Fixed Point Method , Dottie Number

Edgar Valdebenito

abstract

In this note we discuss the problem of finding approximate solutions of the equation:

$$x = \frac{1}{2} \sin^{-1}\left(2 x \sqrt{1 - x^2}\right), \quad 0 \leq x \leq 1$$

via Fixed Point Iteration Method.

1. Introduction

Definition of Fixed Point :

If $p = f(p)$, the we say p is a fixed point for the function $f(x)$.

Fixed Point Iteration Method : Start from any point x_0 and consider the recursive process

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, 3, \ldots \tag{1}$$

If $f(x)$ is continuous and x_n converges to some p then it is clear that p is a fixed point of $f(x)$.

Theorem 1 : Let $f : [a, b] \mapsto [a, b]$ be a differentiable function such that

$$|f'(x)| \leq \alpha < 1 \text{ for all } x \in [a, b] \tag{2}$$

Then $f(x)$ has exactly one fixed point p in $[a, b]$ and the sequence x_n defined by the process (1) , with a starting point , $x_0 \in [a, b]$, converges to p .

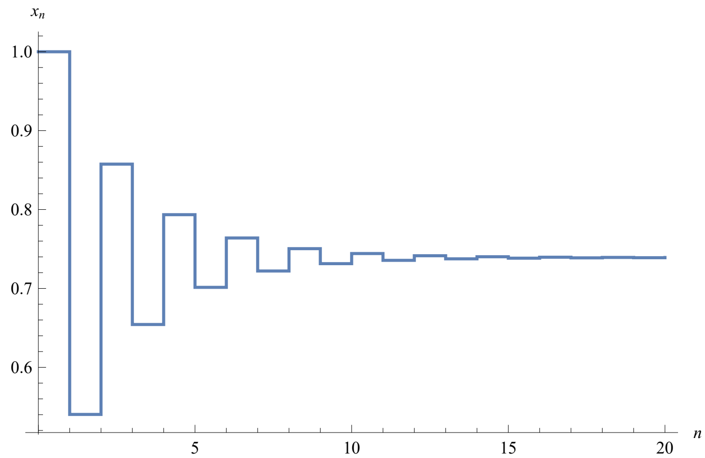
Theorem 2 : let p be a fixed point of $f(x)$, Suppose $f(x)$ is differentiable on $[p - \epsilon, p + \epsilon]$ for some $\epsilon > 0$ and $f(x)$ satisfies the condition $|f'(x)| \leq \alpha < 1$ for all $x \in [p - \epsilon, p + \epsilon]$. Then the sequence x_n defined by (1) , with a starting point $x_0 \in [p - \epsilon, p + \epsilon]$, converges to p .

2. The Equation $\cos x = x$

- The equation $\cos x = x$, $x \in [0, 1]$
- Solution : $x = d = 0.739085133215 \ldots$
- d is the Dottie number (Kaplan 2007)
- d is the unique real root of $\cos x = x$
- Iteration (Fixed Point Method) :

$$x_{n+1} = \cos x_n, \quad x_0 = 1 \Longrightarrow d = \lim_{n \to \infty} x_n \tag{3}$$

- Graphics



- Notation : $d = \cos\cos\cos\ldots\cos 1$
- Integrals

$$d = \int_0^1 \frac{(1 + \sin e^{2\pi i x})\, e^{4\pi i x}}{e^{2\pi i x} - \cos e^{2\pi i x}}\, dx \ , \ i = \sqrt{-1} \tag{4}$$

$$d = \sqrt{1 - \left(1 - \left(\int_{-\infty}^{\infty} \frac{12\pi^2 + 16\,(x - \sinh x)^2}{(3\pi^2 + 4\,(x - \sinh x)^2)^2 + 16\,\pi^2 (x - \sinh x)^2}\, dx\right)^{-1}\right)^2} \tag{5}$$

3. The Equation $x = \frac{1}{2}\sin^{-1}\left(2x\sqrt{1 - x^2}\right)$

- The equation $x = \frac{1}{2}\sin^{-1}\left(2x\sqrt{1 - x^2}\right)$, $x \in [0, 1]$
- Trivial solution : $x = 0$
- Nontrivial solution : $x = d = 0.739085\ldots$, (Dottie number)
- Proof. :

$$\begin{aligned} &\cos d = d \\ \Longrightarrow\ &\sin d = \sqrt{1 - d^2} \\ \Longrightarrow\ &2\cos d \sin d = 2d\sqrt{1 - d^2} \\ \Longrightarrow\ &\sin(2d) = 2d\sqrt{1 - d^2} \\ \Longrightarrow\ &d = \tfrac{1}{2}\sin^{-1}\left(2d\sqrt{1 - d^2}\right) \end{aligned}$$

4. Iteration

- The equation $x = \frac{1}{2}\sin^{-1}\left(2x\sqrt{1 - x^2}\right)$, $0 \le x \le 1$
- Iteration (Fixed Point method) :

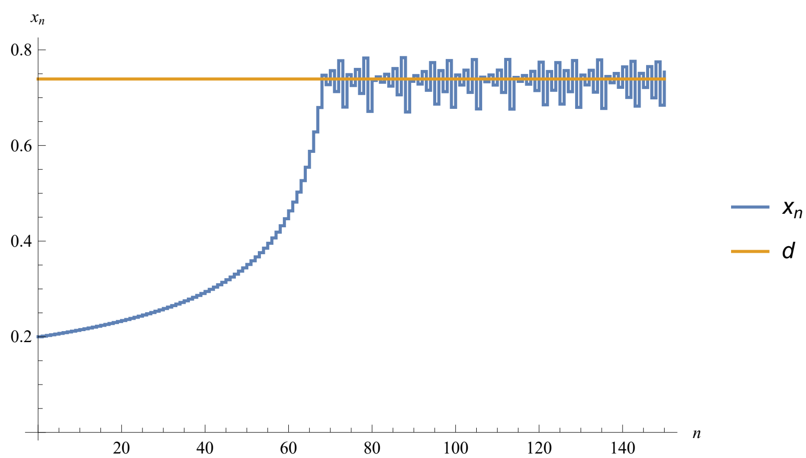
$$x = \frac{1}{2}\sin^{-1}\left(2x\sqrt{1-x^2}\right) = f(x) \quad (6)$$

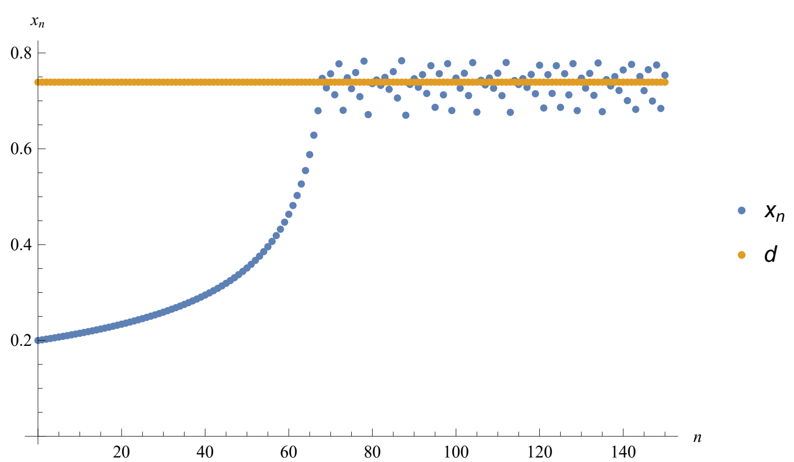
$$x_{n+1} = f(x_n)\,,\ \ 0 \le x_0 \le 1\,,\ \ n = 0,\ 1,\ 2,\ 3,\ \ldots \quad (7)$$

■ Remark : $x_0 = 0 \Longrightarrow x_n = 0$ for all n.

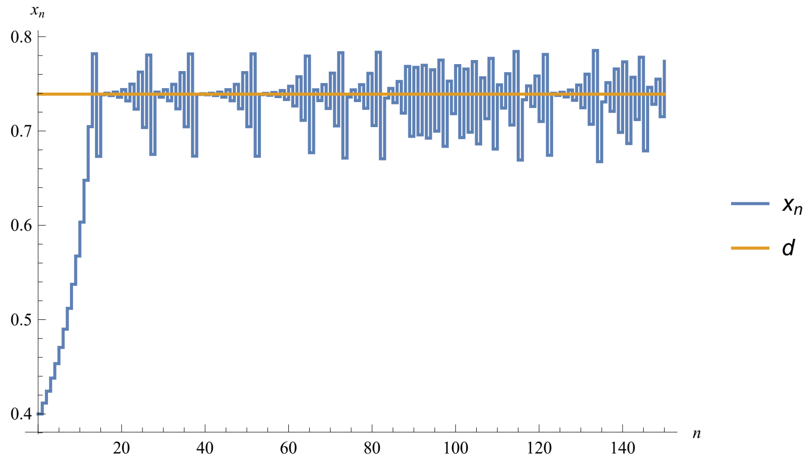
■ Remark : $x_0 = 1 \Longrightarrow x_1 = 0 \Longrightarrow x_n = 0\ \ , n \ge 2$.

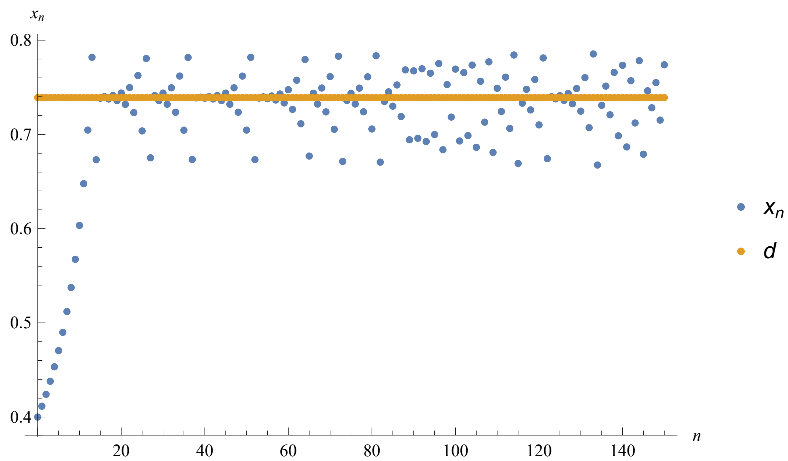
■ Graphics , $x_0 = 0.2$, $d = 0.739085$ ..., Dottie number :



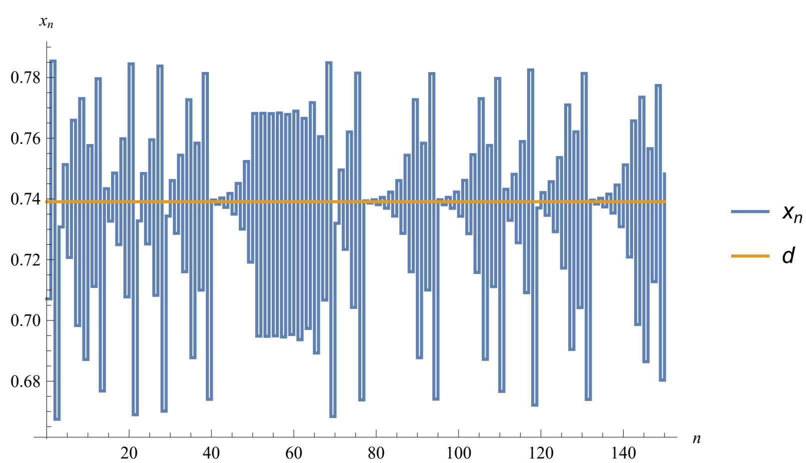


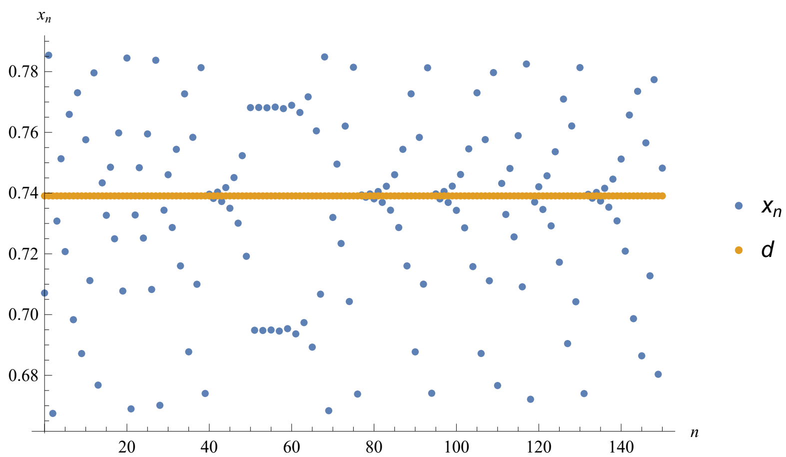
■ Graphics , $x_0 = 0.4$, $d = 0.739085$ ..., Dottie number :



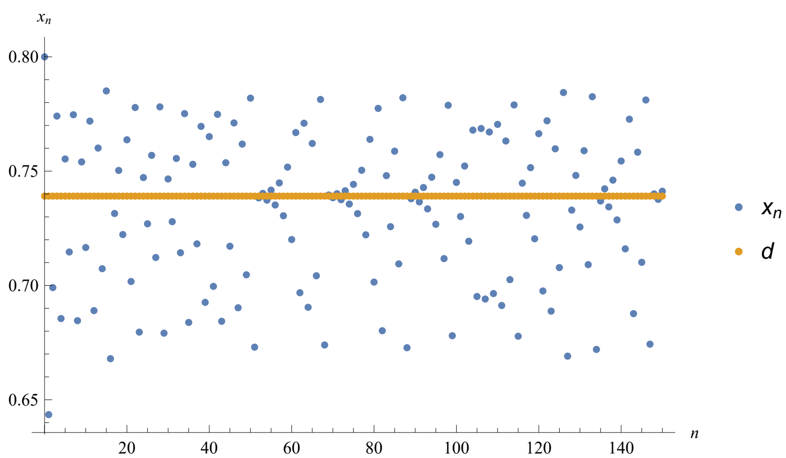


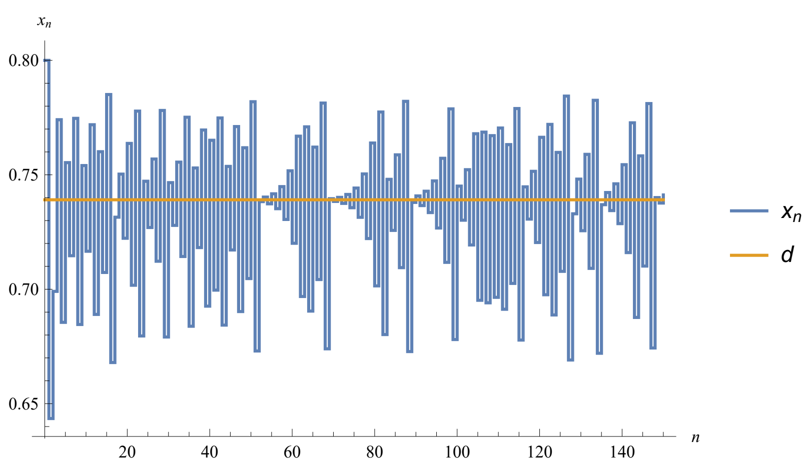
■ Graphics, $x_0 = 1/\sqrt{2} = 0.707106 \ldots$, $d = 0.739085 \ldots$, Dottie number :





■ Graphics , $x_0 = 0.8$, $d = 0.739085$ ..., Dottie number :





■ The function $f(x)$:

$$f([0,\ 1]) \subset [0,\ 1] \tag{8}$$

$$|f'(x)| \geq 1\,,\ \ 0 \leq x \leq 1 \tag{9}$$

■ Let $n \in \mathbb{N} \bigcup \{0\} = \{0, 1, 2, 3, ...\}$, $0 \le u < 1 \Big/ \sqrt{2} < v \le 1$, and assume that

$$u\sqrt{1-u^2} = v\sqrt{1-v^2} \tag{10}$$

$$u_{n+1} = f(u_n), \ u_0 = u, \ n \in \mathbb{N} \bigcup \{0\} \tag{11}$$

$$v_{n+1} = f(v_n), \ v_0 = v, \ n \in \mathbb{N} \bigcup \{0\} \tag{12}$$

then

$$u_n = v_n, \ n \in \mathbb{N} = \{1, 2, 3, ...\} \tag{13}$$

■ Let $0 \le z \le 1$, $u = \sqrt{\dfrac{1-\sqrt{1-z^2}}{2}}$, $v = \sqrt{\dfrac{1+\sqrt{1-z^2}}{2}}$, then $u\sqrt{1-u^2} = v\sqrt{1-v^2}$.

■ Let $n \in \mathbb{N} \bigcup \{0\}$, $x_0 \in [0, 1]$, $x_{n+1} = f(x_n)$, then $0 \le x_n \le \pi/4$ for all $n \in \mathbb{N}$.

■ Conclusion : $0 < x_0 < 1$, $x_0 \ne d$, $x_{n+1} = f(x_n) \Longrightarrow x_n$ is Divergent (oscillation) .

References

A. Kaplan,S.R.: The Dottie Number. Math. Mag. 80,73-74, 2007.

B. Miller, T.H.: On the Numerical values of the Roots of the Equation cos x=x . Proc. Edinburgh Math. Soc. 9, 80-83,1890.

C. Sloane, N.J.A.: Sequence A003957 in the On-Line Encyclopedia of Integer Sequences.

D. Valdebenito, E.: Collected Papers. viXra.org .