question 454: Integrals

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abstract

This note presents some definite integrals

Brief Collection of Integrals

Notation :

The number pi : $\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2\,n+1}$

Catalan' s constant : $G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2\,n+1)^2}$

Euler' s number : $e = \sum_{n=0}^{\infty} \frac{1}{n\,!}$

Psi function : $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$

Incomplete Gamma Function : $\Gamma(a,\, x) = \int_x^{\infty} e^{-t}\, t^{a-1}\, dt$

Lambert Function $W(x)$: $W(x)\, e^{W(x)} = x$

Riemann Zeta Function : $\zeta(z) = \frac{1}{1-2^{1-z}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}$, $\operatorname{Re} z > 0$.

$$\int_0^{\infty} \frac{x \tan^{-1} e^{-x}}{(x^2+\pi^2)^2}\, dx = \frac{1}{8} - \frac{3}{8\,\pi} \tag{1}$$

$$\int_0^{\infty} \frac{x \tan^{-1} e^{x}}{(x^2+\pi^2)^2}\, dx = \frac{5}{8\,\pi} - \frac{1}{8} \tag{2}$$

$$\int_0^{\infty} \frac{\ln x \tan^{-1} x}{x\,(\pi^2+\ln^2 x)^2}\, dx = \frac{1}{\pi} - \frac{1}{4} \tag{3}$$

$$\int_1^{\infty} \frac{\ln x \tan^{-1} x}{x\,(\pi^2+\ln^2 x)^2}\, dx = \frac{5}{8\,\pi} - \frac{1}{8} \tag{4}$$

$$\int_0^{1} \frac{\ln x \tan^{-1} x}{x\,(\pi^2+\ln^2 x)^2}\, dx = \frac{3}{8\,\pi} - \frac{1}{8} \tag{5}$$

$$\int_1^\infty \frac{1}{x^3}\tan^{-1} e^{\pi\sqrt{x^2-1}}\, dx = \frac{5\pi}{8} - \frac{\pi^2}{8} \tag{6}$$

$$\int_\pi^\infty \frac{1}{x^3}\tan^{-1} e^{\sqrt{x^2-\pi^2}}\, dx = \frac{5}{8\pi} - \frac{1}{8} \tag{7}$$

$$\int_0^\infty \frac{\tan^{-1} e^{\sqrt{\pi x}}}{(\pi+x)^2}\, dx = \frac{5}{4} - \frac{\pi}{4} \tag{8}$$

$$\int_\pi^\infty \frac{\tan^{-1} e^{\sqrt{\pi(x-\pi)}}}{x^2}\, dx = \frac{5}{4} - \frac{\pi}{4} \tag{9}$$

$$\int_0^{1/\pi^2} \tan^{-1} e^{-\sqrt{\frac{1}{x}-\pi^2}}\, dx = \frac{1}{4} - \frac{3}{4\pi} \tag{10}$$

$$\int_0^{1/\pi^2} \tan^{-1} e^{\sqrt{\frac{1}{x}-\pi^2}}\, dx = \frac{5}{4\pi} - \frac{1}{4} \tag{11}$$

$$\int_0^{1/\pi^2} \tan^{-1} e^{-\frac{1}{2}\sqrt{\frac{1}{x}-\pi^2}}\, dx = \frac{1-\ln 2}{4\pi} \tag{12}$$

$$\int_0^{1/\pi^2} \tan^{-1} e^{\frac{1}{2}\sqrt{\frac{1}{x}-\pi^2}}\, dx = \frac{1+\ln 2}{4\pi} \tag{13}$$

$$\int_0^{1/\pi^4} \tan^{-1}\left(\exp\left(-\sqrt{\frac{1}{\sqrt{x}}-\pi^2}\right)\right) dx = \frac{1}{8\pi^2} + \frac{G}{\pi^3} - \frac{5}{4\pi^3} \tag{14}$$

$$\int_0^{1/\pi^4} \tan^{-1}\left(\exp\left(\sqrt{\frac{1}{\sqrt{x}}-\pi^2}\right)\right) dx = \frac{7}{4\pi^3} - \frac{G}{\pi^3} - \frac{1}{8\pi^2} \tag{15}$$

$$\int_0^1 \tan^{-1}\left(\cosh^{-1}\left(\frac{1}{x}\right)\right) dx = \frac{1}{2}\left(\psi\left(\frac{3}{4}+\frac{1}{2\pi}\right) - \psi\left(\frac{1}{4}+\frac{1}{2\pi}\right)\right) \tag{16}$$

$$\int_0^{\pi/2} \frac{1}{\cosh(\tan x)}\, dx = \frac{1}{2}\left(\psi\left(\frac{3}{4}+\frac{1}{2\pi}\right) - \psi\left(\frac{1}{4}+\frac{1}{2\pi}\right)\right) \tag{17}$$

$$\int_0^{\pi/2} \frac{1}{\cosh(\cot x)}\, dx = \frac{1}{2}\left(\psi\left(\frac{3}{4}+\frac{1}{2\pi}\right) - \psi\left(\frac{1}{4}+\frac{1}{2\pi}\right)\right) \tag{18}$$

$$\int_0^1 \tan^{-1}\left(\left(\cosh^{-1}\left(\frac{1}{x}\right)\right)^{-1}\right) dx = \frac{\pi}{2} - \frac{1}{2}\left(\psi\left(\frac{3}{4}+\frac{1}{2\pi}\right) - \psi\left(\frac{1}{4}+\frac{1}{2\pi}\right)\right) \tag{19}$$

$$\int_0^1 \tan^{-1}\left(\left(\cosh^{-1}\left(\frac{1}{1-x}\right)\right)^{-1}\right) dx = \frac{\pi}{2} - \frac{1}{2}\left(\psi\left(\frac{3}{4}+\frac{1}{2\pi}\right) - \psi\left(\frac{1}{4}+\frac{1}{2\pi}\right)\right) \tag{20}$$

Let $0 \le u \le 1$, $0 \le v \le \pi/2$, $u = \dfrac{1}{\cosh(\tan v)}$, then

$$\int_0^u \tan^{-1}\left(\cosh^{-1}\left(\frac{1}{x}\right)\right) dx + \int_0^v \frac{1}{\cosh(\tan x)}\, dx = u\,v + \frac{1}{2}\left(\psi\left(\frac{3}{4}+\frac{1}{2\pi}\right) - \psi\left(\frac{1}{4}+\frac{1}{2\pi}\right)\right) \tag{21}$$

$$\int_u^1 \tan^{-1}\left(\cosh^{-1}\left(\frac{1}{x}\right)\right) dx + \int_v^{\pi/2} \frac{1}{\cosh(\tan x)}\, dx = -u\,v + \frac{1}{2}\left(\psi\left(\frac{3}{4} + \frac{1}{2\pi}\right) - \psi\left(\frac{1}{4} + \frac{1}{2\pi}\right)\right) \tag{22}$$

Let $0 \le u \le 1$, $0 \le v \le \pi/2$, $u = 1 - \dfrac{1}{\cosh(\cot v)}$, then

$$\int_0^u \tan^{-1}\left(\left(\cosh^{-1}\left(\frac{1}{1-x}\right)\right)^{-1}\right) dx - \int_0^v \frac{1}{\cosh(\cot x)}\, dx = \frac{\pi}{2} + u\,v - v - \frac{1}{2}\left(\psi\left(\frac{3}{4} + \frac{1}{2\pi}\right) - \psi\left(\frac{1}{4} + \frac{1}{2\pi}\right)\right) \tag{23}$$

$$\int_u^1 \tan^{-1}\left(\left(\cosh^{-1}\left(\frac{1}{1-x}\right)\right)^{-1}\right) dx - \int_v^{\pi/2} \frac{1}{\cosh(\cot x)}\, dx = v - u\,v - \frac{1}{2}\left(\psi\left(\frac{3}{4} + \frac{1}{2\pi}\right) - \psi\left(\frac{1}{4} + \frac{1}{2\pi}\right)\right) \tag{24}$$

Let $0 \le u \le \infty$, $0 \le v \le 1/2$, $v = \left(1 + e^{u^2}\right)^{-1}$, then

$$\int_0^u \frac{1}{1 + e^{x^2}}\, dx + \int_0^v \sqrt{\ln\left(\frac{1}{x} - 1\right)}\, dx = u\,v + \frac{1}{2}\zeta\left(\frac{1}{2}\right)\sqrt{\pi}\left(1 - \sqrt{2}\right) \tag{25}$$

$$\int_u^\infty \frac{1}{1 + e^{x^2}}\, dx + \int_v^{1/2} \sqrt{\ln\left(\frac{1}{x} - 1\right)}\, dx = -u\,v + \frac{1}{2}\zeta\left(\frac{1}{2}\right)\sqrt{\pi}\left(1 - \sqrt{2}\right) \tag{26}$$

$$\int_0^\infty \frac{1}{1 + e^{x^2}}\, dx = \int_0^{1/2} \sqrt{\ln\left(\frac{1}{x} - 1\right)}\, dx = \frac{1}{2}\zeta\left(\frac{1}{2}\right)\sqrt{\pi}\left(1 - \sqrt{2}\right) \tag{27}$$

Let $u = 0.449629\,\ldots$, $u = \left(1 + e^{u^2}\right)^{-1}$, then

$$\int_0^u \left(\frac{1}{1 + e^{x^2}} + \sqrt{\ln\left(\frac{1}{x} - 1\right)}\right) dx = u^2 + \frac{1}{2}\zeta\left(\frac{1}{2}\right)\sqrt{\pi}\left(1 - \sqrt{2}\right) \tag{28}$$

Let $\alpha = e\tanh\left(\frac{e}{2}\tanh\left(\frac{e}{2}\tanh\left(\frac{e}{2}\,\ldots\right)\right)\right) = 2.1521902\,\ldots$, then

$$\pi = \int_0^{\pi/2} \ln\left(\ln^2 \tan x + 2\,\alpha \ln \tan x + \alpha^2 + \frac{\pi^2}{4}\right) dx \tag{29}$$

$$\pi = \int_0^{\pi/2} \ln\left(\frac{\pi^2}{4} + \left(\sinh^{-1}\left(\frac{\cosh\alpha}{\tan x}\right)\right)^2\right) dx \tag{30}$$

$$\frac{\pi}{8}\left(1 - \ln\left(\frac{\pi}{2}\right)\right) = \int_0^\infty \frac{x}{4\,x^2 + \pi^2} \tan^{-1}\left(\frac{\cosh\alpha}{\sinh x}\right) dx \tag{31}$$

Let $\lambda = 1.2926957193733983\,\ldots$, $\cos\lambda = e^{-\lambda}$, then

$$\int_0^1 \cos^{-1}\left(x\, e^{-\cos^{-1}\left(x\, e^{-\cos^{-1}\left(x\, e^{-\cos^{-1}(x\,\ldots)}\right)}\right)}\right) dx = \frac{1}{2}\left(e^{\pi/2} - 1 + 2\,\lambda - \tan\lambda\right) \tag{32}$$

Let $u = \dfrac{e^{\pi/4}}{\sqrt{2}}$, then

$$\int_1^u \ln(x\sec(\ln x\sec(\ln x\sec(\ln x\ \ldots))))\,dx = \frac{1}{2} - \frac{e^{\pi/4}}{\sqrt{2}}\left(1 - \frac{\pi}{4}\right) \tag{33}$$

Let $u = W(2) = 2\,e^{-2\,e^{-2\,e^{-2\ldots}}}$, $W(x)$ is the Lambert Function , then

$$\int_0^{\sqrt{u/2}} \sqrt{-\ln x}\;dx + \int_0^{\sqrt{u/2}} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} + \frac{u}{2} \tag{34}$$

$$\int_0^{\sqrt{u/2}} \sqrt{-\ln x}\;dx = \frac{\sqrt{\pi}}{2} + \frac{u}{2} - \sqrt{\frac{u}{2}}\sum_{n=0}^{\infty}\frac{(-1)^n\,(u/2)^n}{n!\,(2n+1)} \tag{35}$$

$$\int_0^{\sqrt{u/2}} \sqrt{-\ln x}\;dx = \frac{1}{4}\int_0^u \sqrt{\frac{\ln 2 - \ln x}{x}}\;dx = \Gamma\left(\frac{3}{2}, \frac{1}{2}\ln\left(\frac{2}{u}\right)\right) \tag{36}$$

$$\int_0^{\infty} \frac{1}{1 + e^x + e^{2x}}\,dx = \int_0^{1/3} \ln\left(\frac{2 - 2x}{x + \sqrt{4x - 3x^2}}\right)dx = \frac{\ln 3}{2} - \frac{\pi}{6\sqrt{3}} \tag{37}$$

$$\int_0^{1/3}\left(\sqrt[3]{\frac{1}{6}\sqrt{\frac{3x^2 + 14x + 27}{3x^2}} + \frac{7}{54} + \frac{1}{2x}} - \sqrt[3]{\frac{1}{6}\sqrt{\frac{3x^2 + 14x + 27}{3x^2}} - \frac{7}{54} - \frac{1}{2x}}\right)dx = \frac{4}{9} + \frac{\ln 3}{2} - \frac{\pi}{6\sqrt{3}} \tag{38}$$

References

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