A note on a problem in Mishō Sampō

Hiroshi Okumura

Abstract. A problem involving an isosceles triangle with a square and three congruent circles is generalized.

Keywords. 3-4-5 triangle

Mathematics Subject Classification (2010). 01A27, 51M04

1. Introduction

In this note we generalize the following problem, which can be found in [1, 2, 3, 4, 5], where the sangaku with this problem in [4] is undated (see Figure 1).

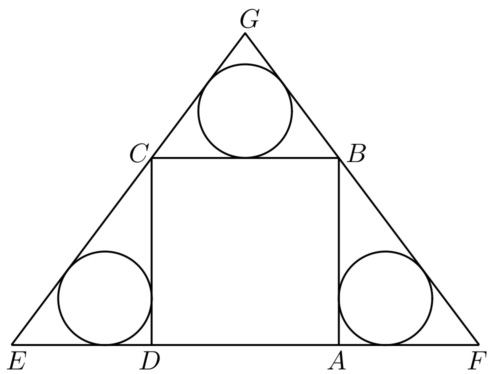

Figure 1.

Problem 1. EFG is an isosceles triangle with base EF. $ABCD$ is a square such that B and C lie on the sides FG and GE, respectively, and D and A lie on the side EF. The incircles of the triangles ABF and BCG are congruent and have radius r. Show that $4r = |AB|$.

We show that the isosceles triangle EFG is formed by a 3-4-5 triangle with its reflected image in the side of length 4, i.e., the ratio of the sides of EFG equals $5 : 5 : 6$.

2. Generalization

Let EFG be a triangle. Let γ_1, γ_2, $\cdots$, γ_n be circles of radius r such that they touch the side EF from the inside of EFG, γ_1 and γ_2 touch, γ_i $(i = 3, 4, \cdots, n)$ touches γ_{i-1} from the side opposite to γ_1, γ_1 touches GE, γ_n touches FG. In this case we say that *EF has n circles of radius r with respect to G* (see Figure 2). This is equivalent to the following equation being true:

$$|EF| = r\cot\frac{\angle E}{2} + r\cot\frac{\angle F}{2} + 2(n-1)r.$$

Problem 1 is generalized as follows (see Figure 3).

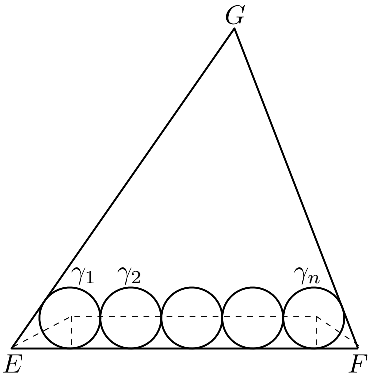

Figure 2: $n = 5$

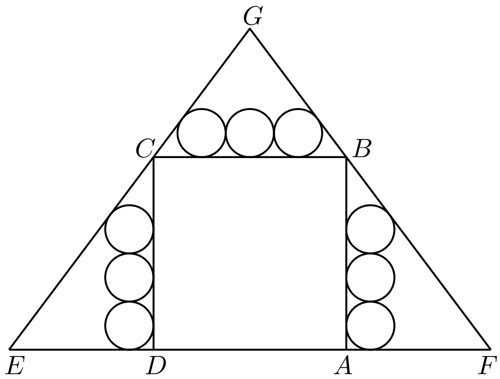

Figure 3: $n = 3$

Theorem 2.1. *EFG is an isosceles triangle with base EF. $ABCD$ is a square such that B and C lie on the sides FG and GE, respectively, D and A lie on the side EF. If AB has n circles of radius r with respect to F and BC has n circles of radius r with respect to G, then the following statements hold.*
(i) $|FG| : |EF| = 5 : 6$.
(ii) $2(n+1)r = |AB|$.
(iii) *If n is odd and expressed as $n = 2k-1$ for a natural number k, EF has $5k-1$ circles of radius r with respect to G.*

Proof. Let $2\theta = \angle ABF$. Then we have

$$|AB| = r\cot\theta + (2n-1)r. \tag{1}$$

While $\angle CBG + 2\theta = 90^\circ$ implies $|BC| = 2r\cot(45^\circ - \theta) + 2(n-1)r$. Therefore we get $\cot\theta = 3$ by $|AB| = |BC|$. Hence $\tan 2\theta = 3/4$, i.e., ABF is a 3-4-5 triangle. This proves (i). The part (ii) follows from (1). We assume $n = 2k-1$. Let $s = |AB|$. Then $s = 4kr$ by (ii). The distance from G to BC equals $(s/2)\cdot(4/3) = 2s/3$. Therefore $|BC| : |EF| = 2s/3 : (s+2s/3) = 2 : 5$, i.e., $|EF| = 5s/2 = 10kr$. Hence $|EF| = 2r\cot(\angle E/2) + 2(5k-1-1)r$, since $\cot(\angle E/2) = 2$. This proves (iii). □

Acknowledgments. The author expresses his thanks to late Professor Toshio Matsuzaki for kindly sending a copy of [4].

References

[1] Ishida (石田保之) et al. ed., Mishō Sampō (未詳算法) volume 9, 1827, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100007110.
[2] Okayu (or Gokayu) (御粥安本), Sampō Senmonshō (算法浅問抄), 1840, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100003116.
[3] Furuya (古谷道生) ed., Sampō Tsūsho (算法通書), 1854, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100005506.
[4] T. Matsuzaki (松崎利雄), The sangaku in Ibaragi (茨城の算額), Ibaragi Tosho (茨城図書), 1997.
[5] Issendai (一千題), 1852, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100003682.

Tohoku Univ. WDB is short for Tohoku University Wasan Material Database.