A boundary value problem of the partial differential-integral equations and their applications

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Abstract

We study the boundary value problem of a partial differential-integral equations that have many applications in finance and insurance. We will solve a boundary value problem of the partial differential-integral equations by using the solution of conjugate equation and reflection method and apply it to determine the probability of company bankruptcy in insurance mathematics.

Keyword: partial differential - integral equation, boundary value problem

Introduction

The partial differential - integral equations is now more in the application of financial and insurance. In many papers have considered the boundary value problem of the differential equations,however,a few papers have considered the boundary value problem of the partial differential-integral equations.

In [1~5], authors considered the unique existence of the partial differential-integral equations by using viscosity analysis and in [5], author considered the maximum value problem of the partial differential-integral equations.

Firstly, we study the boundary value problem of a partial differential-integral equations and secondly, apply its result in insurance mathematics.

1. The boundary value problem of a partial differential-integral equations

Let consider the following boundary value problem of a partial differential-integral equations:

$$\left(\frac{\partial}{\partial s}+L\right)u(s,x)=0, \quad (s,x)\in[0,T]\times R_+^2$$
$$u(T,x)=1, \quad x\in R_+^2, \quad u\left(s,(x_1,0)'\right)=0, \quad s\in[0,T], \quad x_1\in R_+^1 \tag{1}$$

$$u\left(s,\left(0,x_2\right)'\right)=0,\qquad s\in\left[0,T\right],\qquad x_1\in R_+^1, \tag{2}$$

where for $f(x)\in C^2\left(R_+^2\right)$ operator L define as following:

$$Lf(x)=\sum_{i=1}^{2}a_i\cdot\frac{\partial f}{\partial x_i}+\frac{1}{\tau}\sum_{i=1}^{2}\sum_{j=1}^{2}a_{ij}\frac{\partial^2 f(x)}{\partial x_i\partial x_j}+a_0 f(x)$$
$$+\sum_{k=1}^{d}\int_{R^2}\left[f\left(x+c_k(z)\right)-f(x)-\sum_{i=1}^{2}c_{ik}(z)\frac{\partial f}{\partial x_i}I_{\{|z|\le 1\}}(z)\right]\nu_k(dz).$$

Assumption:

(H1) $a_i,\ \ i=0,1,2$ are constant and $c_{ik}(z),\ \ i=1,2,\ \ k=\overline{1,d}$ are satisfyed

$$\int_{R^2}\left|c_{ik}(z)\right|\nu_k(dz)<\infty.$$

(H2) $\forall\theta\in R^2\ (\theta\neq 0)$, $\exists\,\lambda>0$; $\sum_{i=1}^{2}\sum_{j=1}^{2}a_{ij}\theta_i\theta_j\ge\lambda|\theta|^2$.

(H2) $\nu(\cdot)$ is the finit measure on B^2, $\nu(\{0\})=0$ and can express

$$\lambda_k(dz)=\lambda_k p_k(z)dz,\qquad z\in R^2 \tag{3}$$

,where $p_k(z)$ is the hyper-density function on R^2. Namely,

$$p_k(z)\ge 0,\qquad \int_{R^2}p_k(z)dz=1.$$

Then, the operator L can rewrite as following:

$$\left(\overline{L}f\right)(x)=\sum_{i=1}^{2}\widetilde{a}_i\frac{\partial f(x)}{\partial x}+\frac{1}{\tau}\sum_{i=1}^{2}\sum_{j=1}^{2}a_{ij}\frac{\partial^2 f(x)}{\partial x_i\partial x_j}+\widetilde{a}_0 f(x)+\int_R f(z)\cdot\sum_{k=1}^{d}\lambda_k\widetilde{p}_k(z-x)dz \tag{4}$$

$$\left.\begin{aligned} \tilde{a}_i &= a_i - \sum_{k=1}^{d} \lambda_k \int_R c_{ik}(z)\, p_k(z)\, dz \\ \tilde{a}_0 &= a_0 - \sum_{k=1}^{d} \lambda_k \\ \tilde{p}_k(z-x) &= p_k\left(c_k^{-1}(z-x)\right) \frac{d}{dz} c_k^{-1}(z-x) \end{aligned}\right\}. \tag{5}$$

From assumption (H2), we can rewrite the matrix (a_{ij}) into the diagonal-matrix $\sigma = (\sigma_i^2)$ by using appropriate variable transformation.

Comfortably, will express $\tilde{a}_i$ into a_i.

Let's consider following two operators;

$$A_s = \frac{\partial}{\partial t} + \sum a_i \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{2} \sigma_i^2 \frac{\partial^2}{\partial x_i^2} + a_0 , \tag{6}$$

$$A_s^* = -\frac{\partial}{\partial t} - \sum a_i \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{2} \sigma_i^2 \frac{\partial^2}{\partial x_i^2} + a_0 . \tag{7}$$

Theorem 1: For $\tau \le t \le T, \quad x, \xi \in R_+^2 = [0,\infty]\times[0,\infty]$, the solution of the partial differential equation

$$A_s^* k(t,x;\tau,\xi) = 0 \tag{8}$$

$$k(t,(0,x_2)) = k(t,(x_1,0)) = 0$$

$$k(\tau,x;\tau,\xi) = \delta(\xi - x) \tag{9}$$

is as following. Namely,

$$k(t,x;\tau,\xi) = \exp\{\beta(t-\tau) + (\alpha, x-\xi)\} \times \prod_{i=1}^{2} \frac{1}{\sqrt{2\pi(t-\tau)}\sigma_i} \left\{ e^{-\frac{(x_i-\xi_i)^2}{2(t-\tau)\sigma_i{}^2}} - e^{-\frac{(x_i+\xi_i)^2}{2(t-\tau)\sigma_i{}^2}} \right\} \tag{10}$$

Proof: Let's assume that the solution of the equation (8) on boundary condition (9) exist,and apply following transformation:

$$s(t,x;\tau,\xi)=e^{-(\alpha,x)}k(t,x;\tau,\xi).$$

Then for $\alpha=(\alpha_1,\alpha_2),\ a=(a_1,a_2)$ the equation (8) is following:

$$\frac{\partial s}{\partial t}-\left[(a,\alpha)+\frac{1}{2}\alpha^T\sigma^2\alpha+a_0\right]s+\left(-a_1+\sigma_1^2\alpha_1\right)\frac{\partial s}{\partial x_1}+$$
$$+\left(-a_2+\sigma_2^2\alpha_2\right)\frac{\partial s}{\partial x_2}+\left[\sigma_1^2\frac{\partial^2 s}{\partial x_1^2}+\sigma_2^2\frac{\partial^2 s}{\partial x_2^2}\right]=0 . \qquad (11)$$

Now, for $\tau\le t\le T,\quad x,\,\xi\in R_+^2$ instituting α_i and β into (11), (9), we obtain the following equation:

$$\left[-\frac{\partial}{\partial t}+\frac{1}{\tau}\left[\sigma_1^2\frac{\partial^2}{\partial x_1^2}+\sigma_2^2\frac{\partial^2}{\partial x_2^2}\right]+\beta\right]s(t,x;\tau,\xi)=0 \qquad (12)$$

$$s(t,(0,x_2);\tau,\xi)=s(t,(x_1,0);\tau,\xi)=0$$
$$s(\tau,x;\tau,\xi)=e^{-(\alpha,x)}\delta(x-\xi) . \qquad (13)$$

Where $\alpha_i=\dfrac{a_i}{\sigma_i^2}\ ,\quad i=1,\,2\ ,\quad \beta=a_0+(a,\alpha)+\dfrac{1}{2}\alpha^T\sigma^2\alpha$.

Let's obtain the solution of above equation by expanding the domain of equation (12) that is satisfied the boundary condition (13) the R_+ onto R .

Namely, when s is the solution of equation (12), we define the expanded solution Γ as following:

$$\Gamma(t,x;\tau,\xi)=\begin{cases} s(t,(x_1,x_2);\tau,\xi) & x_1\ge 0,\quad x_1\ge 0\\ -s(t,(-x_1,x_2);\tau,\xi) & x_1<0,\quad x_1\ge 0\\ -s(t,(x_1,-x_2);\tau,\xi) & x_1\ge 0,\quad x_1<0\\ s(t,(-x_1,-x_2);\tau,\xi) & x_1<0,\quad x_1<0\end{cases} . \qquad (14)$$

Also, the condition (13) become as following initial condition:

$$\Gamma(\tau,x;\tau,\xi)=\varphi(x,\xi)=\begin{cases} e^{-(\alpha,x)}\delta(x-\xi) \quad , & x_1\ge 0,\quad x_2\ge 0\\ -e^{-\alpha_1x_1+\alpha_2x_2}\delta(x_1-\xi_1,\,x_2+\xi_2), & x_1\ge 0,\quad x_2<0\\ -e^{\alpha_1x_1-\alpha_2x_2}\delta(x_1+\xi_1,\,x_2-\xi_2), & x_1<0,\quad x_2\ge 0\\ e^{\alpha_1x_1+\alpha_2x_2}\delta(x_1+\xi_1,\,x_2+\xi_2), & x_1<0,\quad x_2<0\end{cases} \quad . \quad (15)$$

Therefore, we can consider the boundary value problem (12),(13) to following initial value problem:

$$\left[-\frac{\partial}{\partial t}+\frac{1}{\tau}\left(\sigma_1^2\frac{\partial^2}{\partial x_1^2}+\sigma_2^2\frac{\partial^2}{\partial x_2^2}\right)+\beta\right]\Gamma(t,x;\tau,\xi)=0 \quad (16)$$

$$\Gamma(\tau,x;\tau,\xi)=\varphi(x,\xi) \quad . \quad (17)$$

The solution of this initial value problem is well-known. Namely, the solution is obtained as following:

$$\Gamma(t,x;\tau,\xi)=\int_{R^2}\varphi(x-y,\xi)\prod_{i=1}^{2}\frac{1}{\sqrt{2\pi\sigma_i^2(t-\tau)}}e^{-\frac{y_i^2}{2\sigma_i^2(t-s)}}e^{\beta(t-\tau)}dy_1dy_2=$$

$$=\exp\{-(\alpha,\xi)\}\frac{1}{2\pi(t-\tau)\sigma_1\sigma_2}\left(\exp\left\{-\frac{\frac{(x_1-\xi_1)^2}{\sigma_1^2}+\frac{(x_2-\xi_2)^2}{\sigma_2^2}}{2(t-\tau)}\right\}-\right.$$

$$-\exp\left\{-\frac{\frac{(x_1-\xi_1)^2}{\sigma_1^2}+\frac{(x_2+\xi_2)^2}{\sigma_2^2}}{2(t-\tau)}\right\}-\exp\left\{-\frac{\frac{(x_1+\xi_1)^2}{\sigma_1^2}+\frac{(x_2-\xi_2)^2}{\sigma_2^2}}{2(t-\tau)}\right\}+$$

$$\left.+\exp\left\{-\frac{\frac{(x_1+\xi_1)^2}{\sigma_1^2}+\frac{(x_2+\xi_2)^2}{\sigma_2^2}}{2(t-\tau)}\right\}\right)\exp\{\beta(t-\tau)\} \quad . \quad (18)$$

When reduce to R_+^2 the domain in equation (18), we obtain that

$$k(t,x;\tau,\xi)=e^{(\alpha,x)}s(t,x;\tau,\xi)$$

$$=\exp\{(\alpha,x-\xi)+\beta(t-\tau)\}\prod_{i=1}^{2}\frac{1}{\sqrt{2\pi(t-\tau)}\sigma_i}\left(e^{-\frac{(x_i-\xi_i)^2}{2(t-\tau)\sigma_i^2}}-e^{-\frac{(x_i+\xi_i)^2}{2(t-\tau)\sigma_i^2}}\right).$$

Theorem 2. The differential-integral equation (1) equal to following integral equation:

$$u(\tau,\xi)=\int_{\tau}^{T}\int_{R_+^2}u(t,x)G(t,x;\tau,\xi)dxdt+F(\tau,\xi) \quad (19)$$

Where the notation $F(\tau\ \xi)$, $G(t\ x\ \tau\ \xi)$ are denoted as following:

$$F(\tau,\xi)=\int_{R_+^2}k(T,x;\tau,\xi)dx$$

$$G(t,x;\tau,\xi)=\sum_{l=1}^{d}\lambda_l\int_{R_+^2}k(t,z;\tau,\xi)p_l(z-x)dz, \quad (20)$$

where the notation k is denoted as the equation (10) of theorem 1.

Proof: Let the solution u of equation (1) with respect to operator (4) show equation (19). With respect to operator A, $A*$,we obtain following equation:

$$k(\cdot)A(u(\cdot))=k(\cdot)A(u(\cdot))-uA*(k(\cdot))=$$

$$=\frac{\partial}{\partial t}(u(\cdot)k(\cdot))+\sum_{i=1}^{2}a_i\frac{\partial u}{\partial x_i}+\frac{1}{\tau}\sum_{i=1}^{2}\sigma_i^2\frac{\partial}{\partial x_i}\left(k(\cdot)\frac{\partial u(\cdot)}{\partial x_i}-u(\cdot)\frac{\partial k(\cdot)}{\partial x_i}\right). \quad (21)$$

From the boundary and initial condition of u, k ,each term of equation (21) is expressed as following:

$$\int_{R_+^2}\int_{\tau}^{T}\frac{\partial}{\partial t}(u(\cdot)k(\cdot))dtdx=\int_{R_+^2}u(\cdot)k(\cdot)\Big|_{\tau}^{T}dx=$$

$$=\int_{R_+^2}k(T,x;\tau,\xi)dx-\int_{R_+^2}u(\tau,x)\delta(x-\xi)dx=$$

$$= \int_{R_+^2} k(T,x;\tau,\xi)dx - u(\tau,\xi) \quad , \qquad (22)$$

$$\int_\tau^T \int_{R_+^2} \frac{\partial(k(\cdot)u(\cdot))}{\partial x_1} dxdt = \int_\tau^T \int_0^{+\infty} k(\cdot)u(\cdot)\Big|_0^{+\infty} dtdx_2 = 0 \quad , \qquad (23)$$

$$\int_\tau^T \int_{R_+^2} \frac{\partial}{\partial x_i}\left[k(\cdot)\frac{\partial u(\cdot)}{\partial x_i} - u(\cdot)\frac{\partial k(\cdot)}{\partial x_i}\right]dxdt = 0 \quad . \qquad (24)$$

From (21)~(24) and (1), we obtain

$$\int_\tau^T \int_{R_+^2} k(\cdot)A(u(\cdot))dxdt = \int_{R_+^2} k(T,x;\tau,\xi)dx - u(\tau,\xi) \quad , \qquad (25)$$

$$\int_\tau^T \int_{R_+^2} k(T,x;\tau,\xi)A(u(\cdot))dxdt =$$

$$= \int_\tau^T \int_{R_+^2} k(T,x;\tau,\xi)\left(-\sum \lambda_k \int_{R_+^2} u(t,z)p_l(z-x)dz\right)dxdt =$$

$$= -\int_\tau^T \int_{R_+^2} u(t,x)G(t,x;\tau,\xi)dxdt \quad . \qquad (26)$$

Therefore, by substituting equation (26) to (27), we obtain equation (19).

Namely, the solution of equation (1) is solution of (19).

Conversely, let the solution of (19) show one of (1).

Assume that u is solution of (19). From equation (10),integral kernel G of (19) is as following:

$$\int_\tau^T \int_{R_+^2} G(t,x;\tau,\xi)dxdt \le |\lambda| \int_\tau^T \int_{R_+^2} k(t,x;\tau,\xi)dxdt < \infty \quad ,$$

G is integrable in domain $\Sigma = \{(t \ \ \xi) \in [\tau \ \ T] \times R_+\}$.

Therefore, for arbitrary $\xi \in R_+$, $o \le \tau \le \mathrm{T}$, if u is the solution of (19), it must satisfy following equation:

$$\int_\tau^T \int_{R_+^2} k(t,x;\tau,\xi)\left[A(u(t,x)) + \sum \lambda_k \int_{R_+^2} u(t,z) p_k(z-x) dz\right] dxdt = 0 \quad . \qquad (27)$$

This shows that the braket part of integrade is zero, namely, equation (1) is satisfied.

The proof is completed.

Lemma 1: If $p(x)$ is density function of regular distribution and $p_2(x)$ is arbitrary hyper-density function, $p_1 * p_2(x)$ is infinite time differential in usual meaning. Where $*$ is synthesis multiple symbol.

Lemma 2: F, G of equation (20) in theorem 2 are infinite time differential on domain $\Sigma = \{(t,\xi) \in [\tau, T] \times R_+^2\}$.

We omit proof here.

Theorem 3: On domain Σ the unique solution of integral equation (19) is as following:

$$u(\tau,\xi) = F(\tau,\xi) + \sum_{k=1}^{\infty} \int_\tau^T \int_{R_+^2} G^{*k}(t,x;\tau,\xi) F(t,x) dxdt \quad , \qquad (28)$$

where

$$G^{*1}(t,x;\tau,\xi) = G(t,x;\tau,\xi)$$

$$G^{*k}(t,x;\tau,\xi) = \int_\tau^T \int_{R_+^2} G(s,z;\tau,\xi) G^{*(k-1)}(t,x;s,z) dzds \quad ,$$

G equal to equation (20) of theorem 2.

Proof: By means of lemma 2 and convergence property condition of progressive formula it is sufficient to show that inequality (29) hold:

$$\max_{(\tau,\xi)\in\Sigma}\int_{\tau}^{T}\int_{R_+^2}\left|G(t,x;\tau,\xi)\right|dt\,dx\ \le 1 \quad . \tag{29}$$

From the equation (20) of theorem 2,

$$\left|\int_{\tau}^{T}\int_{R_+^2}G(t,x;\tau,\xi)dtdx\right|\le|a_0|\left|\int_{\tau}^{T}\int_{R_+^2}k(t,x;\tau,\xi)dxdt\right| \quad . \tag{30}$$

On the other hand, the equation (10) can rewrite as following:

$$k(t,y;\tau,\xi)=\frac{e^{-a_0(t-s)}}{2\pi\sigma_1\sigma_2(t-\tau)}\left(\exp\left\{-\frac{(y_1-\xi_1-a_1(t-\tau))^2}{2\sigma_1^2(t-\tau)}-\frac{(y_2-\xi_2-a_2(t-\tau))^2}{2\sigma_2^2(t-s)}\right\}\right.$$

$$-\exp\left\{-\frac{(y_1-\xi_1-a_1(t-\tau))^2}{2\sigma_1^2(t-\tau)}-\frac{(y_2+\xi_2-a_2(t-\tau))^2}{2\sigma_2^2(t-\tau)}\right\}e^{-\frac{2a_2\xi_2}{\sigma_2^2}}-$$

$$-\exp\left\{-\frac{(y_1+\xi_1-a_1(t-\tau))^2}{2\sigma_1^2(t-\tau)}-\frac{(y_2-\xi_2-a_1(t-\tau))^2}{2\sigma_2^2(t-\tau)}\right\}e^{-\frac{2a_1\xi_1}{\sigma_1^2}}+$$

$$+\exp\left\{-\frac{(y_1+\xi_1-a_1(t-\tau))^2}{2\sigma_1^2(t-\tau)}-\frac{(y_2+\xi_2-a_2(t-\tau))^2}{2\sigma_2^2(t-\tau)}\right\}e^{-\frac{2a_1\xi_1}{\sigma_1^2}-\frac{2a_2\xi_2}{\sigma_2^2}} \quad . \tag{31}$$

Now, $N(d)$, $d_j(t\ \tau\ \xi_j)$, $d_j(t\ \tau\ \xi_j)$ are denoted as following:

$$N(d)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d}e^{-\frac{x^2}{\tau}}dx$$

$$d_{1j}(t;\tau,\xi_j)=\frac{-\left(\xi_j+\alpha_j\sigma_j^2(t-\tau)\right)}{\sigma_j\sqrt{t-\tau}}$$

$$d_{2j}(t;\tau,\xi_j)=\frac{-\left(\xi_j-\alpha_j\sigma_j^2(t-\tau)\right)}{\sigma_j\sqrt{t-\tau}} \qquad j=1,\ 2 \quad . \tag{32}$$

By substituting equation (31) into (30), we have

$$\left|\int_{\tau}^{T}\int_{R_+^2} k(t,x;\tau,\xi)dxdt\right| = \left|\int_{\tau}^{T} e^{-a_0(t-z)}\prod_{j=1}^{2}\left[1-N(d_j)-e^{-2\alpha_j\xi_j}\left(1-N(d_{2j})\right)\right]dt\right| . \quad (33)$$

We define $g(\tau\ \xi_j)$ so that $g(\tau,\xi_j)=\left|-N(d_{1j})-e^{-2\alpha_j\xi_j}\left(1-N(d_{2j})\right)\right|$.

Then, function $g(\tau\ \xi_j)$ monotonously increase with respect to ξ_j, $g(\tau,0)=0$ and $g(\tau,+\infty)=1$.

Therefore, from equation (33), we obtain that

$$\left|\int_{\tau}^{T}\int_{R_+^2} k(t,x;\tau,\xi)dxdt\right| \le \left|\int_{\tau}^{T} e^{-a_0(t-\tau)}dt\right| = \frac{1}{a_0}\left(1-e^{-a_0(t-\tau)}\right) \le \frac{1}{a_0} . \quad (34)$$

From equation (30), (34), equation (29) is satisfied. Therefore we can apply progressive formula of second kinds of integral equations.

Let the operator $\Phi : C\left((0,T]\times R_+^2\right)\to C\left((\tau,T]\times R_+^2\right)$ be

$$\Phi(\varphi)=\lambda\int_{\tau}^{T}\int_{R_+^2} G(t,x;\tau,\xi)\varphi(t,x)dxdt + F(\tau,\xi) .$$

Then, equation (19) equal to operation equation $u=\Phi u$.

Now, let $u^0(\tau,\xi)=F(\tau,\xi)$ (35)

be zero-order approximation. By applying

$$u^{(k+1)}=\Phi\left(u^{(k)}\right), \quad (36)$$

we obtain the equation (28).

On the other hand, because the operator Φ is reduced operator, it has the uniquefixed point and therefore, approximation formula (36) converges to the unique solution.

Their applications in insurance mathematics

The two-dimensional risk model we consider in this paper can be formally stated as

$$\begin{pmatrix} R_1(t) \\ R_2(t) \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} t + \begin{pmatrix} \sigma_1 w_\tau(t) \\ \sigma_2 w_\tau(t) \end{pmatrix} + \begin{pmatrix} \sum_{k=1}^{M_1(t)} z_{1k} \\ \sum_{k=1}^{M_2(t)} z_{2k} \end{pmatrix}. \qquad (37)$$

Where $M_1(t),\ M_2(t)$ are the number of claims between time 0 and t, which follows a Poisson process with parameter λ. $\{z_{ik}\}$ are claim size random variables as in the univariate risk model. For simplicity, we assume that {X1k,k= 1,2,...} and {X2k,k= 1,2,...} are independent, and furthermore, both of them are also independent of M1(t), M2(t). u_i the initial surplus of eath insurance company, c_i the rate at which the premiums are received.

$R_i(t)$ is the surplus of i-th insurance company at time t ≥ 0.

In this paper, we consider the following type of ruin:

$T = \inf\{t \mid \min\{R_1(t),\ R_2(t)\} < 0\}$.

With the time of ruin defined, the corresponding probability of ruin is denoted by

$\Phi(u_1, u_2) = P\{T < \infty \mid (R_1(0),\ \ R_2(0)) = (u_1, u_2)\}$.

Set $R(t) = (R_1(t),\ R_2(t))^T$. We can rewrite the equation (37) as following:

$$\begin{cases} dR(t) = c \cdot dt + \sigma \cdot dw(t) + \int_{|z| \le 1} \alpha(z)(\mu - \nu)(dt, dz) + \int_{|z| > 1} \alpha(z)\mu(dt, dz) \\ R(0) = u \end{cases}.$$

Where

$c = (c_1, c_2)^T \quad \sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad \alpha(z) = \begin{pmatrix} \alpha_{11}(z) & \alpha_{12}(z) \\ \alpha_{21}(z) & \alpha_{22}(z) \end{pmatrix} \quad u = (u_1, u_2)^T,$

$w(t) = (w_1(t),\ w_2(t))^T$ are independent brownian motion,

$\mu(dt,dz) = (\mu_1(dt,dz),\cdots,\mu_d(dt,dz))^T$ is Poisson probability measure that μ_i are independent and have adjustment $E\mu_i(dt,dz) = \nu_i(dz)dt$.

Then, $R(t)$ is Levy process. On $0 < s$ and $x \in R_+^2$, let ruin time $T_{s,x}$ define

$T_{s,x} = \inf\{t \ge s,\ R_1(t) < 0 \vee R_2(t) < 0 |\ R(s) = x\}$

and let ruin probabiliy of T time before define $\Psi(s,x;T) = P\{T_{s,x} \le T\}$,

survival probability is expressed as $\Phi(s,x;T) = 1 - \Psi(s,x;T)$.

Φ satisfy the following partial differential-integral equation:

$$\frac{\partial}{\partial s}\Phi(s,x) + L\Phi(s,x) = 0.$$

By means of definition of Φ, it satisfy

$$\Phi(s,(x_1,0)) = 0,\quad \Phi(s,(0,x_2)) = 0,\quad \Phi(T,(x_1,x_2)) = 1,$$

where the operator L is

$$L\Phi(s,x) = \sum c_i \frac{\partial\Phi}{\partial x_i} + \frac{1}{\tau}\sum \sigma_i^2 \frac{\partial^2\Phi}{\partial x_i^{\ 2}} +$$

$$+\sum_{k=1}^{d}\int_{R_+^2}\left[\Phi(s,x+\alpha(z)) - \Phi(s,x) - \sum \alpha_{ik}(z)\frac{\partial^2\Phi}{\partial x_i^{\ 2}} I_{\{|z|\le 1\}}\right]\cdot \nu_k(dz).$$

Therefore, the survival probability Φ until time T can be obtained by means of equation (28) as following:

$$\Phi(\tau,\xi) = F(\tau,\xi) + \sum_{k=1}^{\infty}\int_{\tau}^{T}\int_{R_+^2} G^{(k)}(t,x;\tau,\xi)F(t,x)\,dx\,dt.$$

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