SOLUTION OF THE ERDÖS-MOSER EQUATION
$1 + 2^p + 3^p + ... + (k)^p = (k+1)^p$

DAVID STACHA

Obchodna 32, 906 38 Rohoznik, Slovak Republik

Abstract. I will provide a solution of the Erdös-Moser equation, based on the properties of Bernoulli polynomials, and prove that there is only one solution satisfying the above-mentioned equation.

1. Notation

$1 + 2^p + 3^p + ... + (k)^p = (k+1)^p$ represents the Erdös-Moser equation, where $k, p \in \mathbb{N}$. Let b_n denote Bernoulli numbers and $B_n(x) = \sum_{k=0}^{n} \binom{n}{k} b_{n-k} x^k$ denote Bernoulli polynomials for $n \geq 0$.

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2. Introduction

The Erdös-Moser equation (EM equation), named after Paul Erdös and Leo Moser, has been studied by many number theorists throughout history since combines addition, powers and summation together. An open and very interesting conjecture of Erdös-Moser states that there is no other solution of the EM equation than trivial $1 + 2 = 3$. Investigation of the properties and identities of the EM equation and ultimately prove the conjecture is the main purpose of this article.

3. Solution

Lemma 3.1. *The EM equation is equivalent of*

$$\sum_{k=0}^{x} k^p \equiv \frac{B_{p+1}(x+1)}{p+1} = (x+1)^p \tag{3.1}$$

$x, p \in \mathbb{N} \wedge x > 2 \wedge p > 1$ *since we are seeking other solution than trivial.*

Proof. Sum of pth powers is defined as

$$\sum_{k=0}^{x} k^p = \frac{B_{p+1}(x+1) - B_{p+1}(0)}{p+1}$$

Leo Moser proved that for another solution of the EM equation two must divide p, see [1], which yields that $p+1$ must be odd and $B_{p+1}(0)$ with odd subscripts is equal to zero. □

E-mail address: safiro@centrum.sk.

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Lemma 3.2.

$$B_{p+1}(x+1) - B_{p+1}(x) = (p+1)x^p \tag{3.2}$$

$$B_{p+1}(x+2) - B_{p+1}(x+1) = (p+1)(x+1)^p \tag{3.3}$$

Proof. Relation of Bernoulli polynomials given by Whittaker and Watson, see [2], in general form is defined as $B_n(x+1) - B_n(x) = nx^{n-1}$. □

Lemma 3.3. *Eq. (3.1) in combination with rearranged Eq. (3.2) gives a relation*

$$\frac{B_{p+1}(x+1)}{B_{p+1}(x)} = \frac{(x+1)^p}{(x+1)^p - x^p} \tag{3.4}$$

Proof. Let us express $p+1$ from Eq. (3.2) as

$$\frac{B_{p+1}(x+1)}{x^p} - \frac{B_{p+1}(x)}{x^p} = p+1 \tag{3.5}$$

then by putting LHS of Eq. (3.5) in Eq. (3.1) we get

$$B_{p+1}(x+1) = (x+1)^p \left(\frac{B_{p+1}(x+1)}{x^p} - \frac{B_{p+1}(x)}{x^p} \right)$$

and after elementary rearrangements we can rearrange Eq. (3.1) to the form defined in Lemma (3.3.). □

Theorem 3.4. *The EM equation has other solution than trivial if and only if holds the following equation.*

$$\frac{B_{p+1}(x+2)}{B_{p+1}(x+1)} = 2 \tag{3.6}$$

$x, p \in \mathbb{N} \wedge x > 2 \wedge p > 1$.

Proof. Let us rearrange Eq. (3.1) as

$$B_{p+1}(x+1) = (p+1)(x+1)^p \tag{3.7}$$

the RHS of Eq. (3.3) and Eq. (3.7) are equal, so we can define

$$B_{p+1}(x+2) - B_{p+1}(x+1) = B_{p+1}(x+1)$$
$$B_{p+1}(x+2) = 2B_{p+1}(x+1)$$
$$\frac{B_{p+1}(x+2)}{B_{p+1}(x+1)} = 2$$

□

Lemma 3.5. *Let us define a set*

$$Z = \left\{ \frac{B_{p+1}(x_z+1)}{B_{p+1}(x_z)} = \frac{(x_z+1)^p}{(x_z+1)^p - x_z^p} \mid x_z, p \in \mathbb{N} \wedge p > 1 \right\}$$

which contains Eq. (3.4) defined in Lemma (3.3.)

Example 3.6. $Z = \left\{ \frac{B_{p+1}(1)}{B_{p+1}(0)} = \frac{(1)^p}{(1)^p - 0^p}, \frac{B_{p+1}(2)}{B_{p+1}(1)} = \frac{(2)^p}{(2)^p - 1^p} \cdots \right\}$.

and a set

$$F = \left\{ \frac{B_{p+1}(x_f+2)}{B_{p+1}(x_f+1)} = 2 \mid x_f, p \in \mathbb{N} \wedge x_f > 2 \wedge p > 1 \right\}$$

which contains all Eq.(3.6) with all possible non-trivial solutions x_f satisfying this equation

Example 3.7. Let us assume that $x_f = 4$ is the non-trivial solution. Then $F = \left\{ \frac{B_{p+1}(6)}{B_{p+1}(5)} = 2 \right\}$.

then

$$F \subseteq Z$$

Remark 3.8. From the definitions of the sets in Lemma (3.5.) follows that x_f is a variable of a corresponding element $\frac{B_{p+1}(x_f+2)}{B_{p+1}(x_f+1)} = 2$ and x_z is a variable of a corresponding element $\frac{B_{p+1}(x_z+1)}{B_{p+1}(x_z)} = \frac{(x_z+1)^p}{(x_z+1)^p - x_z^p}$.

Proof. The rules in the sets Z and F are sufficient to prove Lemma (3.5.) since we are seeking other solution than trivial and for $x_f > 2 \wedge p > 1$. It is more than clear that $F \subseteq Z$ since for every variable x_f holds the following relation

$$\forall x_f : x_f = x_z - 1 \tag{3.8}$$

and the corresponding elements of the variables x_z, x_f, which are in relation (3.8), in both sets are equal, and that proves Lemma (3.5.), (see Example 3.9.).

Example 3.9. Similarly as in Example (3.7.), let us assume that $x_f = 4$ would be the non-trivial solution. This example demonstrates the fact that $F \subseteq Z$, which follows from Lemma (3.5.), since the elements in both sets of corresponding variables x_z, x_f, which are in relation (3.8), are equal. In this case when $x_f = 4$ according to relation (3.8) $x_z = 5$ and the corresponding elements are equal (see below).

x_z	Elements of set Z	x_f	Elements of set F
	$\frac{B_{p+1}(x_z+1)}{B_{p+1}(x_z)} = \frac{(x_z+1)^p}{(x_z+1)^p - x_z^p}$		$\frac{B_{p+1}(x_f+2)}{B_{p+1}(x_f+1)} = 2$
3	$\frac{B_{p+1}(4)}{B_{p+1}(3)} = \frac{(4)^p}{(4)^p - 3^p}$		
4	$\frac{B_{p+1}(5)}{B_{p+1}(4)} = \frac{(5)^p}{(5)^p - 4^p}$	4	$\frac{B_{p+1}(6)}{B_{p+1}(5)} = 2$
5	$\frac{B_{p+1}(6)}{B_{p+1}(5)} = \frac{(6)^p}{(6)^p - 5^p}$		
$\vdots$	$\vdots$		

□

Theorem 3.10. *There is no element in the set Z which is equal to two for $x_z > 2 \wedge p > 1$ and since $F \subseteq Z$ the EM equation does not have any other solution than trivial.*

Proof. From Lemma (3.5.) follows $F \subseteq Z$. It is clear that the elements of each set are equations. The elements of corresponding variables x_z, x_f, which are in relation (3.8), are equal and thus these equations must be equal as well. Let us recall that every element in the set Z is defined as $\frac{B_{p+1}(x_z+1)}{B_{p+1}(x_z)} = \frac{(x_z)^p}{(x_z)^p - x_z^p}$ and every element in the set F is defined as $\frac{B_{p+1}(x_f+2)}{B_{p+1}(x_f+1)} = 2$ (see definitions of the sets in Lemma (3.5.)).

Since $F \subseteq Z$ and every element in the set F is equal to two, in order to prove Theorem (3.10.) it is enough to prove that no element in the set Z has an integral solution, equal to two, for $p > 1$ since it will be in contradiction. It is trivial to see that the expression $\frac{(x_z)^p}{(x_z)^p - x_z^p}$ has an integral solutions for $x_z > 1$ if and only if $0 < p < 2$ since by using the binomial expansion of the elements in the set Z we get

$$\frac{B_{p+1}(x_z+1)}{B_{p+1}(x_z)} = \frac{(x_z+1)^p}{(x_z+1)^p - x_z^p} = \frac{x_z^p + px_z^{p-1} + ... + 1}{px_z^{p-1} + ... + 1} = \frac{x_z^p}{px_z^{p-1} + ... + 1} + 1$$

where is clear that $(px_z^{p-1} + ... + 1) \nmid x_z^p$ for $p > 1$. In other words, there is no element in the set Z which is equal to two for $p > 1$ and that is in contradiction with the fact that $F \subseteq Z$. On the basis of this facts we can state that there is only a trivial solution of the EM equation, when $p = 1$, as it follows from the basic formula of summation $\sum_{k=0}^{x} k^1 \equiv \frac{x*(x+1)}{2} = x + 1 \Rightarrow \frac{x}{2} = 1$, where x must be equal to two. All of the above-mentioned facts unconditionally prove Theorem (3.10.) and at the same time the Erdös-Moser conjecture.

□

Example 3.11. Let us assume that $x_f = 4$ is the non-trivial solution. The corresponding Eq.(3.6) (after substitution $\frac{B_{p+1}(6)}{B_{p+1}(5)} = 2$) holds for this x_f and this Eq.(3.6) is an element of the set F. Since $F \subseteq Z$, and thanks to the relation (3.8), we are able to define $x_z = 5$ and the corresponding element of the set Z as $\frac{B_{p+1}(6)}{B_{p+1}(5)} = \frac{(6)^p}{(6)^p - 5^p}$. LHS of the elements in both sets are equal, so RHS must be equal as well, but there is no element in the set Z which is equal to two for $p > 1$, which is in contradiction, and therefore $x_f = 4$ can not be the non-trivial solution.

References

[1] L.Moser, On the Diophantine Equation $1^k + 2^k + ...(m-1)^k = m^k$, *Scripta Math. 19,* (1953), 84-88.

[2] E. T. Whittaker, G. N. Watson, A course of MODERN ANALYSIS, *Cambridge University Press 3rd edition,* (1920), 127.