $y_2''+P_2y_2'+Q_2y_2 = 0$ and:

$$u = \frac{y_2}{y_1}$$

then

$$0 = u''+\left(2\frac{y_1'}{y_1}+P_2\right)u'+\left[(P_2-P_1)\frac{y_1'}{y_1}+Q_2-Q_1\right]u$$

So:

$$0 = u''+\left(2\frac{y_1'}{y_1}+P_2\right)u'+$$
$$+\left[(P_2-P_1)\frac{y_1'}{y_1}+\left\{\left[\left(\tfrac{1}{2}P_2\right)'+\left(\tfrac{1}{2}P_2\right)^2\right]-\left[\left(\tfrac{1}{2}R_2\right)'+\left(\tfrac{1}{2}R_2\right)^2\right]\right\}+\right.$$
$$\left.-\left\{Q_1-\left[\left(\tfrac{1}{2}R_1\right)'+\left(\tfrac{1}{2}R_1\right)^2\right]+\left[\left(\tfrac{1}{2}P_1\right)'+\left(\tfrac{1}{2}P_1\right)^2\right]\right\}\right]u$$

$$\frac{y_1'}{y_1} = \frac{c_{11}e^{-\frac{1}{2}\int(R_1+P_1)dx}+\frac{1}{2}(R_1-P_1)\left(c_{11}\int e^{-\int R_1dx}dx+c_{21}\right)e^{\frac{1}{2}\int(R_1-P_1)dx}}{\left(c_{11}\int e^{-\int R_1dx}dx+c_{21}\right)e^{\frac{1}{2}\int(R_1-P_1)dx}}$$
$$= \frac{c_{11}e^{-\int R_1dx}+\frac{1}{2}(R_1-P_1)\left(c_{11}\int e^{-\int R_1dx}dx+c_{21}\right)}{c_{11}\int e^{-\int R_1dx}dx+c_{21}}$$
$$= \tfrac{1}{2}(R_1-P_1)+\left(\frac{e^{-\int R_1dx}}{\int e^{-\int R_1dx}dx+\frac{c_{21}}{c_{11}}}\right)$$
$$\Rightarrow 0 = u''+\left[P_2+(R_1-P_1)+2\left(\frac{e^{-\int R_1dx}}{\int e^{-\int R_1dx}dx+c}\right)\right]u'+$$

$$+\Big[(P_2-P_1)\Bigg\{\tfrac{1}{2}(R_1-P_1)+\left(\frac{e^{-\int R_1dx}}{\int e^{-\int R_1dx}dx+\frac{c_{21}}{c_{11}}}\right)\Bigg\}+$$
$$+\Big\{\Big[\big(\tfrac{1}{2}P_2\big)'+\big(\tfrac{1}{2}P_2\big)^2\Big]-\Big[\big(\tfrac{1}{2}R_2\big)'+\big(\tfrac{1}{2}R_2\big)^2\Big]\Big\}+$$
$$+\Big\{\Big[\big(\tfrac{1}{2}R_1\big)'+\big(\tfrac{1}{2}R_1\big)^2\Big]-\Big[\big(\tfrac{1}{2}P_1\big)'+\big(\tfrac{1}{2}P_1\big)^2\Big]\Big\}\Big]u$$

□

Now, generalizing using invariants:

$$u=ye^{\frac{1}{2}\int(P-R)dx}\Rightarrow ue^{\frac{1}{2}\int Rdx}=ye^{\frac{1}{2}\int Pdx}$$
$$\Rightarrow\left(ue^{\frac{1}{2}\int Rdx}\right)''=\left(ye^{\frac{1}{2}\int Pdx}\right)''\quad\text{and so on, leading to:}$$

Theorem V.1: For all differentiable u,y,R,P :

$$u=(c_{11}x+c_{12})e^{-\frac{1}{2}\int Rdx}\ \ \&\ \ y=(c_{21}x+c_{22})e^{-\frac{1}{2}\int Pdx}$$
$$\Rightarrow u''+Ru'+\Big[\big(\tfrac{1}{2}R\big)'+\big(\tfrac{1}{2}R\big)^2\Big]u=0=y''+Py'+\Big[\big(\tfrac{1}{2}P\big)'+\big(\tfrac{1}{2}P\big)^2\Big]y$$

Proof:

$$u=ye^{\frac{1}{2}\int(P-R)dx}\Rightarrow ue^{\frac{1}{2}\int Rdx}=ye^{\frac{1}{2}\int Pdx}$$
$$\Rightarrow\begin{cases}\left(ue^{\frac{1}{2}\int Rdx}\right)'=\left(ye^{\frac{1}{2}\int Pdx}\right)'\\ \left(ue^{\frac{1}{2}\int Rdx}\right)''=\left(ye^{\frac{1}{2}\int Pdx}\right)''\end{cases}$$

So:

$$\left(ue^{\frac{1}{2}\int Rdx}\right)'=\big(u'+\tfrac{1}{2}Ru\big)e^{\frac{1}{2}\int Rdx}$$
$$\left(ye^{\frac{1}{2}\int Pdx}\right)'=\big(y'+\tfrac{1}{2}Py\big)e^{\frac{1}{2}\int Pdx}$$
$$\left(ue^{\frac{1}{2}\int Rdx}\right)''=\Big\{u''+Ru'+\Big[\big(\tfrac{1}{2}R\big)'+\big(\tfrac{1}{2}R\big)^2\Big]u\Big\}e^{\frac{1}{2}\int Rdx}$$
$$\left(ye^{\frac{1}{2}\int Pdx}\right)''=\Big\{y''+Py'+\Big[\big(\tfrac{1}{2}P\big)'+\big(\tfrac{1}{2}P\big)^2\Big]y\Big\}e^{\frac{1}{2}\int Pdx}$$
$$\Rightarrow 0=\left(ue^{\frac{1}{2}\int Rdx}\right)''=\Big\{u''+Ru'+\Big[\big(\tfrac{1}{2}R\big)'+\big(\tfrac{1}{2}R\big)^2\Big]u\Big\}e^{\frac{1}{2}\int Rdx}$$
$$=\left(ye^{\frac{1}{2}\int Pdx}\right)''=\Big\{y''+Py'+\Big[\big(\tfrac{1}{2}P\big)'+\big(\tfrac{1}{2}P\big)^2\Big]y\Big\}e^{\frac{1}{2}\int Pdx}=0$$
$$\Rightarrow u''+Ru'+\Big[\big(\tfrac{1}{2}R\big)'+\big(\tfrac{1}{2}R\big)^2\Big]u=0=y''+Py'+\Big[\big(\tfrac{1}{2}P\big)'+\big(\tfrac{1}{2}P\big)^2\Big]y$$

So:

$$0=\left(ue^{\frac{1}{2}\int Rdx}\right)''\Rightarrow c_{11}=\left(ue^{\frac{1}{2}\int Rdx}\right)'\Rightarrow c_{11}x+c_{12}=ue^{\frac{1}{2}\int Rdx}$$
$$\Rightarrow u=(c_{11}x+c_{12})e^{-\frac{1}{2}\int Rdx}\ \ \&\ \ y=(c_{21}x+c_{22})e^{-\frac{1}{2}\int Pdx}$$
$$\Rightarrow u''+Ru'+\Big[\big(\tfrac{1}{2}R\big)'+\big(\tfrac{1}{2}R\big)^2\Big]u=0=y''+Py'+\Big[\big(\tfrac{1}{2}P\big)'+\big(\tfrac{1}{2}P\big)^2\Big]y$$

□

This is in perfect agreement with theorem II.1 , for:

$$\Rightarrow u''+Ru'+\Big\{\Big[\big(\tfrac{1}{2}R\big)'+\big(\tfrac{1}{2}R\big)^2\Big]-\Big[\big(\tfrac{1}{2}P\big)'+\big(\tfrac{1}{2}P\big)^2\Big]\Big\}u=0$$

with: $\big(\tfrac{1}{2}P\big)'+\big(\tfrac{1}{2}P\big)^2=0\Rightarrow\tfrac{1}{2}P=\dfrac{1}{x+c}\Rightarrow\left(c_1\int e^{-\int Pdx}dx+c_2\right)e^{\frac{1}{2}\int(P-R)dx}=(c_{11}x+c_{12})e^{-\frac{1}{2}\int Rdx}$

$$\Rightarrow\big(\tfrac{1}{2}P\big)'+\big(\tfrac{1}{2}P\big)^2=0\Rightarrow\tfrac{1}{2}P=\frac{1}{x+c}\Rightarrow e^{\frac{1}{2}\int\frac{2}{x+c}dx}=x+c$$

Similarly, of course, then:

$$u=\frac{1}{x+c}\left(c_1\int e^{-\int Pdx}dx+c_2\right)e^{\frac{1}{2}\int Pdx}\Rightarrow u''+\frac{2}{x+c}u'+\Big\{-\Big[\big(\tfrac{1}{2}P\big)'+\big(\tfrac{1}{2}P\big)^2\Big]\Big\}u=0$$

Theorem V.2: For all differentiable u,y,R,P :

$$u'+Ru'+\Big[\big(\tfrac{1}{2}(R-P)\big)'+\big(\tfrac{1}{2}(R+P)\big)\big(\tfrac{1}{2}(R-P)\big)\Big]u=0$$
$$\Leftrightarrow\begin{cases}\big[D+\tfrac{1}{2}(R+P)\big]\big[D+\tfrac{1}{2}(R-P)\big]u=0\\ u=\left(c_1\int e^{-\int Pdx}dx+c_2\right)e^{\frac{1}{2}\int(P-R)dx}\end{cases}$$

Proof:

$$u=ye^{\frac{1}{2}\int(P-R)dx}\Rightarrow ue^{\frac{1}{2}\int Rdx}=ye^{\frac{1}{2}\int Pdx}$$
$$\Rightarrow\begin{cases}\left(ue^{\frac{1}{2}\int Rdx}\right)'=\left(ye^{\frac{1}{2}\int Pdx}\right)'\\ \left(ue^{\frac{1}{2}\int Rdx}\right)''=\left(ye^{\frac{1}{2}\int Pdx}\right)''\end{cases}$$

So:

$$\left(u' + \tfrac{1}{2}Ru\right)e^{\frac{1}{2}\int Rdx} = \left(ue^{\frac{1}{2}\int Rdx}\right)' = \left(ye^{\frac{1}{2}\int Pdx}\right)' = \left(y' + \tfrac{1}{2}Py\right)e^{\frac{1}{2}\int Pdx}$$

$$\Rightarrow \left(u' + \tfrac{1}{2}Ru\right)e^{\frac{1}{2}\int (R-P)dx} = \left(y' + \tfrac{1}{2}Py\right)$$

$$\Rightarrow \left(\left(u' + \tfrac{1}{2}Ru\right)e^{\frac{1}{2}\int (R-P)dx}\right)' = \left(y' + \tfrac{1}{2}Py\right)'$$

$$\Rightarrow \left(\left(u'' + \tfrac{1}{2}Ru\right)' + \tfrac{1}{2}(R-P)\left(u' + \tfrac{1}{2}Ru\right)\right)e^{\frac{1}{2}\int (R-P)dx} = \left(\left[ue^{\frac{1}{2}\int (R-P)dx}\right]' + \tfrac{1}{2}P\left[ue^{\frac{1}{2}\int (R-P)dx}\right]\right)'$$

$$\Rightarrow \left[u'' + \left(R - \tfrac{1}{2}P\right)u' + \left[\tfrac{1}{2}R' + \left(\tfrac{1}{2}R\right)^2 - \tfrac{1}{2}R\left(\tfrac{1}{2}P\right)\right]u\right]e^{\frac{1}{2}\int (R-P)dx} =$$
$$= \left(\left[ue^{\frac{1}{2}\int (R-P)dx}\right]' + \tfrac{1}{2}P\left[ue^{\frac{1}{2}\int (R-P)dx}\right]\right)'$$

$$\Rightarrow \left[u'' + \left(R - \tfrac{1}{2}P\right)u' + \left[\tfrac{1}{2}\left(R - \tfrac{1}{2}P\right)' + \tfrac{1}{4}P' + \left(\tfrac{1}{2}R\right)^2 - \tfrac{1}{2}R\left(\tfrac{1}{2}P\right)\right]u\right]e^{\frac{1}{2}\int (R-P)dx} =$$
$$= \left(\left[ue^{\frac{1}{2}\int (R-P)dx}\right]' + \tfrac{1}{2}P\left[ue^{\frac{1}{2}\int (R-P)dx}\right]\right)'$$

$M \equiv R - \frac{1}{2}P$:

$$\Rightarrow \left[u'' + Mu' + \left[\tfrac{1}{2}\left(R - \tfrac{1}{2}P\right)' + \tfrac{1}{4}P' + \left(\tfrac{1}{2}R\right)^2 - \tfrac{1}{2}R\left(\tfrac{1}{2}P\right)\right]u\right]e^{\frac{1}{2}\int (R-P)dx} =$$
$$= \left(\left(\left[ue^{\frac{1}{2}\int (R-P)dx}\right]e^{\frac{1}{2}\int Pdx}\right)' e^{-\frac{1}{2}\int Pdx}\right)'$$

$$\Rightarrow \left[u'' + Mu' + \left[\tfrac{1}{2}M' + \tfrac{1}{4}P' + \tfrac{1}{4}R(R-P)\right]u\right]e^{\frac{1}{2}\int (R-P)dx} =$$
$$= \left(\left(\left[ue^{\frac{1}{2}\int (R-P)dx}\right]e^{\frac{1}{2}\int Pdx}\right)' e^{-\frac{1}{2}\int Pdx}\right)'$$

$$\Rightarrow \left[u'' + Mu' + \left[\tfrac{1}{2}M' + \tfrac{1}{4}P' + \tfrac{1}{4}\left(R - \tfrac{1}{2}P + \tfrac{1}{2}P\right)\left(R - \tfrac{1}{2}P - \tfrac{1}{2}P\right)\right]u\right]e^{\frac{1}{2}\int \left(R-\frac{1}{2}P-\frac{1}{2}P\right)dx} =$$
$$= \left(\left(\left[ue^{\frac{1}{2}\int \left(R-\frac{1}{2}P-\frac{1}{2}P\right)dx}\right]e^{\frac{1}{2}\int Pdx}\right)' e^{-\frac{1}{2}\int Pdx}\right)'$$

$$\Rightarrow \left[u'' + Mu' + \left[\tfrac{1}{2}M' + \tfrac{1}{4}P' + \tfrac{1}{4}\left(M + \tfrac{1}{2}P\right)\left(M - \tfrac{1}{2}P\right)\right]u\right]e^{\frac{1}{2}\int \left(M-\frac{1}{2}P\right)dx} =$$
$$= \left(\left(\left[ue^{\frac{1}{2}\int \left(M-\frac{1}{2}P\right)dx}\right]e^{\frac{1}{2}\int Pdx}\right)' e^{-\frac{1}{2}\int Pdx}\right)'$$

$$\Rightarrow \left[u'' + Mu' + \left[\tfrac{1}{2}\left(M + \tfrac{1}{2}P\right)' + \tfrac{1}{4}\left(M + \tfrac{1}{2}P\right)\left(M - \tfrac{1}{2}P\right)\right]u\right]e^{\frac{1}{2}\int \left(M-\frac{1}{2}P\right)dx} =$$
$$= \left(\left(\left[ue^{\frac{1}{2}\int \left(M-\frac{1}{2}P\right)dx}\right]e^{\frac{1}{2}\int Pdx}\right)' e^{-\frac{1}{2}\int Pdx}\right)'$$

$N = -\frac{1}{2}P$:

$$\Rightarrow \left[u'' + Mu' + \left[\tfrac{1}{2}(M-N)' + \tfrac{1}{4}(M+N)(M-N)\right]u\right]e^{\frac{1}{2}\int (M+N)dx} =$$
$$= \left(\left(\left[ue^{\frac{1}{2}\int (M+N)dx}\right]e^{-\int Ndx}\right)' e^{\int Ndx}\right)'$$

So:

$$u'' + Mu' + \left[\tfrac{1}{2}(M-N)' + \tfrac{1}{4}(M+N)(M-N)\right]u = 0 \Leftrightarrow \left(\left(\left[ue^{\frac{1}{2}\int (M+N)dx}\right]e^{-\int Ndx}\right)' e^{\int Ndx}\right)' = 0$$

or:

$$u'' + Ru' + \left[\left(\tfrac{1}{2}(R-P)\right)' + \left(\tfrac{1}{2}(R+P)\right)\left(\tfrac{1}{2}(R-P)\right)\right]u = 0$$
$$\Leftrightarrow \left(\left(\left[ue^{\int \frac{1}{2}(R+P)dx}\right]e^{-\int Pdx}\right)' e^{\int Pdx}\right)' = 0$$
$$\Leftrightarrow u = \left(c_2e^{\int Pdx} + c_1e^{\int Pdx}\int e^{-\int Pdx}dx\right)e^{-\int \frac{1}{2}(R+P)dx}$$
$$\Leftrightarrow u = \left(c_1\int e^{-\int Pdx}dx + c_2\right)e^{\int \frac{1}{2}(P-R)dx}$$

and:

$$\left(\tfrac{1}{2}(R-P)\right)' + \left(\tfrac{1}{2}(R+P)\right)\left(\tfrac{1}{2}(R-P)\right) = \left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2\right] - \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2\right]$$

so, by theorem III.1:

$$u'' + Ru' + \left[\left(\tfrac{1}{2}(R-P)\right)' + \left(\tfrac{1}{2}(R+P)\right)\left(\tfrac{1}{2}(R-P)\right)\right]u = 0$$
$$\Leftrightarrow \begin{cases} \left[D + \tfrac{1}{2}(R+P)\right]\left[D + \tfrac{1}{2}(R-P)\right]u = 0 \\ u = \left(c_1\int e^{-\int Pdx}dx + c_2\right)e^{\frac{1}{2}\int (P-R)dx} \end{cases}$$

□

Astonishing, how powerful this invariant technique yields the same result as all the previous work. (except for the general elementary solution, and corollary I.3)

Corollary V.2: For all differentiable u, y, R, P :

$$u' + Ru' + \left[\left(\tfrac{1}{2}(R-P)\right)' + \left(\tfrac{1}{2}(R+P)\right)\left(\tfrac{1}{2}(R-P)\right)\right]u = 0$$

$$\Leftrightarrow \begin{cases} \left[D+\tfrac{1}{2}(R+P)\right]\left[D+\tfrac{1}{2}(R-P)\right]u=0 \\ u = \begin{cases} = \left(c_1\int e^{-\int Pdx}dx + c_2\right)e^{\frac{1}{2}\int(P-R)dx} \\ = c_2 e^{-\frac{1}{2}\int(R-P)dx} + c_1 e^{-\frac{1}{2}\int(R+P)dx}\left(\dfrac{\int e^{-\int Pdx}dx}{e^{-\int Pdx}}\right) \end{cases} \end{cases}$$

Proof:

By theorem V.2:

$$u' + Ru' + \left[\left(\tfrac{1}{2}(R-P)\right)' + \left(\tfrac{1}{2}(R+P)\right)\left(\tfrac{1}{2}(R-P)\right)\right]u = 0$$

$$\Leftrightarrow \begin{cases} \left[D+\tfrac{1}{2}(R+P)\right]\left[D+\tfrac{1}{2}(R-P)\right]u=0 \\ u = \left(c_1\int e^{-\int Pdx}dx + c_2\right)e^{\frac{1}{2}\int(P-R)dx} \end{cases}$$

So:

$$\begin{aligned} u &= c_2 e^{-\frac{1}{2}\int Rdx}e^{\frac{1}{2}\int Pdx} + c_1 e^{-\frac{1}{2}\int Rdx}e^{\frac{1}{2}\int Pdx}\int e^{-\int Pdx}dx \\ &= c_2 e^{-\frac{1}{2}\int(R-P)dx} + c_1 e^{-\frac{1}{2}\int Rdx}e^{\frac{1}{2}\int Pdx}e^{-\int Pdx}\left(\frac{\int e^{-\int Pdx}dx}{e^{-\int Pdx}}\right) \\ &= c_2 e^{-\frac{1}{2}\int(R-P)dx} + c_1 e^{-\frac{1}{2}\int Rdx}e^{-\frac{1}{2}\int Pdx}\left(\frac{\int e^{-\int Pdx}dx}{e^{-\int Pdx}}\right) \\ &= c_2 e^{-\frac{1}{2}\int(R-P)dx} + c_1 e^{-\frac{1}{2}\int(R+P)dx}\left(\frac{\int e^{-\int Pdx}dx}{e^{-\int Pdx}}\right) \end{aligned}$$

□

Corollary V.2a: For all differentiable u, y, R, A (A constant) :

$$u' + Ru' + \left[\tfrac{1}{2}R' + \left(\tfrac{1}{2}(R+A)\right)\left(\tfrac{1}{2}(R-A)\right)\right]u = 0$$

$$\Leftrightarrow \begin{cases} \left[D+\tfrac{1}{2}(R+A)\right]\left[D+\tfrac{1}{2}(R-A)\right]u=0 \\ u = \begin{cases} = c_2 e^{-\frac{1}{2}\int(R-A)dx} + c_3 e^{-\frac{1}{2}\int(R+A)dx} & , \; A \neq 0 \\ = c_2 e^{-\frac{1}{2}\int Rdx} + c_3 x e^{-\frac{1}{2}\int Rdx} & , \; A = 0 \end{cases} \end{cases}$$

Proof:

By corollary V.2:

$$u' + Ru' + \left[\left(\tfrac{1}{2}(R-A)\right)' + \left(\tfrac{1}{2}(R+A)\right)\left(\tfrac{1}{2}(R-A)\right)\right]u = 0$$

$$\Leftrightarrow \begin{cases} \left[D+\tfrac{1}{2}(R+A)\right]\left[D+\tfrac{1}{2}(R-A)\right]u=0 \\ u = \begin{cases} = \left(c_1\int e^{-\int Adx}dx + c_2\right)e^{\frac{1}{2}\int(A-R)dx} \\ = c_2 e^{-\frac{1}{2}\int(R-A)dx} + c_1 e^{-\frac{1}{2}\int(R+A)dx}\left(\dfrac{\int e^{-\int Adx}dx}{e^{-\int Adx}}\right) \end{cases} \end{cases}$$

$$e^{-\int Adx} = \begin{cases} \dfrac{e^{-Ax}}{-A} & , \; A \neq 0 \\ 1 & , \; A = 0 \end{cases}$$

So:

$$\frac{\int e^{-\int Adx}dx}{e^{-\int Adx}} = \begin{cases} \dfrac{\frac{1}{-A}\int e^{-Ax}dx}{\frac{e^{-Ax}}{-A}} & , \; A \neq 0 \\ \dfrac{\int 1dx}{1} & , \; A = 0 \end{cases} = \begin{cases} -\dfrac{1}{A} & , \; A \neq 0 \\ x & , \; A = 0 \end{cases}$$

So:

$$u' + Ru' + \left[\left(\tfrac{1}{2}(R-A)\right)' + \left(\tfrac{1}{2}(R+A)\right)\left(\tfrac{1}{2}(R-A)\right)\right]u = 0$$

$$\Leftrightarrow \begin{cases} \left[D+\tfrac{1}{2}(R+A)\right]\left[D+\tfrac{1}{2}(R-A)\right]u=0 \\ u = \begin{cases} = c_2 e^{-\frac{1}{2}\int(R-A)dx} + c_3 e^{-\frac{1}{2}\int(R+A)dx} & , \; A \neq 0 \\ = c_2 e^{-\frac{1}{2}\int Rdx} + c_3 x e^{-\frac{1}{2}\int Rdx} & , \; A = 0 \end{cases} \end{cases}$$

□

Note how the x-factor for the common root ($A = 0$) naturally arives without applying any tricks.

Corollary V.2b: For all differentiable u, R, P :

$$u' + Ru' + Tu = 0 \quad \text{where:} \quad T \equiv \tfrac{1}{2}R' + \left(\tfrac{1}{2}(R+A)\right)\left(\tfrac{1}{2}(R-A)\right)$$

$$\Leftrightarrow \begin{cases} \left[D+\frac{1}{2}(R+A)\right]\left[D+\frac{1}{2}(R-A)\right]u = 0 \\ u = \begin{cases} = c_2 e^{-\frac{1}{2}\int (R-A)dx} + c_3 e^{-\frac{1}{2}\int (R+A)dx} & , \ A = \sqrt{R^2-4T+2R'} \neq 0 \\ = c_2 e^{-\frac{1}{2}\int Rdx} + c_3 x e^{-\frac{1}{2}\int Rdx} & , \ A = 0 \end{cases} \end{cases}$$

Proof:

By corollary V.2a:

$$u' + Ru' + \left[\frac{1}{2}R' + \left(\frac{1}{2}(R+A)\right)\left(\frac{1}{2}(R-A)\right)\right]u = 0$$

$$\Leftrightarrow \begin{cases} \left[D+\frac{1}{2}(R+A)\right]\left[D+\frac{1}{2}(R-A)\right]u = 0 \\ u = \begin{cases} = c_2 e^{-\frac{1}{2}\int (R-A)dx} + c_3 e^{-\frac{1}{2}\int (R+A)dx} & , \ A \neq 0 \\ = c_2 e^{-\frac{1}{2}\int Rdx} + c_3 x e^{-\frac{1}{2}\int Rdx} & , \ A = 0 \end{cases} \end{cases}$$

$$T \equiv \frac{1}{2}R' + \left(\frac{1}{2}(R+A)\right)\left(\frac{1}{2}(R-A)\right)$$

$$\Rightarrow T - \frac{1}{2}R' = \frac{1}{4}(R^2 - A^2) \Rightarrow A = \pm\sqrt{R^2-4T+2R'}$$

$$\Rightarrow u = \begin{cases} = c_2 e^{-\frac{1}{2}\int (R-A)dx} + c_3 e^{-\frac{1}{2}\int (R+A)dx} & , \ A = \pm\sqrt{R^2-4T+2R'} \neq 0 \\ = c_2 e^{-\frac{1}{2}\int Rdx} + c_3 x e^{-\frac{1}{2}\int Rdx} & , \ A = 0 \end{cases}$$

□

Theorem V.1 may be generalized to extend easy solutions a follows:

Theorem V.3: For differentiable u, y, P, R, S :

$$u = \left(c_{11} + c_{12}\int e^{-\int Sdx}dx\right)e^{-\frac{1}{2}\int Rdx} \Leftrightarrow y = \left(c_{21} + c_{22}\int e^{-\int Sdx}dx\right)e^{-\frac{1}{2}\int Pdx}$$

$$\Rightarrow u'' + (R+S)u' + \left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2 + \frac{1}{2}RS\right]u = 0 = y'' + (P+S)y' + \left[\left(\frac{1}{2}P\right)' + \left(\frac{1}{2}P\right)^2 + \frac{1}{2}PS\right]y$$

Proof:

$$u = ye^{\frac{1}{2}\int (P-R)dx} \Rightarrow ue^{\frac{1}{2}\int Rdx} = ye^{\frac{1}{2}\int Pdx}$$

$$\Rightarrow \begin{cases} \left(ue^{\frac{1}{2}\int Rdx}\right)' = \left(ye^{\frac{1}{2}\int Pdx}\right)' \\ \left(ue^{\frac{1}{2}\int Rdx}\right)'' = \left(ye^{\frac{1}{2}\int Pdx}\right)'' \end{cases}$$

So:

$$\left(ue^{\frac{1}{2}\int Rdx}\right)' = \left(u' + \frac{1}{2}Ru\right)e^{\frac{1}{2}\int Rdx}$$

$$\left(ye^{\frac{1}{2}\int Pdx}\right)' = \left(y' + \frac{1}{2}Py\right)e^{\frac{1}{2}\int Pdx}$$

$$\left(ue^{\frac{1}{2}\int Rdx}\right)'' = \left\{u'' + Ru' + \left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2\right]u\right\}e^{\frac{1}{2}\int Rdx}$$

$$\left(ye^{\frac{1}{2}\int Pdx}\right)'' = \left\{y'' + Py' + \left[\left(\frac{1}{2}P\right)' + \left(\frac{1}{2}P\right)^2\right]y\right\}e^{\frac{1}{2}\int Pdx}$$

$$\Rightarrow 0 = \left(ue^{\frac{1}{2}\int Rdx}\right)'' + S\left(ue^{\frac{1}{2}\int Rdx}\right)' = \left\{u'' + (R+S)u' + \left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2 + \frac{1}{2}RS\right]u\right\}e^{\frac{1}{2}\int Rdx}$$

$$= \left(ye^{\frac{1}{2}\int Pdx}\right)'' + S\left(ye^{\frac{1}{2}\int Pdx}\right)' = \left\{y'' + (P+S)y' + \left[\left(\frac{1}{2}P\right)' + \left(\frac{1}{2}P\right)^2 + \frac{1}{2}PS\right]y\right\}e^{\frac{1}{2}\int Pdx} = 0$$

$$\Rightarrow u'' + (R+S)u' + \left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2 + \frac{1}{2}RS\right]u = 0 = y'' + (P+S)y' + \left[\left(\frac{1}{2}P\right)' + \left(\frac{1}{2}P\right)^2 + \frac{1}{2}PS\right]y$$

So:

$$0 = \left(ue^{\frac{1}{2}\int Rdx}\right)'' + S\left(ue^{\frac{1}{2}\int Rdx}\right)' = \left(\left(ue^{\frac{1}{2}\int Rdx}\right)' e^{\int Sdx}\right)' e^{\int Sdx}$$

$$\Rightarrow u = \left(c_1 + c_2\int e^{-\int Sdx}dx\right)e^{-\frac{1}{2}\int Rdx}$$

$$\Rightarrow y = \left(c_1 + c_2\int e^{-\int Sdx}dx\right)e^{-\frac{1}{2}\int Rdx}e^{-\frac{1}{2}\int (P-R)dx} = \left(c_1 + c_2\int e^{-\int Sdx}dx\right)e^{-\frac{1}{2}\int Pdx}$$

$$\Rightarrow u'' + (R+S)u' + \left[\left(\frac{1}{2}R\right)' + \left(\frac{1}{2}R\right)^2 + \frac{1}{2}RS\right]u = 0 = y'' + (P+S)y' + \left[\left(\frac{1}{2}P\right)' + \left(\frac{1}{2}P\right)^2 + \frac{1}{2}PS\right]y$$

□

Corollary V.3a: For differentiable u, y, P & constants k, B :

$$u = \left(c_{11} + c_{12}\int e^{-kx}dx\right)e^{-Bx} \Leftrightarrow y = \left(c_{21} + c_{22}\int e^{-kx}dx\right)e^{-\frac{1}{2}\int Pdx}$$

$$\Rightarrow u'' + (2B+k)u' + [B^2 + kB]u = 0 = y'' + (P+k)y' + \left[\left(\frac{1}{2}P\right)' + \left(\frac{1}{2}P\right)^2 + \frac{1}{2}kP\right]y$$

$$y'' + Py' + \left[\left(\frac{1}{2}P\right)' + \left(\frac{1}{2}P\right)^2\right]y = 0 \quad ; \quad u = e^{\left(-B + \frac{-k \pm k}{2}\right)x} \text{ and } y = e^{\left(\frac{-k \pm k}{2}\right)x - \frac{1}{2}\int Pdx}$$

$$\Rightarrow \begin{cases} y'' + Hy' + \left[\left(\tfrac{1}{2}[H-k]\right)' + \left(\tfrac{1}{2}[H-k]\right)^2 + \tfrac{1}{2}(H-k)k \right]y = 0 \\ y'' + Hy' + \left[\left(\tfrac{1}{2}[H-k]\right)' + \left(\tfrac{1}{2}H\right)^2 - \left(\dfrac{k}{2}\right)^2 \right]y = 0 \\ y'' + Hy' + \left[\left(\tfrac{1}{2}[H-k]\right)' + \left(\tfrac{1}{2}[H-k]\right)\left(\tfrac{1}{2}[H+k]\right) \right]y = 0 \end{cases}$$

where: $H \equiv P + k$

Proof:

From theorem V.3:

$$y'' + Py' + \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2 \right]y = 0 \quad \text{and} \quad u = ye^{\frac{1}{2}\int(P-R)dx}$$

$$\Rightarrow u = \left(c_1 + c_2 \int e^{-\int Sdx} dx\right)e^{-\frac{1}{2}\int Rdx} \Leftrightarrow y = \left(c_1 + c_2 \int e^{-\int Sdx} dx\right)e^{-\frac{1}{2}\int Pdx}$$

$$\Rightarrow u'' + (R+S)u' + \left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2 + \tfrac{1}{2}RS \right]u = 0 = y'' + (P+S)y' + \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2 + \tfrac{1}{2}PS \right]y$$

$S = k$, (k constant):

$$\Rightarrow u'' + (R+k)u' + \left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2 + \tfrac{1}{2}kR \right]u = 0 = y'' + (P+k)y' + \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2 + \tfrac{1}{2}kP \right]y$$

$$\tfrac{1}{2}R = B \Rightarrow \left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2 + \left(\tfrac{1}{2}R\right)k = B^2 + kB \quad , \quad (B \text{ constant})$$

$$\Rightarrow u'' + (2B+k)u' + (B^2+kB)u = 0 \Rightarrow u = e^{mx} \quad , \quad \left(m = \tfrac{1}{2}\left[-(2B+k) \pm \sqrt{(2B+k)^2 - 4(B^2+kB)}\right]\right)$$

$$\Rightarrow u'' + (2B+k)u' + (B^2+kB)u = 0 \Rightarrow u = e^{mx} \quad , \quad \left(m = \tfrac{1}{2}\left[-(2B+k) \pm \sqrt{k^2}\right]\right)$$

$$\Rightarrow u = e^{mx} \Rightarrow y = e^{mx}e^{-\frac{1}{2}\int(P-2B)dx} = e^{\left(\frac{-k\pm k}{2}\right)x - \frac{1}{2}\int Pdx} \quad , \quad \left(m = -B + \frac{-k \pm k}{2}\right)$$

$$\Rightarrow y'' + (P+k)y' + \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2 + \tfrac{1}{2}kP \right]y = 0$$

$$\Rightarrow y'' + Hy' + \left[\left(\tfrac{1}{2}[H-k]\right)' + \left(\tfrac{1}{2}[H-k]\right)^2 + \tfrac{1}{2}(H-k)k \right]y = 0 \quad , \quad (H \equiv P+k)$$

$$\Rightarrow y'' + Hy' + \left[\left(\tfrac{1}{2}[H-k]\right)' + \left(\tfrac{1}{2}H\right)^2 - \left(\dfrac{k}{2}\right)^2 \right]y = 0 \quad , \quad (H \equiv P+k)$$

$$\Rightarrow y'' + Hy' + \left[\left(\tfrac{1}{2}[H-k]\right)' + \left(\tfrac{1}{2}[H-k]\right)\left(\tfrac{1}{2}[H+k]\right) \right]y = 0 \quad , \quad (H \equiv P+k)$$

□

Note: combining Corollary V.3 & Corollary 1.3 yields nothing new:

$$z'' + Fz' + Gz = 0 \text{ and: } w = ze^{\frac{1}{2}\int(F-R)dx}$$

$$\Rightarrow w'' + Rw' + \left\{ G - \left[\left(\tfrac{1}{2}F\right)' + \left(\tfrac{1}{2}F\right)^2 \right] + \left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2 \right] \right\} w = 0$$

And:

$$u = \left(c_{11} + c_{12} \int e^{-\int Sdx} dx\right)e^{-\frac{1}{2}\int Rdx} \Leftrightarrow y = \left(c_{21} + c_{22} \int e^{-\int Sdx} dx\right)e^{-\frac{1}{2}\int Pdx}$$

$$\Rightarrow u'' + (R+S)u' + \left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2 + \tfrac{1}{2}RS \right]u = 0 = y'' + (P+S)y' + \left[\left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2 + \tfrac{1}{2}PS \right]y$$

So, let:

$$F = P + S \quad \& \quad G = \left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2 + \tfrac{1}{2}PS$$

$$\Rightarrow w'' + Rw' + \left\{ G - \left[\left(\tfrac{1}{2}F\right)' + \left(\tfrac{1}{2}F\right)^2 \right] + \left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2 \right] \right\} w = 0$$

$$\Rightarrow z = \left(c_{21} + c_{22} \int e^{-\int Sdx} dx\right)e^{-\frac{1}{2}\int Pdx} \quad \& \quad w = \left(c_{21} + c_{22} \int e^{-\int Sdx} dx\right)e^{-\frac{1}{2}\int Pdx} e^{\frac{1}{2}\int(P+S-R)dx}$$

$$\Rightarrow 0 = w'' + Rw' + \\ + \left\{ \left(\tfrac{1}{2}P\right)' + \left(\tfrac{1}{2}P\right)^2 + \tfrac{1}{2}PS - \left[\left(\tfrac{1}{2}[P+S]\right)' + \left(\tfrac{1}{2}[P+S]\right)^2 \right] + \left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2 \right] \right\} w$$

$$\Rightarrow w = \left(c_{21} + c_{22} \int e^{-\int Sdx} dx\right)e^{\frac{1}{2}\int(S-R)dx}$$

$$\Rightarrow w'' + Rw' + \left\{ -\left[\left(\tfrac{1}{2}S\right)' + \left(\tfrac{1}{2}S\right)^2 \right] + \left[\left(\tfrac{1}{2}R\right)' + \left(\tfrac{1}{2}R\right)^2 \right] \right\} w = 0$$

Another form of the invariant technique is as follows:

Theorem VI.1: For all differentiable $u_1, u_2, R_{11}, R_{12}, R_{21}, R_{22}$, and constants $m_{11}, m_{12}, m_{21}, m_{22}$:

$$\left(u_1 e^{m_{11}\int R_{11}dx}\right)' e^{m_{12}\int R_{12}dx} = \left(u_2 e^{m_{21}\int R_{21}dx}\right)' e^{m_{22}\int R_{22}dx}$$

$$u_2'' + [2m_{21}R_{21} + (m_{22}R_{22} - m_{12}R_{12})]u_2' + \\ + \left[(m_{21}R_{21})' + (m_{21}R_{21})^2 + (m_{21}R_{21})[m_{22}R_{22} - m_{12}R_{12}] \right]u_2 = 0$$

$$[D + (m_{21}R_{21} + m_{22}R_{22} - m_{12}R_{12})][D + (m_{21}R_{21})]u = 0$$

$$\Rightarrow u_2 = e^{-m_{21}\int R_{21}dx}\left(c_1 + c_2 \int e^{-\int(m_{22}R_{22} - m_{12}R_{12})dx} dx\right)$$

Proof:

$$\left(u_1 e^{m_{11}\int R_{11}dx}\right)' e^{m_{12}\int R_{12}dx} = \left(u_2 e^{m_{21}\int R_{21}dx}\right)' e^{m_{22}\int R_{22}dx}$$

$$\Rightarrow \left(u_1 e^{m_{11}\int R_{11}dx}\right)' = \left(u_2 e^{m_{21}\int R_{21}dx}\right)' e^{\int(m_{22}R_{22} - m_{12}R_{12})dx}$$

$$\Rightarrow \left(u_1 e^{m_{11}\int R_{11}dx}\right)'' = \left(\left(u_2 e^{m_{21}\int R_{21}dx}\right)' e^{\int(m_{22}R_{22} - m_{12}R_{12})dx}\right)'$$

$$= \left((u_2' + m_{21}R_{21}u_2)e^{m_{21}\int R_{21}dx} e^{\int(m_{22}R_{22} - m_{12}R_{12})dx}\right)'$$

$$= \left((u_2' + m_{21}R_{21}u_2)e^{\int(m_{21}R_{21} + [m_{22}R_{22} - m_{12}R_{12}])dx}\right)'$$

$$= [u_2'' + m_{21}R_{21}u_2' + m_{21}R_{21}'u_2 + (m_{21}R_{21} + [m_{22}R_{22} - m_{12}R_{12}])(u_2' + m_{21}R_{21}u_2)]e^{\int(m_{21}R_{21} + [m_{22}R_{22} - m_{12}R_{12}])dx}$$

$$= \left\{ u_2'' + [2m_{21}R_{21} + (m_{22}R_{22} - m_{12}R_{12})]u_2' + \right.$$

$$+\left[(m_{21}R_{21})' + (m_{21}R_{21})^2 + (m_{21}R_{21})[m_{22}R_{22} - m_{12}R_{12}]\right]u_2\Big\} e^{\int (m_{21}R_{21} + [m_{22}R_{22} - m_{12}R_{12}])dx}$$

$$\Rightarrow u_2'' + [2m_{21}R_{21} + (m_{22}R_{22} - m_{12}R_{12})]u_2' + \\ + \left[(m_{21}R_{21})' + (m_{21}R_{21})^2 + (m_{21}R_{21})[m_{22}R_{22} - m_{12}R_{12}]\right]u_2 = 0$$

$$\Rightarrow \left(\left(u_2 e^{m_{21}\int R_{21}dx}\right)' e^{\int (m_{22}R_{22} - m_{12}R_{12})dx}\right)' = 0$$

$$\Rightarrow u_2 = e^{-m_{21}\int R_{21}dx}\left(c_1 + c_2 \int e^{-\int (m_{22}R_{22} - m_{12}R_{12})dx} dx\right)$$

$$g = m_{21}R_{21} \quad \& \quad h = m_{21}R_{21} + (m_{22}R_{22} - m_{12}R_{12})$$

$$\Rightarrow u_2'' + [2m_{21}R_{21} + (m_{22}R_{22} - m_{12}R_{12})]u_2' + \\ + \left[(m_{21}R_{21})' + (m_{21}R_{21})(m_{21}R_{21} + [m_{22}R_{22} - m_{12}R_{12}])\right]u_2 = 0$$

□

Generalizing, extending further this invariant technique:

Theorem VI.1: For differentiable $u_1, v_1, w_1, u_2, v_2, w_2, S$:

$$\left\{ \begin{array}{l} (u_1v_1)'w_1 = (u_2v_2)'w_2 \quad \& \quad 0 = \left((u_1v_1)'w_1\right)' + S\left((u_1v_1)'w_1\right) \\ \Rightarrow \left\{ \begin{array}{l} u_1 = v_1^{-1}\left[c_1 + c_2 \int w_1^{-1} e^{-\int S dx} dx\right] = e^{-\int \left(\frac{v_1'}{v_1}\right)dx}\left[c_1 + c_2 \int e^{-\int \left(\frac{w_1'}{w_1} + S\right)dx} dx\right] \\ 0 = u_1'' + \left[2\left(\frac{v_1'}{v_1}\right) + \left(\frac{w_1'}{w_1} + S\right)\right]u_1' + \left[\left(\frac{v_1'}{v_1}\right)' + \left(\frac{v_1'}{v_1}\right)\left[\left(\frac{v_1'}{v_1}\right) + \left(\frac{w_1'}{w_1} + S\right)\right]\right]u_1 \\ u_2 = v_2^{-1}\left[c_1 + c_2 \int w_2^{-1} e^{-\int S dx} dx\right] = e^{-\int \left(\frac{v_2'}{v_2}\right)dx}\left[c_1 + c_2 \int e^{-\int \left(\frac{w_2'}{w_2} + S\right)dx} dx\right] \\ 0 = u_2'' + \left[2\left(\frac{v_2'}{v_2}\right) + \left(\frac{w_2'}{w_2} + S\right)\right]u_2' + \left[\left(\frac{v_2'}{v_2}\right)' + \left(\frac{v_2'}{v_2}\right)\left[\left(\frac{v_2'}{v_2}\right) + \left(\frac{w_2'}{w_2} + S\right)\right]\right]u_2 \end{array} \right. \end{array} \right.$$

Proof:

$$\left((u_1v_1)'w_1\right)' + S\left((u_1v_1)'w_1\right) = (u_1v_1)''w_1 + (u_1v_1)'w_1' + S(u_1v_1)'w_1$$

$$= w_1(u_1v_1)'' + \left(\frac{w_1'}{w_1'} + S\right)(u_1v_1)'w_1$$

$$= \left((u_2v_2)'w_2\right)' + S\left((u_2v_2)'w_2\right)$$

$$\Rightarrow 0 = \left((u_1v_1)'w_1\right)' + S\left((u_1v_1)'w_1\right) = w_1\left[\left((u_1v_1)'\right)' + \left(\frac{w_1'}{w_1} + S\right)(u_1v_1)'\right]$$

$$0 = \left((u_1v_1)'w_1\right)' + S\left((u_1v_1)'w_1\right) = w_1\left[\left((u_1v_1)'\right)' + \left(\frac{w_1'}{w_1} + S\right)(u_1v_1)'\right]$$

$$= \left(\left((u_1v_1)'\right)e^{\int \left(\frac{w_1'}{w_1} + S\right)dx}\right)' e^{-\int \left(\frac{w_1'}{w_1} + S\right)dx} = \left(\left((u_1v_1)'\right)e^{\int \left(\frac{w_1'}{w_1} + S\right)dx}\right)' e^{-\int \left(\frac{w_1'}{w_1} + S\right)dx}$$

$$\Rightarrow u_1 = v_1^{-1}\left[c_1 + c_2 \int e^{-\int \left(\frac{w_1'}{w_1} + S\right)dx} dx\right] = e^{-\int \left(\frac{v_1'}{v_1}\right)dx}\left[c_1 + c_2 \int e^{-\int \left(\frac{w_1'}{w_1} + S\right)dx} dx\right]$$

$$0 = \left((u_1v_1)'w_1\right)' + S\left((u_1v_1)'w_1\right) = \left((u_2v_2)'w_2\right)' + S\left((u_2v_2)'w_2\right)$$

$$= w_2\left[\left((u_2v_2)'\right)' + \left(\frac{w_2'}{w_2} + S\right)(u_2v_2)'\right]$$

$$= \left(\left((u_2v_2)'\right)e^{\int \left(\frac{w_2'}{w_2} + S\right)dx}\right)' e^{-\int \left(\frac{w_2'}{w_2} + S\right)dx} = \left(\left((u_2v_2)'\right)e^{\int \left(\frac{w_2'}{w_2} + S\right)dx}\right)' e^{-\int \left(\frac{w_2'}{w_2} + S\right)dx}$$

$$\Rightarrow u_2 = v_2^{-1}\left[c_1 + c_2 \int e^{-\int \left(\frac{w_2'}{w_2} + S\right)dx} dx\right] = e^{-\int \left(\frac{v_2'}{v_2}\right)dx}\left[c_1 + c_2 \int e^{-\int \left(\frac{w_2'}{w_2} + S\right)dx} dx\right]$$

$$\Rightarrow \left\{ \begin{array}{l} 0 = \left((u_1v_1)'w_1\right)' + S\left((u_1v_1)'w_1\right) = (u_1v_1)''w_1 + (u_1v_1)'w_1' + S\left((u_1v_1)'w_1\right) \\ \quad = \left((u_2v_2)'w_2\right)' + S\left((u_2v_2)'w_2\right) = (u_2v_2)''w_2 + (u_2v_2)'w_2' + S\left((u_2v_2)'w_2\right) \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} 0 = w_1\left[(u_1v_1)'' + \left(\frac{w_1'}{w_1} + S\right)(u_1v_1)'\right] \\ \quad = w_2\left[(u_2v_2)'' + \left(\frac{w_2'}{w_2} + S\right)(u_2v_2)'\right] \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} 0 = w_1\left[\left(v_1\left[u_1' + \left(\frac{v_1'}{v_1}\right)u_1\right]\right)' + \left(\frac{w_1'}{w_1} + S\right)\left(v_1\left[u_1' + \left(\frac{v_1'}{v_1}\right)u_1\right]\right)\right] \\ \quad = w_2\left[\left(v_2\left[u_2' + \left(\frac{v_2'}{v_2}\right)u_2\right]\right)' + \left(\frac{w_2'}{w_2} + S\right)\left(v_2\left[u_2' + \left(\frac{v_2'}{v_2}\right)u_2\right]\right)\right] \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} 0 = w_1\left[v_1'\left[u_1' + \left(\frac{v_1'}{v_1}\right)u_1\right] + v_1\left[u_1' + \left(\frac{v_1'}{v_1}\right)u_1\right]' + v_1\left(\frac{w_1'}{w_1} + S\right)u_1' + v_1\left(\frac{w_1'}{w_1} + S\right)\left(\frac{v_1'}{v_1}\right)u_1\right] \\ \quad = w_2\left[v_2'\left[u_2' + \left(\frac{v_2'}{v_2}\right)u_2\right] + v_2\left[u_2' + \left(\frac{v_2'}{v_2}\right)u_2\right]' + v_2\left(\frac{w_2'}{w_2} + S\right)u_2' + v_2\left(\frac{w_2'}{w_2} + S\right)\left(\frac{v_2'}{v_2}\right)u_2\right] \end{array} \right.$$

$$
\Rightarrow \begin{cases} 0 = w_1 v_1 \left[\dfrac{v_1'}{v_1}\left[u_1' + \left(\dfrac{v_1'}{v_1}\right)u_1 \right] + \left[u_1' + \left(\dfrac{v_1'}{v_1}\right)u_1 \right]' + \left(\dfrac{w_1'}{w_1} + S\right)u_1' + \left(\dfrac{w_1'}{w_1} + S\right)\left(\dfrac{v_1'}{v_1}\right)u_1 \right] \\ \quad = w_2 v_2 \left[\dfrac{v_2'}{v_2}\left[u_2' + \left(\dfrac{v_2'}{v_2}\right)u_2 \right] + \left[u_2' + \left(\dfrac{v_2'}{v_2}\right)u_2 \right]' + \left(\dfrac{w_2'}{w_2} + S\right)u_2' + \left(\dfrac{w_2'}{w_2} + S\right)\left(\dfrac{v_2'}{v_2}\right)u_2 \right] \end{cases}
$$

$$
\Rightarrow \begin{cases} 0 = \left(\dfrac{v_1'}{v_1}\right)u_1' + \left(\dfrac{v_1'}{v_1}\right)^2 u_1 + u_1'' + \left(\dfrac{v_1'}{v_1}\right)u_1' + \left(\dfrac{v_1'}{v_1}\right)' u_1 + \left(\dfrac{w_1'}{w_1} + S\right)u_1' + \left(\dfrac{w_1'}{w_1} + S\right)\left(\dfrac{v_1'}{v_1}\right)u_1 \\ \quad = \left(\dfrac{v_2'}{v_2}\right)u_2' + \left(\dfrac{v_2'}{v_2}\right)^2 u_2 + u_2'' + \left(\dfrac{v_2'}{v_2}\right)u_2' + \left(\dfrac{v_2'}{v_2}\right)' u_2 + \left(\dfrac{w_2'}{w_2} + S\right)u_2' + \left(\dfrac{w_2'}{w_2} + S\right)\left(\dfrac{v_2'}{v_2}\right)u_2 \end{cases}
$$

$$
\Rightarrow \begin{cases} 0 = u_1'' + \left[2\left(\dfrac{v_1'}{v_1}\right) + \left(\dfrac{w_1'}{w_1} + S\right) \right]u_1' + \left[\left(\dfrac{v_1'}{v_1}\right)^2 + \left(\dfrac{v_1'}{v_1}\right)' + \left(\dfrac{w_1'}{w_1} + S\right)\left(\dfrac{v_1'}{v_1}\right) \right]u_1 \\ \quad = u_2'' + \left[2\left(\dfrac{v_2'}{v_2}\right) + \left(\dfrac{w_2'}{w_2} + S\right) \right]u_2' + \left[\left(\dfrac{v_2'}{v_2}\right)^2 + \left(\dfrac{v_2'}{v_2}\right)' + \left(\dfrac{w_2'}{w_2} + S\right)\left(\dfrac{v_2'}{v_2}\right) \right]u_2 \end{cases}
$$

$$
\Rightarrow \begin{cases} 0 = u_1'' + \left[2\left(\dfrac{v_1'}{v_1}\right) + \left(\dfrac{w_1'}{w_1} + S\right) \right]u_1' + \left[\left(\dfrac{v_1'}{v_1}\right)' + \left(\dfrac{v_1'}{v_1}\right)^2 + \left(\dfrac{w_1'}{w_1} + S\right)\left(\dfrac{v_1'}{v_1}\right) \right]u_1 \\ \quad = u_2'' + \left[2\left(\dfrac{v_2'}{v_2}\right) + \left(\dfrac{w_2'}{w_2} + S\right) \right]u_2' + \left[\left(\dfrac{v_2'}{v_2}\right)' + \left(\dfrac{v_2'}{v_2}\right)^2 + \left(\dfrac{w_2'}{w_2} + S\right)\left(\dfrac{v_2'}{v_2}\right) \right]u_2 \end{cases}
$$

$$
\Rightarrow \begin{cases} (u_1v_1)'w_1 = (u_2v_2)'w_2 \;\;\&\;\; 0 = \big((u_1v_1)'w_1\big)' + S\big((u_1v_1)'w_1\big) \\ \Rightarrow \begin{cases} u_1 = v_1^{-1}\left[c_1 + c_2 \int w_1^{-1} e^{-\int S dx} dx \right] = e^{-\int\left(\frac{v_1'}{v_1}\right)dx}\left[c_1 + c_2 \int e^{-\int\left(\frac{w_1'}{w_1} + S\right)dx} dx \right] \\ 0 = u_1'' + \left[2\left(\dfrac{v_1'}{v_1}\right) + \left(\dfrac{w_1'}{w_1} + S\right) \right]u_1' + \left[\left(\dfrac{v_1'}{v_1}\right)' + \left(\dfrac{v_1'}{v_1}\right)^2 + \left(\dfrac{w_1'}{w_1} + S\right)\left(\dfrac{v_1'}{v_1}\right) \right]u_1 \\ u_2 = v_2^{-1}\left[c_1 + c_2 \int w_2^{-1} e^{-\int S dx} dx \right] = e^{-\int\left(\frac{v_2'}{v_2}\right)dx}\left[c_1 + c_2 \int e^{-\int\left(\frac{w_2'}{w_2} + S\right)dx} dx \right] \\ 0 = u_2'' + \left[2\left(\dfrac{v_2'}{v_2}\right) + \left(\dfrac{w_2'}{w_2} + S\right) \right]u_2' + \left[\left(\dfrac{v_2'}{v_2}\right)' + \left(\dfrac{v_2'}{v_2}\right)^2 + \left(\dfrac{w_2'}{w_2} + S\right)\left(\dfrac{v_2'}{v_2}\right) \right]u_2 \end{cases} \end{cases}
$$

$$
\Rightarrow \begin{cases} \begin{cases} P_1 \equiv 2\left(\dfrac{v_1'}{v_1}\right) + \left(\dfrac{w_1'}{w_1} + S\right) \\ Q_1 \equiv \left(\dfrac{v_1'}{v_1}\right)' + \left(\dfrac{v_1'}{v_1}\right)\left[\left(\dfrac{v_1'}{v_1}\right) + \left(\dfrac{w_1'}{w_1} + S\right) \right] \\ g_1 \equiv \left(\dfrac{v_1'}{v_1}\right) \;\;\&\;\; h_1 \equiv \left(\dfrac{v_1'}{v_1}\right) + \left(\dfrac{w_1'}{w_1} + S\right) \end{cases} \\ \begin{cases} P_2 \equiv 2\left(\dfrac{v_2'}{v_2}\right) + \left(\dfrac{w_2'}{w_2} + S\right) \\ Q_2 \equiv \left(\dfrac{v_2'}{v_2}\right)' + \left(\dfrac{v_2'}{v_2}\right)\left[\left(\dfrac{v_2'}{v_2}\right) + \left(\dfrac{w_2'}{w_2} + S\right) \right] \\ g_2 \equiv \left(\dfrac{v_2'}{v_2}\right) \;\;\&\;\; h_2 \equiv \left(\dfrac{v_2'}{v_2}\right) + \left(\dfrac{w_2'}{w_2} + S\right) \end{cases} \end{cases}
$$

□

Corollary VI.1: For differentiable $u_1, g_1, h_1, u_2, g_2, h_2, S$:

$$
\Rightarrow \begin{cases} \left(u_1 e^{\int g_1 dx}\right)' e^{\int (h_1 - g_1 - S)dx} = \left(u_2 e^{\int g_2 dx}\right)' e^{\int (h_2 - g_2 - S)dx} \\ \;\&\;\; 0 = \left(\left(u_1 e^{\int g_1 dx}\right)' e^{\int (h_1 - g_1 - S)dx}\right)' + S\left(u_1 e^{\int g_1 dx}\right)' e^{\int (h_1 - g_1 - S)dx} \\ \Rightarrow \begin{cases} u_1 = e^{-\int g_1 dx}\left(c_1 + c_2 \int e^{-\int (h_1 - g_1)dx} dx \right) \\ 0 = u_1'' + (g_1 + h_1)u_1' + (g_1' + g_1 h_1)u_1 \\ u_2 = e^{-\int g_2 dx}\left(c_1 + c_2 \int e^{-\int (h_2 - g_2)dx} dx \right) \\ 0 = u_2'' + (g_2 + h_2)u_2' + (g_2' + g_2 h_2)u_2 \end{cases} \end{cases}
$$

Proof:

From theorem VI.1:

For differentiable $u_1, v_1, w_1, u_2, v_2, w_2, S$:

$$\left\{\begin{array}{l}(u_1v_1)'w_1 = (u_2v_2)'w_2 \ \ \& \ \ 0 = \Big((u_1v_1)'w_1\Big)' + S\Big((u_1v_1)'w_1\Big) \\ \Rightarrow \left\{\begin{array}{l} u_1 = v_1^{-1}\left[c_1 + c_2\int w_1^{-1}e^{-\int Sdx}dx\right] = e^{-\int\left(\frac{v_1'}{v_1}\right)dx}\left[c_1 + c_2\int e^{-\int\left(\frac{w_1'}{w_1}+S\right)dx}dx\right] \\ 0 = u_1'' + \left[2\left(\frac{v_1'}{v_1}\right) + \left(\frac{w_1'}{w_1} + S\right)\right]u_1' + \left[\left(\frac{v_1'}{v_1}\right)' + \left(\frac{v_1'}{v_1}\right)\left[\left(\frac{v_1'}{v_1}\right) + \left(\frac{w_1'}{w_1} + S\right)\right]\right]u_1 \\ u_2 = v_2^{-1}\left[c_1 + c_2\int w_2^{-1}e^{-\int Sdx}dx\right] = e^{-\int\left(\frac{v_2'}{v_2}\right)dx}\left[c_1 + c_2\int e^{-\int\left(\frac{w_2'}{w_2}+S\right)dx}dx\right] \\ 0 = u_2'' + \left[2\left(\frac{v_2'}{v_2}\right) + \left(\frac{w_2'}{w_2} + S\right)\right]u_2' + \left[\left(\frac{v_2'}{v_2}\right)' + \left(\frac{v_2'}{v_2}\right)\left[\left(\frac{v_2'}{v_2}\right) + \left(\frac{w_2'}{w_2} + S\right)\right]\right]u_2 \end{array}\right.\end{array}\right.$$

So:

$$\left\{\begin{array}{l} \left\{\begin{array}{l} v_1 \equiv e^{\int g_1dx} \ \ \& \ \ w_1 \equiv e^{\int(h_1-g_1-S)dx} \\ P_1 \equiv g_1 + h_1 \ \ \& \ \ Q_1 \equiv g_1' + g_1h_1 \end{array}\right. \\ \left\{\begin{array}{l} g_2 \equiv \left(\frac{v_2'}{v_2}\right) \ \ \& \ \ h_2 \equiv \left(\frac{v_2'}{v_2}\right) + \left(\frac{w_2'}{w_2} + S\right) \\ P_2 \equiv g_2 + h_2 \ \ \& \ \ Q_2 \equiv g_2' + g_2h_2 \end{array}\right. \end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{l} \left(u_1e^{\int g_1dx}\right)'e^{\int(h_1-g_1-S)dx} = \left(u_2e^{\int g_2dx}\right)'e^{\int(h_2-g_2-S)dx} \\ \quad \& \ \ 0 = \left(\left(u_1e^{\int g_1dx}\right)'e^{\int(h_1-g_1-S)dx}\right)' + S\left(u_1e^{\int g_1dx}\right)'e^{\int(h_1-g_1-S)dx} \\ \Rightarrow \left\{\begin{array}{l} u_1 = e^{-\int g_1dx}\left(c_1 + c_2\int e^{-\int(h_1-g_1)dx}dx\right) \\ 0 = u_1'' + (g_1 + h_1)u_1' + (g_1' + g_1h_1)u_1 = (D + g_1)(D + g_1h_1)u_1 \\ u_2 = e^{-\int g_2dx}\left(c_1 + c_2\int e^{-\int(h_2-g_2)dx}dx\right) \\ 0 = u_2'' + (g_2 + h_2)u_2' + (g_2' + g_2h_2)u_2 = (D + g_2)(D + g_2h_2)u_2 \end{array}\right. \end{array}\right.$$

□

References

[1] Kamke, E.; *Differentialgleichungen Lōsungsmethoden Und Lōsungen, 3rd Ed.*, Chelsea Publishing Company, New York, N. Y.; 1959.
[2] Nagle, R.K. , & Saff, E.B.; *Fundamentals of Differential Equations and Boundary Value Problems*; Addison Wesley Publishing Company, Inc.; Reading, MA; 1994.
[3] Nagle, R.K. , & Saff, E.B., & Snider, A.D.; *Fundamentals of Differential Equations, 5th Ed.*; Addison Wesley Longman, Inc.; Reading, MA; 2000.
[4] Polyanin, Andrei D. & Zaitsev, Valentin F.; *Handbook of Exact Solutions for Ordinary Differential Equations, 2nd. Ed.*; Chapman & Hall/CRC; New York, NY; 2003.
[5] Zill, Dennis G.; *A First Course in Differential Equations with Applications, 4th Ed.*; PWS-KENT Publishing Company; Boston, MA; 1989.
[6] Cassano, Claude M.; http://www.dnatube.com/video/6899/A-Particular-Solutions-Inhomogeneous-2nd-Order-ODE-Formula
[7] Cassano, Claude M.; http://www.dnatube.com/video/6967/A-Particular-Solutions-Inhomogeneous-3rd-Order-ODE-Formula
[8] Cassano, Claude M.; Factoring any Second Order Homogeneous Linear Ordinary Differenial Equation; http://vixra.org/abs/1708.0224
[9] Cassano, Claude M.; Even More Ordinary Differential Equations Easy Way; http://vixra.org/abs/1510.0426

Visit some of my site pages to find many of my books, articles, papers, videos and links, here:

https://independent.academia.edu/CLAUDEMICHAELCASSANO

https://www.pinterest.com/cloudmichael/?eq=cassano&etslf=14806

https://www.youtube.com/user/cloudmichael11
https://www.youtube.com/channel/UCvOBUg1YCPWuF2PZAvdIDWg

http://vixra.org/author/claude_michael_cassano

Visit my author page on amazon.com to find my books available on Kindle in digital and some in print at the website, here:

http://www.amazon.com/Claude-Michael-Cassano/e/B008MD6CVS/ref=ntt_athr_dp_pel_1

Visit:

http://www.barnesandnoble.com/s/claude-michael-cassano?keyword=claude+michael+cassano&store=allproducts

for my books available on NOOK and some in print at Barnes & Noble.