Neutrosophic Semi-Continuous Multifunctions

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ABSTRACT

In this paper we introduce the concepts of neutrosophic upper and neutrosophic lower semi-continuous multifunctions and study some of their basic properties.

KEYWORDS: Neutrosophic topological space, semi-continuous multifunctions.

1 INTRODUCTION

There is no doubt that the theory of multifunctions plays an important role in functional analysis and fixed point theory. It also has a wide range of applications in economic theory, decision theory, non-cooperative games, artificial intelligence, medicine and information sciences. Inspired by the research works of Smarandache (1999; 2001; 2007), we introduce and study the notions of neutrosophic upper and neutrosophic lower semi-continuous multifunctions in this paper. Further, we present some characterizations and properties of such notions.

2 PRELIMINARIES

Throughout this paper, by (X, τ) or simply by X we will mean a topological space in the classical sense, and (Y, τ_1) or simply Y will stand for a neutrosophic topological space as defined by Salama and Alblowi (2012).

Definition 1. *Smarandache (1999, 2001, 2007) Let X be a non-empty fixed set. A neutrosophic set A is an object having the form $A =< x, \mu_A(x), \sigma_A(x), \gamma_A(x) >$, where $\mu_A(x)$, $\sigma_A(x)$ and $\gamma_A(x)$ are represent the degree of member ship function, the degree of indeterminacy, and the degree of non-membership, respectively of each element $x \in X$ to the set A.*

Definition 2. *(Salama & Alblowi, 2012) A neutrosophic topology on a nonempty set X is a family τ of neutrosophic subsets of X which satisfies the following three conditions:*

1. $0, 1 \in \tau$,
2. *If $g, h \in \tau$, their $g \wedge h \in \tau$,*
3. *If $f_i \in \tau$ for each $i \in I$, then $\vee_{i \in I} f_i \in \tau$.*

The pair (X, τ) is called a neutrosophic topological space.

Definition 3. *Members of τ are called neutrosophic open sets, denoted by $NO(X)$, and complement of neutrosophic open sets are called neutrosophic closed sets, where the complement of a neutrosophic set A, denoted by A^c, is $1 - A$.*

Neutrosophic sets in Y will be denoted by $\lambda, \gamma, \delta, \rho$, etc., and although subsets of X will be denoted by A, B, U, V, etc. A neutrosophic point in Y with support $y \in Y$ and value $\alpha (0 < \alpha \leq 1)$ is denoted by y_α. A neutrosophic set λ in Y is said to be quasi-coincident (q-coincident) with a neutrosophic set μ, denoted by $\lambda q \mu$, if and only if there exists $y \in Y$ such that $\lambda(y) + \mu(y) > 1$. A neutrosophic set λ of Y is called a neutrosophic neighbourhood of a fuzzy point y_α in Y if there exists a neutrosophic open set μ in Y such that $y_\alpha \in \mu \leq \lambda$. The intersection of all neutrosophic closed sets of Y containing λ is called the neutrosophic closure of λ and is denoted by $\mathrm{Cl}(\lambda)$. The union of all neutrosophic open sets contained in λ is called the neutrosophic interior of λ and is denoted by $\mathrm{Int}(\lambda)$. The family of all open sets of a topological space X is denoted by $O(X)$ and $O(X, x)$ denoted the family $\{A \in O(X) | x \in A\}$, where x is a point of X.

Definition 4. *Let (X, τ) be a topological space in the classical sense and (Y, τ_1) be an neutrosophic topological space. $F : (X, \tau) \to (Y, \tau_1)$ is called a neutrosophic multifunction if and only if for each $x \in X, F(x)$ is a neutrosophic set in Y.*

Definition 5. *For a neutrosophic multifunction $F : (X, \tau) \to (Y, \tau_1)$, the upper inverse $F^+(\lambda)$ and lower inverse $F^-(\lambda)$ of a neutrosophic set λ in Y are defined as follows:*
$F^+(\lambda) = \{x \in X | F(x) \leq \lambda\}$ *and* $F^-(\lambda) = \{x \in X | F(x) q \lambda\}$.

Lemma 1. *For a neutrosophic multifunction $F : (X, \tau) \to (Y, \tau_1)$, we have $F^-(1 - \lambda) = X - F^+(\lambda)$, for any neutrosophic set λ in Y.*

3 NEUTROSOPHIC SEMICONTINUOUS MULTI–FUNCTIONS

Definition 6. *A neutrosophic multifunction $F : (X, \tau) \to (Y, \tau_1)$ is said to be*

1. *neutrosophic upper semicontinuous at a point $x \in X$ if for each $\lambda \in NO(Y)$ containing $F(x)$ (therefore, $F(x) \leq \lambda$), there exists $U \in O(X, x)$ such that $F(U) \leq \lambda$ (therefore $U \subset F^+(\lambda)$).*

2. *neutrosophic lower semicontinuous at a point $x \in X$ if for each $\lambda \in NO(Y)$ with $F(x) q \lambda$, there exists $U \in O(X, x)$ such that $U \subseteq F^-(\lambda)$.*

3. *neutrosophic upper semicontinuous (neutrosophic lower semicontinuous) if it is neutrosophic upper semicontinuous (neutrosophic lower semicontinuous) at each point $x \in X$.*

Theorem 1. *The following assertions are equivalent for a neutrosophic multifunction $F : (X, \tau) \to (Y, \tau_1)$:*

1. *F is neutrosophic upper semicontinuous;*

2. *For each point x of X and each neutrosophic neighbourhood λ of $F(x)$, $F^+(\lambda)$ is a neighbourhood of x;*

3. *For each point x of X and each neutrosophic neighbourhood λ of $F(x)$, there exists a neighbourhood U of x such that $F(U) \leq \lambda$;*

4. *$F^+(\lambda) \in O(X)$ for oeach $\lambda \in NO(Y)$;*

5. *$F^-(\delta)$ is a closed set in X for each neutrosophic closed set δ of Y;*

6. *$\mathrm{Cl}(F^-(\mu)) \subseteq F^-(\mathrm{Cl}(\mu))$ for each neutrosophic set μ of Y.*

Proof. (1)⇒(2) Let $x \in X$ and μ be a neutrosophic neighbourhood of $F(x)$. Then there exists $\lambda \in NO(Y)$ such that $F(x) \leq \lambda \leq \mu$, By (1), there exists $U \in O(X, x)$ such that $F(U) \leq \lambda$. Therefore $x \in U \subseteq F^+(\mu)$ and hence $F^+(\mu)$ is a neighbourhood of x.
(2)⇒(3) Let $x \in X$ and λ be a neutrosophic neighbourhood of $F(x)$. Put $U = F^+(\lambda)$. Then

by (2), U is neighbourhood of x and $F(U) = \bigvee_{x \in U} F(x) \leq \lambda$.
(3)⇒(4) Let $\lambda \in NO(Y)$, we want to show that $F^+(\lambda) \in O(X)$. So let $x \in F^+(\lambda)$. Then there exists a neighbourhood G of x such that $F(G) \leq \lambda$. Therefore for some $U \in O(X, x), U \subseteq G$ and $F(U) \leq \lambda$. Therefore we get $x \in U \subseteq F^+(\lambda)$ and hence $F^+(\lambda) \in O(X)$.
(4)⇒(5) Let δ be a neutrosophic closed set in Y. So, we have $X \backslash F^-(\delta) = F^+(1-\delta) \in O(X)$ and hence $F^-(\delta)$ is closed set in X.
(5)⇒(6) Let μ be any neutrosophic set in Y. Since $\mathrm{Cl}(\mu)$ is neutrosophic closed set in Y, $F^-(\mathrm{Cl}(\mu))$ is closed set in X and $F^-(\mu) \subseteq F^-(\mathrm{Cl}(\mu))$. Therefore, we obtain $\mathrm{Cl}(F^-(\mu)) \subseteq F^-(\mathrm{Cl}(\mu))$.
(6)⇒(1) Let $x \in X$ and $\lambda \in NO(Y)$ with $F(x) \leq \lambda$. Now $F^-(1-\lambda) = \{x \in X | F(x) q (1-\lambda)\}$. So, for x not belongs to $F^-(1-\lambda)$. Then, we must have $F(x) \hbar (1-\lambda)$ and this implies $F(x) \leq 1-(1-\lambda) = \lambda$ which is true. Therefore $x \notin F^-(1-\lambda)$ by (6), $x \notin \mathrm{Cl}(F^-(1-\lambda))$ and there exists $U \in O(X, x)$ such that $U \cap F^-(1-\lambda) = \emptyset$. Therefore, we obtain $F(U) = \bigvee_{x \in U} F(x) \leq \lambda$. This proves F is neutrosophic upper semicontinuous. □

Theorem 2. *The following statements are equivalent for a neutrosophic multifunction $F : (X, \tau) \to (Y, \tau_1)$:*

1. *F is neutrosophic lower semicontinuous;*
2. *For each $\lambda \in NO(Y)$ and each $x \in F^-(\lambda)$, there exists $U \in O(X, x)$ such that $U \subseteq F^-(\lambda)$;*
3. *$F^-(\lambda) \in O(X)$ for every $\lambda \in NO(Y)$.*
4. *$F^+(\delta)$ is a closed set in X for every neutrosophic closed set δ of Y;*
5. *$\mathrm{Cl}(F^+(\mu)) \subseteq F^+(\mathrm{Cl}(\mu))$ for every neutrosophic set μ of Y;*
6. *$F(\mathrm{Cl}(A)) \leq \mathrm{Cl}(F(A))$ for every subset A of X;*

Proof. (1)⇒(2) Let $\lambda \in NO(Y)$ and $x \in F^-(\lambda)$ with $F(x) q \lambda$. Then by properties–1, there exists $U \in O(X, x)$ such that $U \subseteq F^-(\lambda)$.
(2)⇒(3) Let $\lambda \in NO(Y)$ adn $x \in F^-(\lambda)$. Then by (2), there exists $U \in O(X, x)$ such that $U \subseteq F^-(\lambda)$. Therefore, we have $x \in U \subseteq \mathrm{Cl}\,\mathrm{Int}(U) \subseteq \mathrm{Cl}\,\mathrm{Int}(F^-(\lambda))$ and hence $F^-(\lambda) \in O(X)$.
(3)⇒(4) Let δ be a neutrosophic closed in Y. So we have $X \backslash F^+(\delta) = F^-(1-\delta) \in O(X)$ and hence $F^+(\delta)$ is closed set in X.
(4)⇒(5) Let μ be any neutrosophic set in Y. Since $\mathrm{Cl}(\mu)$ is neutrosophic closed set in Y, then by (4), we have $F^+(\mathrm{Cl}(\mu))$ is closed set in X and $F^+(\mu) \subseteq F^+(\mathrm{Cl}(\mu))$. Therefore, we obtain $\mathrm{Cl}(F^+(\mu)) \subseteq F^+(\mathrm{Cl}(\mu))$.
(5)⇒(6) Let A be any subset of X. By (5), $\mathrm{Cl}(A) \subseteq \mathrm{Cl}\, F^+(F(A)) \subseteq F^+(\mathrm{Cl}(F(A)))$.

Therefore we obtain $\mathrm{Cl}(A) \subseteq F^+(\mathrm{Cl}\,F(A))$. This implies that $F(\mathrm{Cl}(A)) \leq \mathrm{Cl}\,F(A)$.
(6)⇒(5) Let μ be any neutrosophic set in Y. By (6), $F(\mathrm{Cl}\,F^+(\mu)) \leq \mathrm{Cl}(F(F^+(\mu)))$ and hence $\mathrm{Cl}(F^+(\mu)) \subseteq F^+(\mathrm{Cl}(F(F^+(\mu)))) \subseteq F^+(\mathrm{Cl}(\mu))$. Therefore $\mathrm{Cl}(F^+(\mu)) \subseteq F^+(\mathrm{Cl}(\mu))$.
(5)⇒(1) Let $x \in X$ and $\lambda \in NO(Y)$ with $F(x)q\lambda$. Now, $F^+(1-\lambda) = \{x \in X | F(x) \leq 1-\lambda\}$. So, for x not belongs to $F^+(1-\lambda)$, then we have $F(x) \nleq 1-\lambda$ and this implies that $F(x)q\lambda$. Therefore, $x \notin F^+(1-\lambda)$. Since $1-\lambda$ is neutrosophic closed set in Y, by (5), $x \notin \mathrm{Cl}(F^+(1-\lambda))$ and there exists $U \in O(X,x)$ such that $\emptyset = U \cap F^+(1-\lambda) = U \cap (X \backslash F^-(\lambda))$. Therefore, we obtain $U \subseteq F^-(\lambda)$. This proves F is neutrosophic lower semicontinuous. □

Definition 7. *For a given neutrosophic multifunction $F : (X,\tau) \to (Y,\tau_1)$, a neutrosophic multifunction $\mathrm{Cl}(F) : (X,\tau) \to (Y,\tau_1)$ is defined as $(\mathrm{Cl}\,F)(x) = \mathrm{Cl}\,F(x)$ for each $x \in X$.*

We use $\mathrm{Cl}\,F$ and the following Lemma to obtain a characterization of lower neutrosophic semicontinuous multifunction.

Lemma 2. *If $F : (X,\tau) \to (Y,\tau_1)$ is a neutrosophic multifunction, then $(\mathrm{Cl}\,F)^-(\lambda) = F^-(\lambda)$ for each $\lambda \in NO(Y)$.*

Proof. Let $\lambda \in NO(Y)$ and $x \in (\mathrm{Cl}\,F)^-(\lambda)$. This means that $(\mathrm{Cl}\,F)(x)q\lambda$. Since $\lambda \in NO(Y)$, we have $F(x)q\lambda$ and hence $x \in F^-(\lambda)$. Therefore $(\mathrm{Cl}\,F)^-(\lambda) \subseteq F^-(\lambda) - - - (*)$.

Conversely, let $x \in F^-(\lambda)$ since $\lambda \in NO(Y)$ then $F(x)q\lambda \subseteq (\mathrm{Cl}\,F)(x)q\lambda$ and hence $x \in (\mathrm{Cl}\,F)^-(\lambda)$. Therefore $F^-(\lambda) \subseteq (\mathrm{Cl}\,F)^-(\lambda) - - - - (**)$.
From $(*)$ and $(**)$, we get $(\mathrm{Cl}\,F)^-(\lambda) = F^-(\lambda)$. □

Theorem 3. *A neutrosophic multifunction $F : (X,\tau) \to (Y,\tau_1)$ is neutrosophic lower semicontinuous if and only if $\mathrm{Cl}\,F : (X,\tau) \to (Y,\tau_1)$ is neutrosophic lower semicontinuous.*

Proof. Suppose F is neutrosophic lower semicontinuous. Let $\lambda \in NO(Y)$ and $F(x)q\lambda$. This means that $x \in F^-(\lambda)$. Then there exists $U \in O(X,x)$ such that $U \subseteq F^-(\lambda)$. Therefore, we have $x \in U \subseteq \mathrm{Int}(U) \subseteq \mathrm{Int}\,F^-(\lambda)$ and hence $F^-(\lambda) \in O(X)$. Then by Lemma 2, we have $U \subseteq F^-(\lambda) = (\mathrm{Cl}\,F)^-(\lambda)$ and $(\mathrm{Cl}\,F)^-(\lambda) \in O(X)$, and hence $(\mathrm{Cl}\,F)(x)q\lambda$. Therefore $\mathrm{Cl}\,F$ is fuzy lower semicontinuous. Conversely, suppose $\mathrm{Cl}\,F$ is neutrosophic lower semicontinuous. If for each $\lambda \in NO(Y)$ with $(\mathrm{Cl}\,F)(x)q\lambda$ and $x \in (\mathrm{Cl}\,F)^-(\lambda)$ then there exists $U \in O(X,x)$ such that $U \subseteq (\mathrm{Cl}\,F)^-(\lambda)$. By Lemma 2 and Theorem 2, we have $U \subseteq (\mathrm{Cl}\,F^-(\lambda)) = F^-(\lambda)$ and $F^-(\lambda) \in O(X)$. Therefore F is neutrosophic lower semicontinuous. □

Definition 8. *Given a family $\{F_i : (X,\tau) \to (Y,\sigma) : i \in I\}$ of neutrosophic multifunctions, we define the union $\underset{i\in I}{\vee} F_i$ and the intersection $\underset{i\in I}{\wedge} F_i$ as follows: $\underset{i\in I}{\vee} F_i : (X,\tau) \to (Y,\sigma)$, $(\underset{i\in I}{\vee} F_i)(x) = \underset{i\in I}{\vee} F_i(x)$ and $\underset{i\in I}{\wedge} F_i : (X,\tau) \to (Y,\sigma)$, $(\underset{i\in I}{\wedge} F_i)(x) = \underset{i\in I}{\wedge} F_i(x)$.*

Theorem 4. *If $F_i : X \to Y$ are neutrosophic upper semi-continuous multifunctions for $i = 1,2,...,n$, then $\overset{n}{\underset{i\in I}{\vee}} F_i$ is a neutrosophic upper semi-continuous multifunction.*

Proof. Let A be a neutrosophic open set of Y. We will show that $(\overset{n}{\underset{i\in I}{\vee}} F_i)^+(A) = \{x \in X : \overset{n}{\underset{i\in I}{\vee}} F_i(x) \subset A\}$ is open in X. Let $x \in (\overset{n}{\underset{i\in I}{\vee}} F_i)^+(A)$. Then $F_i(x) \subset A$ for $i = 1, 2, ..., n$. Since $F_i : X \to Y$ is neutrosophic upper semi-continuous multifunction for $i = 1, 2, ..., n$, then there exists an open set U_x containing x such that for all $z \in U_x$, $F_i(z) \subset A$. Let $U = \overset{n}{\underset{i\in I}{\cup}} U_x$. Then $U \subset (\overset{n}{\underset{i\in I}{\vee}} F_i)^+(A)$. Thus, $(\overset{n}{\underset{i\in I}{\vee}} F_i)^+(A)$ is open and hence $\overset{n}{\underset{i\in I}{\vee}} F_i$ is a neutrosophic upper semi-continuous multifunction. □

Lemma 3. *Let $\{A_i\}_{i\in I}$ be a family of neutrosophic sets in a neutrosophic topological space X. Then a neutrosophic point x is quasi-coincident with $\vee A_i$ if and only if there exists an $i_0 \in I$ such that xqA_{i_0}.*

Theorem 5. *If $F_i : X \to Y$ are neutrosophic lower semi-continuous multifunctions for $i = 1, 2, ..., n$, then $\overset{n}{\underset{i\in I}{\vee}} F_i$ is a neutrosophic lower semi-continuous multifunction.*

Proof. Let A be a neutrosophic open set of Y. We will show that $(\overset{n}{\underset{i\in I}{\vee}} F_i)^-(A) = \{x \in X : (\overset{n}{\underset{i\in I}{\vee}} F_i)(x)qA\}$ is open in X. Let $x \in (\overset{n}{\underset{i\in I}{\vee}} F_i)^-(A)$. Then $(\overset{n}{\underset{i\in I}{\vee}} F_i)(x)qA$ and hence $F_{i0}(x)qA$ for an i_0. Since $F_i : X \to Y$ is neutrosophic lower semi-continuous multifunction, there exists an open set U_x containing x such that for all $z \in U$, $F_{i0}(z)qA$. Then $(\overset{n}{\underset{i\in I}{\vee}} F_i)(z)qA$ and hence $U \subset (\overset{n}{\underset{i\in I}{\vee}} F_i)^-(A)$. Thus, $(\overset{n}{\underset{i\in I}{\vee}} F_i)^-(A)$ is open and hence $\overset{n}{\underset{i\in I}{\vee}} F_i$ is a neutrosophic lower semi-continuous multifunction. □

Theorem 6. *Let $F : (X, \tau) \to (Y, \sigma)$ be a neutrosophic multifunction and $\{U_i : i \in I\}$ be an open cover for X. Then the following are equivalent:*

1. *$F_i = F_{|U_i}$ is a neutrosophic lower semi-continuous multifunction for all $i \in I$,*
2. *F is neutrosophic lower semi-continuous.*

Proof. (1) $\Rightarrow$ (2): Let $x \in X$ and A be a neutrosophic open set in Y with $x \in F^-(A)$. Since $\{U_i : i \in I\}$ is an open cover for X, then $x \in U_{i0}$ for an $i_0 \in I$. We have $F(x) = F_{i0}(x)$ and hence $x \in F^-_{i0}(A)$. Since $F_{|U_i0}$ is neutrosophic lower semi-continuous, there exists an open set $B = G \cap U_{i0}$ in U_{i0} such that $x \in B$ and $F^-(A) \cap U_{i_0} = F_{|U_i}(A) \supset B = G \cap U_{i0}$, where G is open in X. We have $x \in B = G \cap U_{i0} \subset F^-_{|U_i0}(A) = F^-(A) \cap U_{i0} \subset F^-(A)$. Hence, F is neutrosophic lower semi-continuous.
(2) $\Rightarrow$ (1): Let $x \in X$ and $x \in U_i$. Let A be a neutrosophic open set in Y with $F_i(x)qA$. Since F is lower semi-continuous and $F(x) = F_i(x)$, there exists an open set U containing x such that $U \subset F^-(A)$. Take $B = U_i \cap U$. Then B is open in U_i containing x. We have $B \subset F^-i(A)$. Thus F_i is a neutrosophic lower semi-continuous. □

Theorem 7. *Let $F : (X, \tau) \to (Y, \sigma)$ be a neutrosophic multifunction and $\{U_i : i \in I\}$ be an open cover for X. Then the following are equivalent:*

1. *$F_i = F_{|U_i}$ is a neutrosophic upper semi-continuous multifunction for all $i \in I$,*

2. *F is neutrosophic upper semi-continuous.*

Proof. It is similar to that of Theorem 6. □

Remark 8. *A subset A of a topological space (X, τ) can be considered as a neutrosophic set with characteristic function defined by*

$$A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Let (Y, σ) be a neutrosophic topological space. The neutrosophic sets of the form $A \times B$ with $A \in \tau$ and $B \in \sigma$ form a basis for the product neutrosophic topology $\tau \times \sigma$ on $X \times Y$, where for any $(x, y) \in X \times Y$, $(A \times B)(x, y) = min\{A(x), B(y)\}$.

Definition 9. *For a neutrosophic multifunction $F : (X, \tau) \to (Y, \sigma)$, the neutrosophic graph multifunction $G_F : X \to X \times Y$ of F is defined by $G_F(x) = x_1 \times F(x)$ for every $x \in X$.*

Theorem 9. *If the neutrosophic graph multifunction G_F of a neutrosophic multifunction $F : (X, \tau) \to (Y, \sigma)$ is neutrosophic lower semi-continuous, then F is neutrosophic lower semi-continuous.*

Proof. Suppose that G_F is neutrosophic lower semi-continuous and $x \in X$. Let A be a neutrosophic open set in Y such that $F(x)qA$. Then there exists $y \in Y$ such that $(F(x))(y) + A(y) > 1$. Then $(G_F(x))(x, y) + (X \times A)(x, y) = (F(x))(y) + A(y) > 1$. Hence, $G_F(x)q(X \times A)$. Since G_F is neutrosophic lower semi-continuous, there exists an open set B in X such that $x \in B$ and $G_F(b)q(X \times A)$ for all $b \in B$. Let there exists $b_0 \in B$ such that $F(b_0)qA$. Then for all $y \in Y$, $(F(b_0))(y) + A(y) < 1$. For any $(a, c) \in X \times Y$, we have $(G_F(b_0))(a, c) \subset (F(b_0))(c)$ and $(X \times A)(a, c) \subset A(c)$. Since for all $y \in Y$, $(F(b_0))(y) + A(y) < 1$, $(G_F(b_0))(a, c) + (X \times A)(a, c) < 1$. Thus, $G_F(b_0)q(X \times A)$, where $b_0 \in B$. This is a contradiction since $G_F(b)q(X \times A)$ for all $b \in B$. Hence, F is neutrosophic lower semi-continuous. □

Theorem 10. *If the neutrosophic graph multifunction G_F of a neutrosophic multifunction $F : X \to Y$ is neutrosophic upper semi-continuous, then F is neutrosophic upper semi-continuous.*

Proof. Suppose that G_F is neutrosophic upper semi-continuous and let $x \in X$. Let A be neutrosophic open in Y with $F(x) \subset A$. Then $G_F(x) \subset X \times A$. Since G_F is neutrosophic upper semi-continuous, there exists an open set B containing x such that $G_F(B) \subset X \times A$. For any $b \in B$ and $y \in Y$, we have $(F(b))(y) = (G_F(b))(b, y) \subset (X \times A)(b, y) = A(y)$. Then $(F(b))(y) \subset A(y)$ for all $y \in Y$. Thus, $F(b) \subset A$ for any $b \in B$. Hence, F is neutrosophic upper semi-continuous. □

Theorem 11. *Let $F : (X, \tau) \to (Y, \sigma)$ be a neutrosophic multifunction. Then the following are equivalent:*

1. *F is neutrosophic lower semi-continuous,*
2. *For any $x \in X$ and any net $(x_i)_{i\in I}$ converging to x in X and each neutrosophic open set B in Y with $x \in F^-(B)$, the net $(x_i)_{i\in I}$ is eventually in $F^-(B)$.*

Proof. (1) $\Rightarrow$ (2): Let (x_i) be a net converging to x in X and B be any neutrosophic open set in Y with $x \in F^-(B)$. Since F is neutrosophic lower semi-continuous, there exists an open set $A \subset X$ containing x such that $A \subset F^-(B)$. Since $x_i \to x$, there exists an index $i_0 \in I$ such that $x_i \in A$ for every $i \geq i_0$. We have $x_i \in A \subset F^-(B)$ for all $i \geq i_0$. Hence, $(x_i)_{i\in I}$ is eventually in $F^-(B)$.
(2) $\Rightarrow$ (1): Suppose that F is not neutrosophic lower semi-continuous. There exists a point x and a neutrosophic open set A with $x \in F^-(A)$ such that $B \not\subseteq F^-(A)$ for any open set $B \subset X$ containing x. Let $x_i \in B$ and $x_i \notin F^-(A)$ for each open set $B \subset X$ containing x. Then the neighborhood net (x_i) converges to x but $(x_i)_{i\in I}$ is not eventually in $F^-(A)$. This is a contradiction. □

Theorem 12. *Let $F : (X, \tau) \to (Y, \sigma)$ be a neutrosophic multifunction. Then the following are equivalent:*

1. *F is neutrosophic upper semi-continuous,*
2. *For any $x \in X$ and any net (x_i) converging to x in X and any neutrosophic open set B in Y with $x \in F^+(B)$, the net (x_i) is eventually in $F^+(B)$.*

Proof. The proof is similar to that of Theorem 11. □

Theorem 13. *The set of all points of X at which a neutrosophic multifunction $F : (X, \tau) \to (Y, \sigma)$ is not neutrosophic upper semi-continuous is identical with the union of the frontier of the upper inverse image of neutrosophic open sets containing $F(x)$.*

Proof. Suppose F is not neutrosophic upper semi-continuous at $x \in X$. Then there exists a neutrosophic open set A in Y containing $F(x)$ such that $A \cap (X \backslash F^+(B)) \neq \emptyset$ for every open set A containing x. We have $x \in \mathrm{Cl}(X \backslash F^+(B)) = X \backslash \mathrm{Int}(F^+(B))$ and $x \in F^+(B)$. Thus, $x \in Fr(F^+(B))$. Conversely, let B be a neutrosophic open set in Y containing $F(x)$ with $x \in Fr(F^+(B))$. Suppose that F is neutrosophic upper semi-continuous at x. There exists an open set A containing x such that $A \subset F^+(B)$. We have $x \in \mathrm{Int}(F^+(B))$. This is a contradiction. Thus, F is not neutrosophic upper semi-continuous at x. □

Theorem 14. *The set of all points of X at which a neutrosophic multifunction $F : (X, \tau) \to (Y, \sigma)$ is not neutrosophic lower semi-continuous is identical with the union of the frontier of the lower inverse image of neutrosophic closed sets which are quasi-coincident with $F(x)$.*

Proof. It is similar to that of Theorem 13. □

Definition 10. *A neutrosophic set λ of a neutrosophic topological space Y is said to be neutrosophic compact relative to Y if every cover $\{\lambda_\alpha\}_{\alpha\in\Delta}$ of λ by neutrosophic open sets of Y has a finite subcover $\{\lambda_i\}_{i=1}^n$ of λ.*

Definition 11. *A neutrosophic set λ of a neutrosophic topological space Y is said to be neutrosophic Lindelof relative to Y if every cover $\{\lambda_\alpha\}_{\alpha\in\Delta}$ of λ by neutrosophic open sets of Y has a countable subcover $\{\lambda_n\}_{n\in N}$ of λ.*

Definition 12. *A neutrosophic topological space Y is said to be neutrosophic compact if χ_Y (characteristic function of Y) is neutrosophic compact relative to Y.*

Definition 13. *A neutrosophic topological space Y is said to be neutrosophic Lindelof if χ_Y (characteristic function of Y) is neutrosophic Lindelof relative to Y.*

Definition 14. *A neutrosophic multifunction $F : (X, \tau) \to (Y, \tau_1)$ is said to be punctually neutrosophic compact (resp. punctually neutrosophic Lindelof) if for each $x \in X, F(x)$ is neutrosophic compact (resp. neutrosophic Lindelof).*

Theorem 15. *Let the neutrosophic multifunction $F : (X, \tau) \to (Y, \tau_1)$ be a neutrosophic upper semicontinuous and F is punctually neutrosophic compact. If A is compact relative to X, then $F(A)$ is neutrosophic compact relative to Y.*

Proof. Let $\{\lambda_\alpha|\alpha \in \Delta\}$ be any cover of $F(Z)$ by neutrosophic copen sets of Y. We claim that $F(A)$ is neutrosophic compact relative to Y. For each $x \in A$, there exists a finite subset $\Delta(x)$ of Δ such that $F(x) \leq \cup\{\lambda_\alpha|\alpha \in \Delta(x)\}$. Put $\lambda(x) = \cup\{\lambda_\alpha|\alpha \in \Delta(x)\}$. Then $F(x) \leq \lambda(x) \in NO(Y)$ and there exists $U(x) \in O(X, x)$ such that $F(U(x)) \leq \lambda(x)$. Since $\{U(x)|x \in A\}$ is an open cover of A there exists a finite number of A, say, $x_1, x_2, .., x_n$ such that $A \subseteq \cup\{U(x_i)|i = 1, 2, .., n\}$. Therefore we obtain $F(A) \leq F(\overset{n}{\underset{i=1}{\cup}} U(x_i)) \leq \overset{n}{\underset{i=1}{\cup}} F(U(x_i)) \leq \overset{n}{\underset{i=1}{\cup}} \lambda(x_i) \leq \overset{n}{\underset{i=1}{\cup}} (\underset{\alpha\in\Delta(x_i)}{\cup} \lambda_\alpha)$. This shows that $F(A)$ is neutrosophic compact relative to Y. □

Theorem 16. *Let the neutrosophic multifunction $F : (X, \tau) \to (Y, \tau_1)$ be a neutrosophic upper semicontinuous and F is punctually neutrosophic Lindelof. If A is Lindelof relative to X, then $F(A)$ is neutrosophic Lindelof relative to Y.*

Proof. The proof is similar to that of Theorem 15 □

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