Pythagorean Triangles, Ellipse & Fermat

- a mathematical essay; shorter version
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Abstract

The paper presents an attempt to solve a 300-year-old mathematical problem with minimalistic means of high-school mathematics [1]. As introduction, the Pythagorean equation of right angle triangles $\mathbf{a}^{\wedge 2} + \mathbf{b}^{\wedge 2} = \mathbf{c}^{\wedge 2}$ inscribed in the semicircle is reviewed; then, in an analogue way, the equation $\mathbf{a}^{\wedge 3} + \mathbf{b}^{\wedge 3} = \mathbf{c}^{\wedge 3}$ (and then $\mathbf{a}^{\wedge \mathbf{n}} + \mathbf{b}^{\wedge \mathbf{n}} = \mathbf{c}^{\wedge \mathbf{n}}$) represented by a triangle inscribed in the (vertical) ellipse with its basis **c** making the minor axis of the ellipse and the sides of the triangle made by the factors {**a**,**b**}. Should the inscribed triangles $\mathbf{a}^{\wedge 3} + \mathbf{b}^{\wedge 3} = \mathbf{c}^{\wedge 3}$ (and then $\mathbf{a}^{\wedge \mathbf{n}} + \mathbf{b}^{\wedge \mathbf{n}} = \mathbf{c}^{\wedge \mathbf{n}}$) represent the integer equations - with {**a, b, c, n**} positive integers, **n > 2 -** their sides must be rational to each other; they must form so called **integer triangles**. In such triangles, the square of altitude $\mathbf{y}^{\wedge 2}$ (or the altitude **y**) must be rational to the sides. An assumption is made that at least one of the inscribed triangles may be an integral one. A **unit** is derived from **c** by dividing it by a natural number **m**; if the assumption is true, the **unit** will measure (= divide) $\mathbf{y}^{\wedge 2}$ (or **y**) without leaving an irrational rest behind. The value of $\mathbf{y}^{\wedge 2}$ (or **y**) is taken from the equation of the ellipse. Conducted calculations show that $\mathbf{y}^{\wedge 2}$ (or **y**) divided by the **unit** leave always an irrational rest behind incompatible with **c**; this proves that $\mathbf{y}^{\wedge 2}$ (or **y**) is irrational with the basis **c** what excludes the existence of the assumed integral triangles and, in consequence, of the discussed integral equations.

"… **one of a host of mathematicians who combined numbers and general nonsense**…" Albert H. Beiler on some less talented mathematicians in Recreations in the Theory of Numbers, Dover Publications, 1966

"Ubi materia, ibi Geometria." Where there is matter, there is geometry. — Johannes Kepler; from https://todayinsci.com/

"Cuius rei demonstrationem mirabilem sane detexi hanc marginis exiguitas non caperet." - I have discovered a truly remarkable proof of this theorem which this margin is too small to contain. Note written [by Fermat] on the margins of his copy Arithmetica of Diophantus…(Wikiquots; https://en.wikiquote.org/wiki/Pierre_de_Fermat)

…*So, Fermat's original proof is still out there somewhere?* Andrew Wiles: "**I don't believe Fermat had a proof. I think he fooled himself into thinking he had a proof. But what has made this problem special for amateurs is that there's a tiny possibility that there does exist an elegant 17th-century proof… "**
(Nov 2000; NOVA; http://www.pbs.org/wgbh/nova/physics/andrew-wiles-fermat.html)

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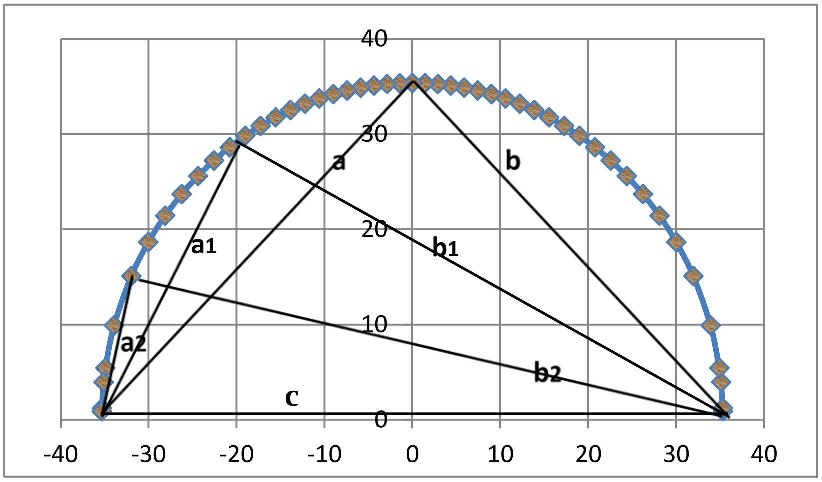
[1]] **MSC 2010**: Primary classification: **97A80** - Popularization of mathematics; Secondary classification: **11D41** -Higher degree equations; Fermat's equation

1. $a^{\wedge 2} + b^{\wedge 2} = c^{\wedge 2}$ and a triangle inscribed in a semi-circle

Recall the eternal Pythagorean equation : $\mathbf{a^{\wedge 2} + b^{\wedge 2} = c^{\wedge 2}}$ and representing (modeling) it the right angle triangle **abc**. Let the right side of the equation ($\mathbf{c^{\wedge 2}}$) be kept constant and change **a** & **b** (left side); the **changes** must fulfill then the requirements (constraints) of the equation. Let initially **a = b**, consequently, $\mathbf{a^{\wedge 2} = b^{\wedge 2}}$ and $\mathbf{a^{\wedge 2} + b^{\wedge 2} = 2a^{\wedge 2}}$; then $\mathbf{2a^{\wedge 2} = c^{\wedge 2}}$ and $\mathbf{c = a\sqrt{2}}$. The equilateral (and right angle) triangle **abc** is built now by the two sides **a** and a hypotenuse **c** equal to $\mathbf{a\sqrt{2}}$. Subsequently, **b** will be changed in relation to **a**: **b > a**, then **b >> a** until **b** will approach **c** and **a** will approach **0**. After reversing sides, **a** will be growing and **b** decreasing: **a > b**, **a >> b** until **a** approaches **c** and **b** approaches **0**. All resulting in such a way triangles are $\mathbf{90^0}$ degree at the apex and all apices together delineate a semicircle whose a diameter is the (constant) hypotenuse **c**; evidently, all the triangles are inscribed in the (semi)circle.



2. Trying to model the equation $a^{\wedge 3} + b^{\wedge 3} = c^{\wedge 3}$ by a triangle inscribed in an ellipse

In a similar way to squares, the equation of cubes: $\mathbf{a^{\wedge 3} + b^{\wedge 3} = c^{\wedge 3}}$ also can be modeled by a triangle: in a triangle **abc** two sides (**a & b**) **-** raised to the power of **3** and added together - are equal to the third side (basis **c**) raised to the power of three ($\mathbf{c^{\wedge 3}}$).

At the start **a = b** (and $\mathbf{a^{\wedge 3} = b^{\wedge 3}}$); consequently, $\mathbf{c^{\wedge 3} = 2{*}a^{\wedge 3}}$ and $\mathbf{c = a{*}\sqrt[3]{2}}$. Let **c** (as a basis of the triangle) stay **constant** ($\mathbf{c = a{*}\sqrt[3]{2}}$), **b** grow and **a** decrease - from **b = a** to **b > a**, then **b >> a** until **b** approaches **c** and **a** approaches **0.** While growing **b** is given, **a** is calculated from $\mathbf{c^{\wedge 3} - b^{\wedge 3} = a^{\wedge 3}}$.

It is to observe that the apices of the triangles draw an upper (left) quarter of a (vertical) **ellipse**; after reversing sides **a** is growing and **b** decreasing - from **a = b** to **a > b** and, later on, to **a >> b** and up to **a =** (almost) **c** - while **b** approaches **0;** the second quarter of the ellipse is drawn (on the right side).

To stress the difference to the power of **2** (and, later on, to the established labeling of the equation of the ellipse), the labeling of the triangle sides will be changed: the equal two sides raised to the power of **3** will be called now $\mathbf{r_0^{\wedge 3}}$; thus, at the beginning, **(left)** $\mathbf{r_0}$ **= (right)** $\mathbf{r_0}$ and $\mathbf{r_0^{\wedge 3} + r_0^{\wedge 3} = 2r_0^{\wedge 3} = [r_0{*}\sqrt[3]{2}]^{\wedge 3}}$; $\mathbf{c = r_0{*}\sqrt[3]{2}}$ corresponds to the old hypotenuse **c** when the power **n** was **n = 2**. Keeping $\mathbf{c = r_0{*}\sqrt[3]{2}}$ constant, the ratio of the two remaining sides is successively changing; the decreasing side of the triangle (former **a**) is now $\mathbf{r_1}$, the increasing side $\mathbf{r_2}$. So, first $\mathbf{r_2 = r_1}$ (= $\mathbf{r_0}$)**,** then $\mathbf{r_2 > r_1}$ and $\mathbf{r_2 >> r_1}$ until $\mathbf{r_2}$ approaches **c** and $\mathbf{r_1}$ approaches **0**.

All the time the sum of the cubes of the two sides is the same: $\mathbf{c^{\wedge 3}}$. Again, $\mathbf{r_1^{\wedge 3} + r_2^{\wedge 3} = c^{\wedge 3}}$; $\mathbf{c^{\wedge 3} = [r_0{*}\sqrt[3]{2}]^{\wedge 3}}$.

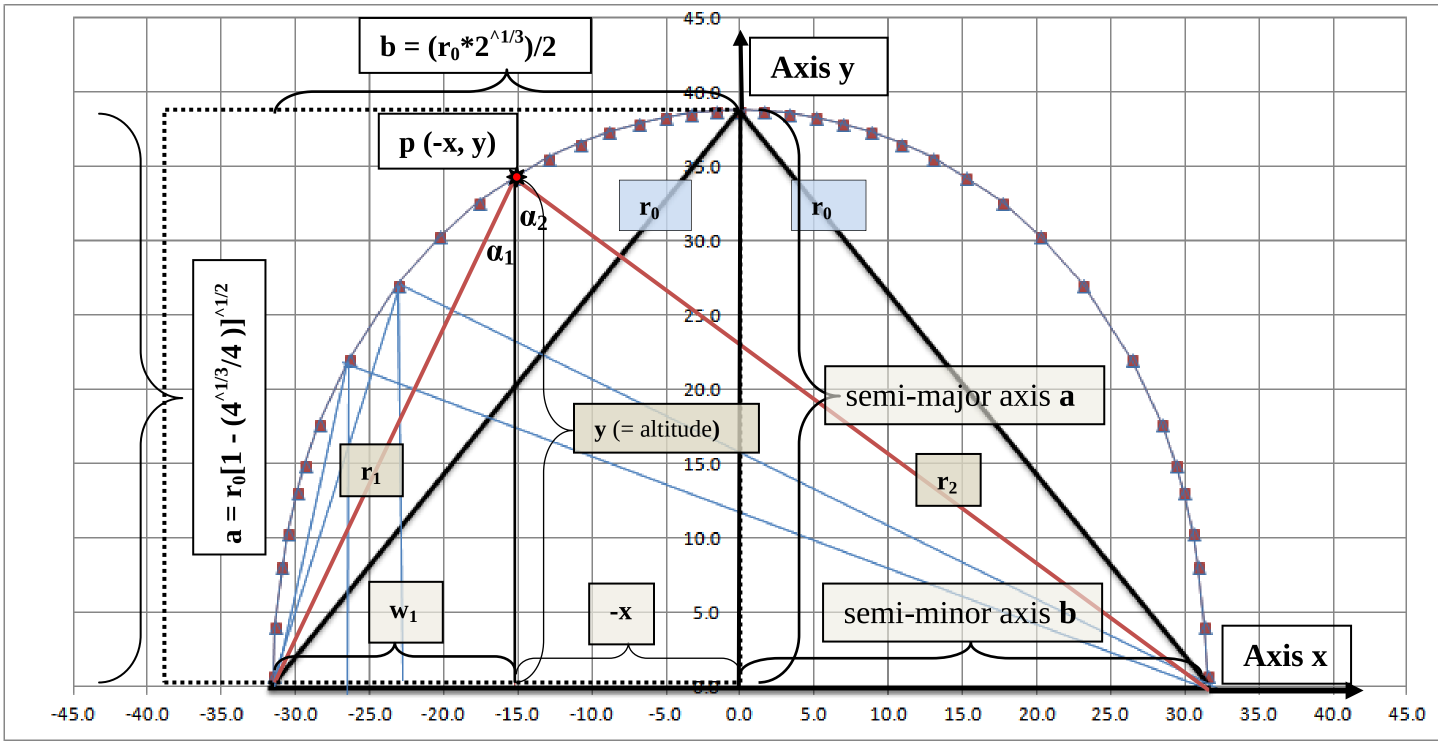
The situation is mirrored when the $\mathbf{r_1}$ is growing, $\mathbf{r_2}$ decreasing and - when $\mathbf{r_1}$ is approaching **c -** then $\mathbf{r_2}$ is approaching **0**. As mentioned, varying ratio of $\mathbf{r_2/r_1}$ will cause the apices of the triangles draw the curve of a (vertical) ellipse.

As an example, a calculating table (in Excel) and a matching chart have been created with the starting value $\mathbf{r_0 = 50}$.
So, first $\mathbf{r_1 = r_2 = r_0 = 50}$; $\mathbf{c = r_0 * \sqrt[3]{2} = 62.9960524947436} \cong \mathbf{63}$; **c** will be constant; $\mathbf{r_2}$ is generated by the table (or inserted manually) and $\mathbf{r_1}$ is calculated fron $\mathbf{r_1{}^{\wedge 3} = c^{\wedge 3} - r_2{}^{\wedge 3}}$.
From the triangle sides $\mathbf{\{r_1, r_2, c\}}$ **-** after Heron's formula - the **area** is calculated hence the triangle's **altitude**.
The top **angle α** (at **apex**) is the sum of two adjacent angles $\mathbf{\alpha_1 + \alpha_2 = \alpha}$ where $\mathbf{\alpha_1}$ is the angle between the side $\mathbf{r_1}$ and the **altitude** and $\mathbf{\alpha_2}$ (is) the angle between the **altitude** and the side $\mathbf{r_2}$.
The ratios $\mathbf{(altitude/r_1) = \cos \alpha_1}$; $\mathbf{(altitude/r_2) = \cos \alpha_2}$. An auxiliary value $\mathbf{w_1}$ is calculated - it is the **distance** on **c** between the point where the **altitude** crosses the basis **c** and the nearest corner on **c**. Further on, $\mathbf{|-x| = (c/2) - (w_1)}$; **(y) = altitude**; all calculated by the programmed table.
On the ellipse, the Cartesian coordinates with **0** (origin) in the middle of **c** is inserted; the **apex p** of each generated triangle has coordinates **p(-x, y)** or **p(x, y)**. The ellipse is generated by the apices of all triangles…

(c)	$\mathbf{r_1}$	$\mathbf{r_2}$	**area of Δ**	**y (altitude)**	$\mathbf{w_1}$	**-x**	**+x**	$\mathbf{\alpha_1}$	$\mathbf{\alpha_2}$	**α**
63	50.0	50	1223.1	38.8	31.5	0.0	0.0	39.0	39.0	78.1
63	49.0	51	1221.6	38.8	29.9	-1.6	1.6	37.6	40.5	78.1
63	47.8	52	1216.9	38.6	28.2	-3.3	3.3	36.1	42.0	78.1
63	46.6	53	1208.5	38.4	26.4	-5.1	5.1	34.6	43.6	78.2
63	45.2	54	1195.7	38.0	24.6	-6.9	6.9	32.9	45.3	78.3
63	43.7	55	1177.9	37.4	22.7	-8.8	8.8	31.2	47.2	78.4
63	42.1	56	1154.1	36.6	20.6	-10.9	10.9	29.4	49.1	78.5
63	40.2	57	1122.7	35.6	18.5	-13.0	13.0	27.5	51.3	78.7
63	38.0	58	1081.9	34.3	16.3	-15.2	15.2	25.3	53.7	79.0
63	35.5	59	1028.4	32.7	13.9	-17.6	17.6	23.0	56.4	79.4
63	32.4	60	956.9	30.4	11.3	-20.2	20.2	20.3	59.6	79.9
63	28.4	61	856.2	27.2	8.4	-23.1	23.1	17.1	63.5	80.7
63	22.7	62	696.4	22.1	5.1	-26.4	26.4	12.9	69.1	82.0
63	18.0	62.5	559.5	17.8	3.1	-28.4	28.4	9.8	73.5	83.3
63	15.2	62.7	473.9	15.0	2.1	-29.4	29.4	8.1	76.1	84.2
63	13.3	62.8	414.4	13.2	1.6	-29.9	29.9	6.9	77.9	84.8
63	10.5	62.9	327.8	10.4	1.0	-30.5	30.5	5.3	80.5	85.8
63	8.2	62.95	257.1	8.2	0.6	-30.9	30.9	4.0	82.5	86.6
63	4.2	62.99	131.0	4.2	0.1	-31.4	31.4	2.0	86.2	88.2
63	0.9	62.996	26.9	0.9	0.0	-31.5	31.5	0.4	89.2	89.6

The marks on the ellipse below correspond to the altitude of the triangle (**y**) from the table. [The curve is smoothed by Excel.] The basis of the triangle is $\mathbf{c = r_0 * \sqrt[3]{2}}$. The sides $\mathbf{r_1}$ **&** $\mathbf{r_2}$ on the graph below were taken randomly as an example.

Almost all details are seen here: a triangle $\mathbf{r_1 r_2 c}$ taken into consideration, its apex **p(-x, y)** lying on the ellipse; there is **c**, [$\mathbf{c = r_0 * \sqrt[3]{2}}$], **y = altitude**, further the value $\mathbf{w_1}$ and **(-x)**.
On the left side of the chart a dotted square is inserted to show that the curve is not a circle but rather an ellipse.
Calculations of the **semi-axes a & b**: **b** is **1/2** of **c;** $\mathbf{b = (r_0 * \sqrt[3]{2})/2}$; $\mathbf{b^{\wedge 2} = (r_0{}^{\wedge 2}) * \sqrt[3]{4}/4}$
the **semi-axis a:** $\mathbf{r_0{}^{\wedge 2} = a^{\wedge 2} + b^{\wedge 2}}$; $\mathbf{a^{\wedge 2} = r_0{}^{\wedge 2} - (r_0{}^{\wedge 2}) * \sqrt[3]{4}/4}$; $\mathbf{a^{\wedge 2} = r_0{}^{\wedge 2} * (1 - \sqrt[3]{4}/4)}$; $\mathbf{a = r_0 * [1 - \sqrt[3]{4}/4]^{\wedge 1/2}}$



3. Testing whether the model triangle inscribed in the ellipse can be rational

Recall again the old Pythagorean equation (of the rectangular triangle) $\mathbf{a}^{\wedge 2} + \mathbf{b}^{\wedge 2} = \mathbf{c}^{\wedge 2}$ (where **c** is a hypotenuse).
If **a** is assumed **rational** and **a = b,** then $\mathbf{c} = \mathbf{a}*\sqrt{\mathbf{2}}$; it makes **c** clearly irrational to **a**.
Now, if - exactly as on page 1 - to keep the hypotenuse **c** constant and change the **a/b** ratio (still fulfilling the equation) - could a point be reached where all 3 factors of the equation {**a, b, c**} (= all sides of the triangle) are rational to each other ? How could **c,** which is **constant** & **irrational**, change unexpectedly to **rational** ?

For more than 2500 years it has been known as possible; namely, after **a** & **b** have changed to sides with values $\mathbf{a_1}*\sqrt{\mathbf{2}}$ & $\mathbf{b_1}*\sqrt{\mathbf{2}}$ [also see [2]], then the sides {**a, b, c**} have values {$\mathbf{a_1}*\sqrt{\mathbf{2}}$, $\mathbf{b_1}*\sqrt{\mathbf{2}}$, $\mathbf{a}*\sqrt{\mathbf{2}}$}, respectively.
Dividing the sides of such triangle by $\sqrt{\mathbf{2}}$ makes the irrational factor $\sqrt{\mathbf{2}}$ be cancelled off; it leaves sides rational/integral to each other.

The irrationality of the hypotenuse **c** [$\mathbf{c} = \mathbf{a}*\sqrt{\mathbf{2}}$] must be therefore assessed in relation to the two remaining sides {$\mathbf{a_1}$, $\mathbf{b_1}$}. If all 3 sides {$\mathbf{a_1}, \mathbf{b_1}, \mathbf{c}$} have the same irrational factor and if it is possible to measure them with a **common unit** (be it even an irrational one) without leaving a reminder, then those segments (here: sides) can (and even must !) be considered as rational to each other. Again, if **a** is assumed as rational, $\mathbf{a}*\sqrt{\mathbf{2}}$ will be irrational to **a**; but $\mathbf{a}*\sqrt{\mathbf{2}}$ and $\mathbf{a_1}*\sqrt{\mathbf{2}}$ will be rational to each other when **a** and $\mathbf{a_1}$ (without the factor $\sqrt{\mathbf{2}}$) are.

Therefore - in the triangle $\mathbf{r_1 r_2 c}$ whose three sides stand for the equation $\mathbf{r_1}^{\wedge 3} + \mathbf{r_2}^{\wedge 3} = \mathbf{c}^{\wedge 3}$
and whose constant (& irrational) basis **c** is $\mathbf{c} = \mathbf{r_0}*\sqrt[3]{\mathbf{2}}$ - it seems that the only way to make all three sides of the triangle rational to each other is to find similar irrational values for the sides $\mathbf{r_1}$ &, $\mathbf{r_2}$:
$\mathbf{r_1}*\sqrt[3]{\mathbf{2}}$ and $\mathbf{r_2}*\sqrt[3]{\mathbf{2}}$ [**stipulation**: $\mathbf{r_1}$ and $\mathbf{r_2}$ alone, without the factor $\sqrt[3]{\mathbf{2}}$**,** must be rational to $\mathbf{r_0}$].

So, the **assumption** has been made that such a triangle, called the "**integer triangle**", exists.

[2]] However, $\mathbf{a_1}$ and $\mathbf{b_1}$ alone (= without the factor $\sqrt{\mathbf{2}}$) as well as **a** must be rational (to each other)…

Rationality in this triangle would mean divisibility by $\mathbf{r_0}*\sqrt[3]{\mathbf{2}}$ without [an irrational] reminder; but - since $\mathbf{r_0}*\sqrt[3]{\mathbf{2}}$ is the greatest of the three sides - a smaller **unit** will be rather used [for division and/or measuring]:
- a **rational part** of $\mathbf{r_0}*\sqrt[3]{\mathbf{2}}$. Therefore measuring unit is: **unit** = $\mathbf{([r_0*\sqrt[3]{2}]/m)}$; [**m** being a natural number, **m > 1**]. In order to accept the quotient as rational, the result of the division by the unit $\mathbf{[(r_0/m)*\sqrt[3]{2}]}$ must be an integer or rational number and no an irrational [with one small exception to this rule - see below].

Should measuring (= dividing by the **unit**) show rationality, the result will be written as **rational** (**rat.** or $\mathbf{rat_1}$, $\mathbf{rat_2}$ etc.); thus $\mathbf{(r_1*\sqrt[3]{2})/[(r_0/m)*\sqrt[3]{2}]}$, $\mathbf{(r_2*\sqrt[3]{2})/[(r_0/m)*\sqrt[3]{2}]}$ including $\mathbf{(r_0*\sqrt[3]{2})/[(r_0/m)*\sqrt[3]{2}]}$ will result in **rat.** or $\mathbf{rat_1}$, $\mathbf{rat_2}$, $\mathbf{rat_3}$ - if it is an integral triangle.

If the assumed as "rational" value $\mathbf{[rat.\ r_x*\sqrt[3]{2}]}$ is squared, the measuring will give: $\mathbf{[rat.\ r_x*\sqrt[3]{2}]^{\wedge2}/[(r_0/m)*\sqrt[3]{2}] = [rat.*\sqrt[3]{2}]}$. Going slightly ahead, that could be the case in Heronian triangles [4] where the altitude **y** is rational to the sides and - if squared to $\mathbf{y^{\wedge2}}$ - and then measured with the unit would give as result $\mathbf{[rat.*\sqrt[3]{2}]}$ and not **[rat.]** [see [4] and later on in text]. The last result is still acceptable as "rational" (to the other segments).

Back to the main issue: to examine if the assumption (of an instance of an integral triangle) is correct, it is necessary to test whether the analyzed triangle has all **properties of integral triangles** [4]**.** Among others, the **altitude** or the **square of altitude** of such triangle has to be rational [see also [3, 4]]. Note that the **apex** of any triangle inscribed in the way specified above must lie on the ellipse itself; call it a point **p**; this point **p** can be described by Cartesian coordinates as **p(x, y)** [or **p(-x, y)** when $\mathbf{r_1 < r_2}$]. Note farther the obvious: **y** is the altitude of the inscribed triangle. Again, for an integer triangle, the altitude **y** or the squared altitude $\mathbf{y^{\wedge2}}$ has to be rational to all three sides of the triangle; consequently, $\mathbf{y^{\wedge2}}$ of the inscribed triangles will be tested.

From the equation of the ellipse [3] $\mathbf{(y/a)^{\wedge2} + (x/b)^{\wedge2} = 1}$, **y** and $\mathbf{y^{\wedge2}}$ will be solved**:**

$\mathbf{(y/a)^{\wedge2} + (x/b)^{\wedge2} = 1 \ \ | * a^{\wedge2}}$

$\mathbf{y^{\wedge2} + a^{\wedge2} *(x/b)^{\wedge2} = a^{\wedge2}}$; $\qquad \mathbf{y^{\wedge2} = a^{\wedge2} - a^{\wedge2} *(x/b)^{\wedge2}}$

$\mathbf{y^{\wedge2} = a^{\wedge2}}$ $[1 - \mathbf{(x/b)^{\wedge2}}]$ **;** $\quad \mathbf{y^{\wedge2} = a^{\wedge2}\, [(b^{\wedge2} - x^{\wedge2})/b^{\wedge2}]}$

$\mathbf{y^{\wedge2} = (a^{\wedge2}/b^{\wedge2})*(b^{\wedge2} - x^{\wedge2})}$; $\mathbf{y = (a/b)*\sqrt{(b^{\wedge2} - x^{\wedge2})}}$

The most convenient form of $\mathbf{y^{\wedge2}}$ for a discussion seems to be $\mathbf{y^{\wedge2} = a^{\wedge2}*[(b^{\wedge2} - x^{\wedge2})/b^{\wedge2}]}$. In an integer triangle the altitude always divides the basis into rational parts; the segments **b** & **x** have thus to be **rational**.
So, **x** is a rational part of **b**: $\mathbf{x = (k/l)*b}$ or $\mathbf{x = (k/l)*r_0\sqrt[3]{2}/2}$ where {**k,l**} are integers, **k < l**.

$\mathbf{x = (k/l)*r_0\sqrt[3]{2}/2;} \quad \mathbf{x^{\wedge2} = (k/l)^{\wedge2}*(r_0^{\wedge2})*\sqrt[3]{4}/4}$

A part of $\mathbf{y^{\wedge2}}$, $\mathbf{[(b^{\wedge2} - x^{\wedge2})/b^{\wedge2}]}$ is equal to:

$\mathbf{[(b^{\wedge2} - x^{\wedge2})/b^{\wedge2}] = [(r_0^{\wedge2})*\sqrt[3]{4}/4 - (k/l)^{\wedge2}*(r_0^{\wedge2})*\sqrt[3]{4}/4]/[\,(r_0^{\wedge2})*\sqrt[3]{4}/4] = 1 - (k/l)^{\wedge2}}$

Thus, $\mathbf{y^{\wedge2} = [a^{\wedge2}]*[1 - (k/l)^{\wedge2}] = [r_0^{\wedge2}*(1 - \sqrt[3]{4}/4)]*[1 - (k/l)^{\wedge2}]}$

Now, $\mathbf{y^{\wedge2}}$ measured with the **unit** is: $\mathbf{y^{\wedge2}/unit = y^{\wedge2}/[(r_0/m)*\sqrt[3]{2}\,] = [r_0^{\wedge2}*(1 - \sqrt[3]{4}/4)]*[1 - (k/l)^{\wedge2}]/\,[(r_0/m)*\sqrt[3]{2}\,]}$

$\mathbf{y^{\wedge2}/unit = m*r_0*[1 - (k/l)^{\wedge2}]*[1 - \sqrt[3]{4}/4]/\sqrt[3]{2} = [m*r_0]*[1 - (k/l)^{\wedge2}]*[1/\sqrt[3]{2} - \sqrt[3]{2}/4]}$

Simplified, $\mathbf{y^{\wedge2}/unit = [rat_1]*[rat_2]*[irr.\ part\ of\ type\ (1/\sqrt[3]{2} - \sqrt[3]{2}/4)]}$. However, for $\mathbf{y^{\wedge2}}$, to show a **rational** relationship with the sides of an integral triangle, the result should have been: **[rat]**
- when $\mathbf{y^{\wedge2}}$ is rational to the sides and **y** is not - or $\mathbf{[rat]*[irr.\ of\ type\ (\sqrt[3]{2})]}$ - when **y** (and $\mathbf{y^{\wedge2}}$ **!**) are rational with sides of the integer triangle - as it is the case in Heron's triangles.

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[3]] The terms **integral** and **rational** are used **interchangeably**; any rational not integral values (= rational and fractional at the same time) can be easily changed to integral by multiplying by denominators of rational fractions.

[4]] In so called **Heron's integral triangles** all three sides and (additionally) the altitude have to be rational [4]; it's a more stringent requirement than that for the square of the altitude being rational… Recall that all rationals - if squared - remain rational but only some irrationals of the type $\sqrt{\textbf{not squared rationals}}$ become rational/integral when squared: for example: $[\sqrt{2}]^{\wedge2}$ = **2**.

Since the confirmation of mutual rationality of the triangle sides has failed, the existence of such integral triangle has to be rejected. In consequence, it is impossible for all 3 sides of the (inscribed) triangle - all 3 factors of the equation $\mathbf{r_1^{\wedge 3} + r_2^{\wedge 3} = c^{\wedge 3}}$ - to be rational/integral at the same time.

4. General equation $\mathbf{a^{\wedge n} + b^{\wedge n} = c^{\wedge n}}$ when (integer n) > 2

Let the equation $\mathbf{a^{\wedge n} + b^{\wedge n} = c^{\wedge n}}$ be represented by the triangle with the sides {**a**, **b**} and the basis constant **c**, at **n integral**, **n > 2**. To check, whether {**a**, **b**, **c**} can be all rational at the same time, a discussion like above for $\mathbf{a^{\wedge 3} + b^{\wedge 3} = c^{\wedge 3}}$ will be carried out.

Similar labels will be used like in the case for **n = 3**: $\mathbf{r_0^{\wedge n} + r_0^{\wedge n} = 2r_0^{\wedge n} = c^{\wedge n}}$
$\mathbf{c^{\wedge n} = 2r_0^{\wedge n}} \mid \mathbf{(\ )^{\wedge 1/n}}$; $\mathbf{c} = r_0\mathbf{*2^{\wedge 1/n}}$; $\mathbf{c = r_0*\sqrt[n]{2}}$; $\mathbf{a \rightarrow r_1}$; $\mathbf{b \rightarrow r_2}$; $\mathbf{r_1^{\wedge n} + r_2^{\wedge n} = 2r_0^{\wedge n} = c^{\wedge n}}$

Actually, any real positive value can be ascribed to $\mathbf{c^{\wedge n}}$; then it suffices to calculate what $\mathbf{r_0^{\wedge n}}$ and - later on - what $\mathbf{r_1^{\wedge n}}$ **&** $\mathbf{r_2^{\wedge n}}$ can build it. So, first $\mathbf{r_0^{\wedge n} + r_0^{\wedge n} = 2r_0^{\wedge n} = c^{\wedge n}}$ [or $\mathbf{c^{\wedge n} = r_0^{\wedge n} + r_0^{\wedge n} = 2r_0^{\wedge n}}$] and then $\mathbf{c^{\wedge n} = r_1^{\wedge n} + r_2^{\wedge n} = 2r_0^{\wedge n}}$. While $\mathbf{r_1}$ and $\mathbf{r_2}$ act like variables, **c** stays constant; there can be an infinite number of the sides $\mathbf{r_1}$ & $\mathbf{r_2}$ and, consequently, an infinite number of the triangles $\mathbf{r_1r_2c}$ with the same **c** and staying within constraints of the equation $\mathbf{r_1^{\wedge n} + r_2^{\wedge n} = c^{\wedge n}}$. These triangles delineate with their apices a continuous curve which forms a vertical ellipse.

If at least one of these innumerable triangles has got all sides rational (to each other) at the same time, i.e. if it happens to be an integral triangle $\mathbf{r_1r_2c}$, the **altitude y** or the **square** of **altitude** $\mathbf{y^{\wedge 2}}$ of this triangle must be rational to all sides. [See also [5] and [6]]

Relevant segments and relations between them (see the graph below):
$\mathbf{b = c/2 = [r_0*2^{\wedge 1/n}]/2 = r_0*2^{\wedge(1-n)/n}}$; $\mathbf{b^{\wedge 2} = r_0^{\wedge 2}*4^{\wedge(1-n)/n}}$
$\mathbf{a^{\wedge 2} = r_0^{\wedge 2} - b^{\wedge 2} = r_0^{\wedge 2} - r_0^{\wedge 2}*4^{\wedge(1-n)/n} = r_0^{\wedge 2}[1 - 4^{\wedge(1-n)/n}]}$
$\mathbf{a^{\wedge 2} = r_0^{\wedge 2}*[1 - 4^{\wedge(1-n)/n}]}$; $\mathbf{a = r_0*[1 - 4^{\wedge(1-n)/n}]^{\wedge 1/2}}$

Additionally, $\mathbf{r_0^{\wedge 2}}$ is irrational towards (square of) the semi-major axis: $\mathbf{a^{\wedge 2}}$ ($\mathbf{a^{\wedge 2} = r_0^{\wedge 2}*[1 - 4^{\wedge(1-n)/n}]}$) and (square of) the semi-minor axis: $\mathbf{b^{\wedge 2}}$ ($\mathbf{b^{\wedge 2} = r_0^{\wedge 2}*4^{\wedge(1-n)/n}}$); farther, $\mathbf{a^{\wedge 2}}$ and $\mathbf{b^{\wedge 2}}$ are irrational to each other.
If so, then $\mathbf{r_0}$ is irrational to **a** ($\mathbf{a = r_0*[1 - 4^{\wedge(1-n)/n}]^{\wedge 1/2}}$) and **b** ($\mathbf{b = r_0*2^{\wedge(1-n)/n}}$); also **a** ($\mathbf{a = r_0*[1 - 4^{\wedge(1-n)/n}]^{\wedge 1/2}}$) and **b** ($\mathbf{b = r_0*2^{\wedge(1-n)/n}}$) are irrational to each other.

Rationality of $\mathbf{y^{\wedge 2}}$ is tested by dividing it with the designated rationality **unit**: a small part of the basis **c**;
unit = c/m; $\mathbf{c/m = [r_0(2)^{\wedge 1/n}/m]}$ (**m** being natural number, **m > 1**).

First, the equation of the ellipse (see [3] and [7]) is solved for $\mathbf{y^{\wedge 2}}$; $\mathbf{y^{\wedge 2} = (a^{\wedge 2}/b^{\wedge 2})*(b^{\wedge 2} - x^{\wedge 2})}$. Next, all factors of $\mathbf{(a^{\wedge 2}/b^{\wedge 2})*(b^{\wedge 2} - x^{\wedge 2})}$ are examined for their rationality. The factors $\mathbf{(b^{\wedge 2} - x^{\wedge 2})}$ & $\mathbf{(b^{\wedge 2})}$ will be all **rational** (to the sides of assumed integer triangle…) since **b = c/2** [= half of the basis] and **|- x|** is determined by the altitude crossing the basis **c** and in the **integral triangles** this segment must be rational; thus: $\mathbf{(b^{\wedge 2} - x^{\wedge 2})/(b^{\wedge 2}) = [rat]}$.
$\mathbf{y^{\wedge 2} = (a^{\wedge 2}/b^{\wedge 2})*(b^{\wedge 2} - x^{\wedge 2}) = [a^{\wedge 2}]*[(b^{\wedge 2} - x^{\wedge 2})/(b^{\wedge 2})] = (a^{\wedge 2})*[rat]}$
$\mathbf{y^{\wedge 2}/unit = (a^{\wedge 2})*[rat.]/[r_0(2)^{\wedge 1/n}/m] = r_0^{\wedge 2}*[1 - 4^{\wedge(1-n)/n}]*[rat.]/[r_0(2)^{\wedge 1/n}/m] = = [m*r_0*rat.]*[2^{\wedge -1/n} - 2^{\wedge(1-2n)/n}]}$
$\mathbf{y^{\wedge 2}/[r_0(2)^{\wedge 1/n}/m] = [m*r_0*rat.]*[2^{\wedge -1/n} - 2^{\wedge(1-2n)/n}]}$

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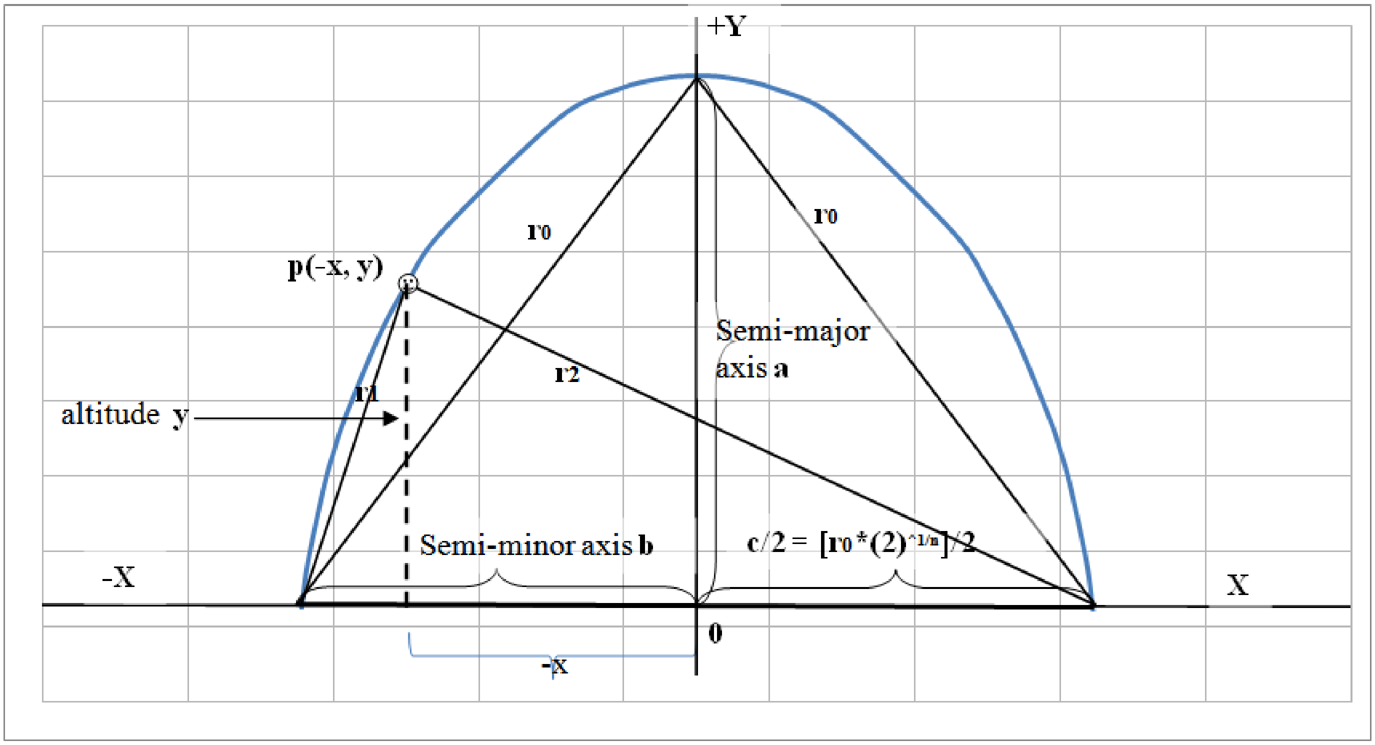
[5] **y** (the altitude) is, evidently, a perpendicular segment from the **apex** to the basis **c**.
[6] If **n = 2**, the apices of the inscribed triangles draw a circle; if **1 < n < 2** there will be a horizontal ellipse; at **n > 2 , n** real and positive (not only integer), there will be a vertical ellipse.
[7] Equation of the ellipse [3]: $\mathbf{(y/a)^{\wedge 2} + (x/b)^{\wedge 2} = 1}$



Conclusion: $\mathbf{y}^{\wedge 2}$ divided by the unit $\mathbf{[r_0(2)^{\wedge 1/n}/m]}$ renders $\mathbf{[m*r_0*rat]*[2^{\wedge -1/n} - 2^{\wedge (1-2n)/n}]}$ which means **[rat]*[irr.] = irrational**. Had the result been "rational" (i.e. compatible with **c** or **unit**), it would have been: **[rat. number]** or **[rat. number]*[(2)$^{\wedge 1/n}$]** what would have meant that $\mathbf{y}^{\wedge 2}$ or **y** (respectively) are rational with the sides of the assumed rational triangle $\mathbf{r_1 r_2 c}$.
(Note that the squares of the semi-axes $\mathbf{a}^{\wedge 2}$ and $\mathbf{b}^{\wedge 2}$ of the ellipse used here shall always be irrational to each other.)
[See also [8]]

Ergo, since $\mathbf{y}^{\wedge 2}$ (or **y**) measured with the **unit** derived from **c** is never rational to the sides of the assumed integral triangle $\mathbf{r_1,r_2,c}$ **-** that integral triangle cannot exist.

In the end, **Pierre de Fermat** (1607 - 1665) was right twice: in $\mathbf{a}^{\wedge \mathbf{n}} + \mathbf{b}^{\wedge \mathbf{n}} = \mathbf{c}^{\wedge \mathbf{n}}$ the factors {**a, b, c**}- at **n** > **2** and integral - cannot be all rational at the same time and a proof for that is a bit larger than a margin (of a book) but not much larger [1], [2].

5. Bibliography

[1] Recreations in the Theory of Numbers, Albert H. Beiler, Dover Publications, 1966
[2] Fermat's enigma, Simon Singh, Anchor Books, 1998
[3] http://www.mathwarehouse.com/ellipse/equation-of-ellipse.php
[4] https://en.wikipedia.org/wiki/Integer_triangle
[5] https://en.wikipedia.org/wiki/Irrational_number

[8]] Note that **n** is here **n <> 2**; if **n = 2**, then **semi-major axis a = semi-minor axis b = radius** and the ellipse changes to a circle where $\mathbf{c = r_0*(2)^{\wedge 1/2} = 2*radius}$; so, **c** & $\mathbf{r_0}$ are irrational to each other, but their squares are not: $\mathbf{r_0}^{\wedge 2}$ versus $\mathbf{2*r_0}^{\wedge 2}$ is rational.