$b^{\#}$D - Sets and Associated Separation Axioms

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Abstract

In this paper the notion of $b^{\#}$D-sets is introduced. Some weak separation axioms namely $b^{\#}-D_k$, $b^{\#}-R_0$, $b^{\#}$-R_1 and $b^{\#}$-S_0 are introduced and studied. Some lower separation axioms are characterized by using these separation axioms.

Keywords : $b^{\#}$-open set, $b^{\#}$D-sets, $b^{\#}$-R_0, $b^{\#}$-R_1, $b^{\#}$-S_0, $b^{\#}$-D_k.

1 Introduction and Preliminaries

In the year 1996, Andrijivic [1] initiated the study of b-open sets in topology. Recently Usha Parameswari et.al.[6] introduced the notion of $b^{\#}$-open sets and $b^{\#}$-closed sets. The concept of D-sets was introduced by Tong [5] using open sets and some separation axioms are studied using this notion. Following this Keskin and Noiri [2] introduced the notion of bD sets and their properties are investigated. In this paper, the notion of $b^{\#}$D sets and the associated separation axioms are investigated. Some new types of separation axioms namely $b^{\#}$-R_0, $b^{\#}$-R_1, $b^{\#}$-S_0 are also introduced using $b^{\#}$-closure operator. The relationships with analog concepts that are available in the literature of topology are discussed.
Throughout this paper X denotes a topological space on which no separation axiom is assumed. For any subset A of X, $cl(A)$ denotes the closure of A and $int(A)$ denotes the interior of A in X.

Definition 1.1 *A subset A of a space X is said to be*
(i) b-open [1] if $A \subseteq cl(int(A)) \cup int(cl(A))$.
(ii)$b^{\#}$-open [6] if $A = cl(int(A)) \cup int((cl(A))$.

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The complements of b-open and $b^{\#}$-open sets are respectively called b-closed and $b^{\#}$-closed sets.

Definition 1.2 *[2] A subset A of a topological space X is called a bD-set if there are $U, V \in bO(X,\tau)$ such that $U \neq X$ and $S = U - V$.*

Definition 1.3 *[4] A collection m_X of subsets of X is called a minimal structure on X (briefly m-structure) if $\phi \in m_X$ and $X \in m_X$.*

Definition 1.4 *[3] A collection μ of subsets of X is called a supra topology on X if $\phi, X \in \mu$ and μ is closed under arbitrary union in which case (X,μ) is called a supra topological space or supra space.*

The collection B of subsets of X is a supra basis for some supra topology μ on X if every non empty member of μ is a union of members of B. The collection of $b^{\#}$-open sets in (X,τ) is a supra basis for a supra topology on X. We identify this supra topology on X by $\tau^{\#}$ induced by τ. The members of $\tau^{\#}$ are called *supra*$^{\#}$-open sets in (X,τ).

The intersection of all $b^{\#}$-closed sets of X containing A is called the $b^{\#}$-closure of A denoted by $b^{\#}cl(A)$ and the union of all $b^{\#}$-open sets in X contained in A is called the $b^{\#}$-interior of A and is denoted by $b^{\#}int(A)$. The notations $bO(X,\tau)$, $bC(X,\tau)$, $b^{\#}O(X,\tau)$, $b^{\#}C(X,\tau)$, $bD(X,\tau)$ respectively denote the family of all b-open sets, b-closed sets, $b^{\#}$-open sets, $b^{\#}$-closed sets, bD-sets in X.

2 $b^{\#}$-T_k Spaces

Recently Usha et.al., introduced $b^{\#}$-T_k Spaces. In this section these separation axioms are further investigated.

Definition 2.1 *[7] Let (X,τ) be a topological space. Then X is said to be*

(i).$b^{\#}$-T_0 if for any two distinct points x and y of X there exists a $b^{\#}$-open set G such that ($x \in G$ and $y \notin G$) or ($y \in G$ and $x \notin G$).

(ii).$b^{\#}$-T_1 if for any two distinct points x and y of X there exist $b^{\#}$-open sets G and H such that $x \in G$ but $y \notin G$ and $y \in H$ but $x \notin H$.

(iii).$b^{\#}$-T_2 if for any two distinct points x and y of X there exist disjoint $b^{\#}$-open sets G and H such that $x \in G$ and $y \in H$.

Proposition 2.2 *Let X be a topological space. If X is $b^{\#}$-T_1, then every singleton subset of X is supra$^{\#}$-closed in (X,τ).*

Proof.

Suppose that X is $b^{\#}$-T_1 and x is any point in X. Let $y \in X - \{x\}$. Then $x \neq y$. Therefore there exists a $b^{\#}$-open set U such that $y \in U$ but $x \notin U$. Thus for each $y \in X - \{x\}$, there exists a $b^{\#}$-open set U_y such that $y \in U_y \subseteq X - \{x\}$. Therefore $\bigcup\{U_y : y \neq x\} \subseteq X - \{x\}$ which implies that $X - \{x\} = \bigcup\{U_y : y \neq x\}$. This implies that $X - \{x\}$ is supra$^{\#}$-open and so $\{x\}$ is supra$^{\#}$-closed.

Theorem 2.3 *(X, τ) is $b^{\#}$-T_1 if and only if $b^{\#}cl\,\{x\} = \{x\}$ for every $x \in X$.*
Proof. *Assume (X, τ) is $b^{\#}$-T_1. Fix $x \in X$. For $y \neq x$, there exist a $b^{\#}$-open set H with $y \in H$, $x \notin H$ that implies $H \cap \{x\} = \phi$. This proves that $y \notin b^{\#}cl\,\{x\}$ for every $y \neq x$. Therefore $b^{\#}cl\,\{x\}$ $= x$. Suppose $b^{\#}cl\,\{x\} = \{x\}$ for every $x \in X$. Let $x \neq y$. Then $y \notin b^{\#}cl\,\{x\}$. Then there exist a $b^{\#}$-open set H with $y \in H$,$x \notin H$. Also $y \neq x$ implies $x \notin b^{\#}cl\,\{y\}$. Then there exist a $b^{\#}$-T_1-open set G with $x \in G$, $y \notin G$. Therefore (X, τ) is $b^{\#}$-T_1.*

Theorem 2.4 *Let X be a topological space. Consider the following statements.*

(i) X is $b^{\#}$-T_2.

(ii) Given x_0, for $x \neq x_0$ in X, there is a $b^{\#}$-open set U in X containing x_0 such that $x \notin b^{\#}cl(U)$.

(iii) For each $x \in X$, $\bigcap\{\, b^{\#}cl(U)$: U is $b^{\#}$-open in X containing $x = \{x\}$. Then the implications (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) hold.

Proof. *To prove (i) $\Rightarrow$ (ii). Assume X is $b^{\#}$-T_2. Fix $x_0 \in X$. Let $x \neq x_0$. Since X is a $b^{\#}$-T_2, there exist disjoint $b^{\#}$-open sets U and V such that $x_0 \in U$ and $x \in V$. Then $X - V$ is $b^{\#}$-closed. Then $U \subseteq X - V$. Since $X - V$ is $b^{\#}$-closed, $b^{\#}cl(U) \subseteq X - V$. If $x \in b^{\#}cl(U)$ then $x \in X - V$ which is not possible. Therefore $x \notin b^{\#}cl(U)$. Now to prove (ii) $\Rightarrow$ (iii). Suppose that for each $x \neq y$, there exist a $b^{\#}$-open set U such that $x \in U$ and $y \notin b^{\#}cl(U)$. Then $\bigcap\{\, b^{\#}cl(U)$: U is $b^{\#}$-open in X containing $x = \{x\}$. (iii) $\Rightarrow$ (ii) follows easily.*

Remark 2.5 *The implication (iii) $\Rightarrow$ (i) of Theorem 2.4 does not hold as given in the following example.*

Example 2.6 *In the real line R, for $a < b$, $[a, b]$ is $b^{\#}$-open, (a, b) is $b^{\#}$-closed and all other intervals are neither $b^{\#}$-open nor $b^{\#}$-closed.By Example 4.10 of [6], it is easy to see that the conditions (ii) and (iii) holds in R, but R is not $b^{\#}$-T_2.*

Definition 2.7 *A topological space (X, τ) is called $b^{\#}$-symmetric if for all x and y in X, $x \in b^{\#}cl(y)$ implies $y \in b^{\#}cl(x)$.*

Proposition 2.8 *If X is $b^{\#}$-symmetric and $b^{\#}$-T_0, then X is $b^{\#}$-T_1.*
Proof. *Let x, y be such that $x \neq y$. Since X is $b^{\#}$-T_2, there is a $b^{\#}$-open set U such that $x \in U \subseteq X - \{y\}$. Then $x \notin b^{\#}cl(\{y\})$. Since X is $b^{\#}$-symmetric, $y \notin b^{\#}cl(\{x\})$. Therefore there is a $b^{\#}$-open set V such that $y \in V \subseteq X - \{x\}$. This shows that X is $b^{\#}$-T_1.*

3 $b^{\#}D$-Sets

Definition 3.1 *A subset S of a topological space X is called a $b^{\#}D$-set if there are $b^{\#}$-open sets U, V such that $S = U - V$, U is a proper subset of X.*

It follows that every proper $b^{\#}$-open subset S of X is a $b^{\#}$D-set and a $b^{\#}$D-set need not be $b^{\#}$-open. In particular, the empty set is a $b^{\#}$D-set and the whole set X is not a $b^{\#}$D-set. Therefore the collection of all $b^{\#}$D-sets is not a minimal structure [4] on X. The family of all $b^{\#}$D-sets of X is denoted by $b^{\#}D(X, \tau)$. The next example shows that a $b^{\#}$D-set need not be $b^{\#}$-open.

Example 3.2 *Let* $X = \{a, b, c\}$ *and* $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. *Then* $b^{\#}O(X, \tau) = \{\phi, \{a, c\}, \{b, c\}, X\}$ *and* $b^{\#}D(X, \tau) = \{\phi, \{a\}, \{b\}, \{a, c\}, \{b, c\}\}$. *Then it is clear that* $\{a\}$ *is a* $b^{\#}D$*-set but not* $b^{\#}$*-open. Also the set* $\{a, c\}$ *is both a* $b^{\#}D$*-set and a* $b^{\#}$*-open set.*

Remark 3.3 *Every* $b^{\#}D$*-set is a* bD*-set. But, the converse is not true as seen from the next example.*

Example 3.4 *Let* $X = \{a, b, c\}$ *and* $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$. *Then* $b^{\#}D(X, \tau) = \{\phi, \{a, b\}\}$ *and* $bD(X, \tau) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. *Then it is clear that* $\{a\}$ *is a* bD*-set but not a* $b^{\#}D$*-set.*

The following implication diagram holds.

b-open $\Leftarrow$ b$^{\#}$-open
$\Downarrow$ $\Downarrow$
bD-set $\Leftarrow$ b$^{\#}$D-set

Definition 3.5 *A topological space* (X, τ) *is said to be*

(i) $b^{\#}$*-*D_0 *if for any pair of distinct points* x *and* y *of* X *there exists a* $b^{\#}D$*-set of* X *containing* x *but not* y *or a* $b^{\#}D$*-set of* X *containing* y *but not* x.

(ii) $b^{\#}$*-*D_1 *if for any pair of distinct points* x *and* y *of* X *there exists a* $b^{\#}D$*-set of* X *containing* x *but not* y *and a* $b^{\#}D$*-set of* X *containing* y *but not* x.

(iii) $b^{\#}$*-*D_2 *if for any pair of distinct points* x *and* y *of* X *there exist disjoint* $b^{\#}D$*-sets* F *and* G *of* X *such that* $x \in F$ *and* $y \in G$.

Proposition 3.6 *For a topological space* (X, τ)*, the following properties hold*

(i) If (X, τ) *is* $b^{\#}$*-*T_k*, then it is* $b^{\#}$*-*D_k*, for k = 0, 1, 2.*

(ii) If (X, τ) *is* $b^{\#}$*-*D_k*, then it is* $b^{\#}$*-*D_{k-1}*, for k = 1,2.*

Proof. *(i) follows from the fact that every proper* $b^{\#}$*-open subset* S *of* X *is a* $b^{\#}D$*-set. (ii) Suppose* (X, τ) *is* $b^{\#}$*-*D_1. *Let* $x \neq y in X$. *Since* X *is* $b^{\#}$*-*D_1*, there exist* $b^{\#}D$*-sets* G *and* H *such that* $x \in G$ *but* $y \notin G$ *and* $y \in H$ *but* $x \notin H$. *Clearly* $x \in G$ *but* $y \notin G$ *or* $y \in H$ *but* $x \notin H$. *Therefore* (X, τ) *is* $b^{\#}$*-*D_0. *Suppose* (X, τ) *is* $b^{\#}$*-*D_2. *Let* $x \neq y in X$. *Since* X *is* $b^{\#}$*-*D_1 *there exist disjoint* $b^{\#}D$*-sets* G *and* H *such that* $x \in G$ *and* $y \in H$. *Clearly* $x \in G$ *but* $y \notin G$ *and* $y \in H$ *but* $x \notin H$. *Therefore* $(X, \tau) is b^{\#}$*-*D_1.

Remark 3.7 *The next examples show that the converses in the above proposition are not true.*

Example 3.8 *Let* $X = \{a, b, c\}$ *and* $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. $b^{\#}O(X, \tau) = \{\phi, \{a, c\}, \{b, c\}, X\}$ *and* $b^{\#}D(X, \tau) = \{\phi, \{a\}, \{b\}, \{a, c\}, \{b, c\}\}$. *Then* (X, τ) *is* $b^{\#}$*-*D_1 *but not* $b^{\#}$*-*T_1.

Example 3.9 *Let* $X = \{a, b, c\}$ *and* $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$. $b^{\#}O(X, \tau) = \{\phi, \{a, b\}, \{b, c\}, X\}$ *and* $b^{\#}D(X, \tau) = \{\phi, \{a\}, \{c\}, \{a, b\}, \{b, c\}\}$. *Then* (X, τ) *is* $b^{\#}$*-*D_2 *but not* $b^{\#}$*-*T_2.

Proposition 3.10 *A space (X, τ) is $b^{\#}$-D_0 if and only if it is $b^{\#}$-T_0.*
Proof. *Suppose that (X, τ) is $b^{\#}$-D_0. Let $x \neq y$ in X. Since X is $b^{\#}$-D_0, there is a $b^{\#}D$-set G with $x \in G$ and $y \notin G$. Since G is a $b^{\#}D$-set we have $G = U_1 - U_2$ where $U_1 \neq X$ and $U_1, U_2 \in b^{\#}O(X, \tau)$. Then we have two cases : Case(i) $y \notin U_1$ where U_1 is $b^{\#}$-open with $x \in U_1$ and $y \notin U_1$. Case (ii) $y \in U_1$ and $y \in U_2$ where U_2 is $b^{\#}$-open with $y \in U_2$ and $x \notin U_2$. Thus in both the cases, (X, τ) is $b^{\#}$-T_0. Conversely, if (X, τ) is $b^{\#}$-T_0, by Proposition 3.6(i),(X, τ) is $b^{\#}$-D_0.*

Proposition 3.11 *A space (X, τ) is $b^{\#}$-D_1 if and only if it is $b^{\#}$-D_2.*
Proof. *Suppose that (X, τ) is $b^{\#}$-D_1. Let $x \neq y$ in X. Since X is $b^{\#}$-D_1, there exist $b^{\#}D$-sets G_1, G_2 in X such that $x \in G_1$, $y \notin G_1$ and $y \in G_2$,$x \notin G_2$. Since G_1 and G_2 are $b^{\#}D$-sets we have $G_1 = U_1 - U_2$ and $G_2 = U_3 - U_4$, where U_1, U_2, U_3 and U_4 are $b^{\#}$-open sets in X. Then $x \in G_1$ implies $x \in U_1$ and $x \notin U_2$. Since $y \notin G_1$ we have Case (i) : $y \notin U_1$ Case (ii) : $y \in U_1$ and $y \in U_2$. Now $y \in G_2$ implies $y \in U_3$ and $y \notin U_4$. Since $x \notin G_2$, we have case (iii) $x \notin U_3$ case (iv) $x \in U_3$ and $x \in U_4$. From case(i) and (iii), $x \in U_1 - U_3$ and $y \in U_3 - U_1$ implies $(U_1 - U_3) \cap (U_3 - U_1) = \phi$. From case (ii),$x \in U_1 - U_2$ implies $(U_1 - U_2) \cap U_2 = \phi$. From case (iv), $x \in U_3 - U_4$ implies $(U_3 - U_4) \cap U_4 = \phi$. Therefore (X, τ) is $b^{\#}$-D_2. Converse follows from Proposition 3.6(ii).*

Corollary 3.12 *If (X, τ) is $b^{\#}$-D_1 then it is $b^{\#}$-T_0.*
Proof. *Follows from Proposition 3.6(ii) and Proposition 3.10.*

Lemma 3.13 *Let (X, τ) be a topological space and $x \in X$. Then (i) if A is regular closed then A is $b^{\#}$-open. (ii) if A is regular open then A is $b^{\#}$-closed.*
Proof. *If A is regular closed then $A = cl(int(A))$ that implies $cl(A) = cl(int(A))$ and $int(cl(A)) = int(cl(int(A))) \subseteq cl(int(A))$. (ie.) $int(cl(A)) \cup cl(int(A)) = cl(int(A)) = A$. This proves (i). If A is regular open then $X - A$ is regular closed that implies $X - A$ is $b^{\#}$-open and A is $b^{\#}$-closed. This proves (ii).*

Proposition 3.14 *Let $x \neq y$ in (X, τ). Suppose there exists regular closed sets G and H such that*
(i) $x \in G$, $y \notin G$ (or) $y \in H$, $x \notin H$.
(ii) $x \in G$, $y \notin G$ and $y \in H$, $x \notin H$.
(iii) $x \in G$, $y \notin G$, $y \in H$, $x \notin H$ and $G \cap H = \phi$. Then
a) if (i) holds for any pair of distinct points x and y of X then (X, τ) is $b^{\#}$-T_0.
b) if (ii) holds for any pair of distinct points x and y of X then (X, τ) is $b^{\#}$-T_1.
c) if (iii) holds for any pair of distinct points x and y of X then (X, τ) is $b^{\#}$-T_2.
Proof. *Follows from Lemma 3.13.*

4 $b^{\#}$-R_k-Spaces

In this section, separation axioms namely $b^{\#}$-R_0 and $b^{\#}$-R_1 are introduced and characterized.

Definition 4.1 *A topological space (X, τ) is called $b^{\#}$-R_0 if for every $b^{\#}$-open set U, $b^{\#}cl(\{x\}) \subseteq U$ for all $x \in U$.*

Example 4.2 *Let* $X = \{a, b, c\}$ *and* $\tau = \{\phi, \{c\}, \{b\}, \{a, b\}, \{b, c\}, X\}$. *Then* $b^{\#}O(X, \tau) = \{\phi, \{c\}, \{a, b\}, X\}$ *and* $b^{\#}C(X) = \{\phi, X, \{a, b\}, \{c\}\}$. *It can be proved that* (X, τ) *is* $b^{\#}$-R_0.

Example 4.3 *Let* $X = \{a, b, c\}$ *and* $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. *Then* $b^{\#}O(X, \tau) = \{\phi, \{a, c\}, \{b, c\}, X\}$ *and* $b^{\#}C(X) = \{\phi, X, \{a\}, \{b\}\}$. *It can be proved that* (X, τ) *is not* $b^{\#}$-R_0.

Definition 4.4 *A space* X *is a* $b^{\#}$-R_1 *if for any* x, y *in* X *with* $b^{\#}cl\,\{x\} \cap b^{\#}cl\,\{y\} = \phi$, *there exist disjoint* $b^{\#}$*-open sets* U *and* V *such that* $b^{\#}cl\,\{x\}$ *is a subset of* U *and* $b^{\#}cl\,\{y\}$ *is a subset of* V.

Proposition 4.5 *If* (X, τ) *is* $b^{\#}$-R_1, *then* (X, τ) *is* $b^{\#}$-R_0.
Proof. *Suppose* (X, τ) *is* $b^{\#}$-R_1. *Let* U *be* $b^{\#}$*-open such that* $x \in U$. *If* $y \notin U$, *since* $x \notin b^{\#}cl\,\{y\}$, *we have* $b^{\#}cl\,\{x\} \neq b^{\#}cl\,\{y\}$. *So, there exists a* $b^{\#}$*-open set* V *such that* $b^{\#}cl\,\{y\} \subseteq V$ *and* $x \notin V$ *implies* $y \notin b^{\#}cl\,\{x\}$. *Thus* $b^{\#}cl\,\{x\} \subseteq U$. *Therefore* (X, τ) *is* $b^{\#}$-R_0.

Theorem 4.6 *Let* X *be a topological space. Then the following statements are equivalent. (i)* X *is* $b^{\#}$-R_0. *(ii) Any two distinct points of* X *are* $b^{\#}$*-symmetric.*
Proof. *Suppose* X *is* $b^{\#}$-R_0 *and* $x \neq y$. *Let* $x \in b^{\#}cl(\{y\})$ *and* U *be any* $b^{\#}$*-open set such that* $y \in U$. *Then* $x \in U$. *Since* X *is* $b^{\#}$-R_0, *every* $b^{\#}$*-open set containing* y *contains* x. *This implies* $y \in b^{\#}cl(\{x\})$ *and therefore* (X, τ) *is* $b^{\#}$*-symmetric. This proves (i)* $\Rightarrow$ *(ii). Conversely suppose* X *is* $b^{\#}$*-symmetric. Let* V *be a* $b^{\#}$*-open set such that* $x \in V$. *If* $y \in b^{\#}cl(\{x\})$ *then* $x \in b^{\#}cl(y)$. *This implies* $V \cap \{y\} = \phi$. *That is* $y \in V$. *Therefore* $b^{\#}cl(\{x\}) \subseteq V$ *for every* $x \in V$. *This proves that* X *is* $b^{\#}$-R_0. *Therefore (ii)* $\Rightarrow$ *(i).*

The notions of $b^{\#}$-symmetric and $b^{\#}$-R_0 are equivalent from Definition 2.7 and Theorem 4.6.

Proposition 4.7 *A topological space* (X, τ) *is* $b^{\#}$-R_0 *if and only if for every point* x, y *in* X, $b^{\#}cl\,\{x\} \neq b^{\#}cl\,\{y\}$ *implies* $b^{\#}cl\,\{x\} \cap b^{\#}cl\,\{y\} = \phi$.
Proof. *Suppose that* (X, τ) *is* $b^{\#}$-R_0. *Let* $x, y \in X$ *such that* $b^{\#}cl\,\{x\} \neq b^{\#}cl\,\{y\}$. *Then, there exists* $z \in X$ *with* $z \in b^{\#}cl\,\{x\}$ *and* $z \notin b^{\#}cl\,\{y\}$. *There exists* $V \in b^{\#}O(X)$ *such that* $y \notin V$, $z \in V$, $x \in V$. *Therefore* $x \notin b^{\#}cl\,\{y\}$. *Thus* $x \in \left[X - b^{\#}cl\,\{y\}\right]$ *which implies* $b^{\#}cl\,\{x\} \subseteq \left[X - b^{\#}cl\,\{y\}\right]$ *and* $b^{\#}cl\,\{x\} \cap b^{\#}cl\,\{y\} = \phi$. *Conversely, let* $V \in b^{\#}O(X)$ *and* $x \in V$ *we have to show that* $b^{\#}cl\,\{x\} \subseteq V$. *Let* $y \notin V$. *This implies* $y \in X - V$. *Then* $x \neq y$ *and* $x \notin b^{\#}cl\,\{y\}$. *This shows that* $b^{\#}cl\,\{x\} \neq b^{\#}cl\,\{y\}$. *Since* $b^{\#}cl\,\{x\} \cap b^{\#}cl\,\{y\} = \phi$, $y \notin b^{\#}cl\,\{x\}$ *and* $b^{\#}cl\,\{x\} \subseteq V$. *Therefore* (X, τ) *is* $b^{\#}$-R_0.

Proposition 4.8 *If* (X, τ) *is* $b^{\#}$-R_1 *then for* $x, y \in X$ *with* $b^{\#}cl\,\{x\} \cap b^{\#}cl\,\{y\} = \phi$ *there exist* $b^{\#}$*-closed sets* F_1 *and* F_2 *such that* $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ *and* $X = F_1 \cup F_2$.
Proof. *Suppose* (X, τ) *is* $b^{\#}$-R_1. *By Definition 4.4, for any* $x, yinX$ *with* $b^{\#}cl\,\{x\} \cap b^{\#}cl\,\{y\} = \phi$, *there exist disjoint* $b^{\#}$*-open sets* U *and* V *such that* $b^{\#}cl\,\{x\} \subseteq U$ *and* $b^{\#}cl\,\{y\} \subseteq V$. *Taking complement,* $X - b^{\#}cl\,\{x\} \supseteq X - U$ *and* $X - b^{\#}cl\,\{y\} \supseteq X - V$. *Let* $X - U = F_1$ *and* $X - V = F_2$. *That is* $F_1 \subseteq X - b^{\#}cl\,\{x\}$ *and* $F_2 \subseteq X - b^{\#}cl\,\{y\}$. *This shows that* $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$. *Then* $U \cap V = \phi$ *implies* $X = F_1 \cup F_2$.

Proposition 4.9 *Suppose (X,τ) is $b^{\#}$-R_1. Then for every x and y with $x \in X - b^{\#}cl\{y\}$,x and y have disjoint $b^{\#}$-open neighbourhoods.*
Proof. *Assume (X,τ) is $b^{\#}$-R_1. Fix $x \neq y$. Let $x \in X - b^{\#}cl\{y\}$. Then $b^{\#}cl\{x\} \neq b^{\#}cl\{y\}$. By using Proposition 4.7, $b^{\#}cl\{x\} \cap b^{\#}cl\{y\} = \phi$. Since (X,τ) is $b^{\#}$-R_1, $b^{\#}cl\{x\}$ and $b^{\#}cl\{y\}$ have disjoint $b^{\#}$-open neighbourhoods. This implies x and y have disjoint $b^{\#}$-open neighbourhoods.*

5 $b^{\#}$-S_0-Spaces

In this section we introduce a new separation axiom $b^{\#}$-S_0 and it is characterized with other separation axioms.

Definition 5.1 *A topological space (X,τ) is $b^{\#}$-S_0 if $b^{\#}cl(A)$ is $b^{\#}$-closed for every subset A of X.*

The example for a space which is $b^{\#}$-S_0 is given below.

Example 5.2 *Let $X = \{a,b,c\}$ and $\tau = \{\phi, \{a,c\}, \{b,c\}, \{b\}, \{c\}, X\}$. Then $b^{\#}C(X,\tau) = \{\phi, \{b\}, \{a,c\}, X\}$. It can be proved that (X,τ) is $b^{\#}$-S_0.*

The next example shows that there is a space which is not $b^{\#}$-S_0.

Example 5.3 *In the real line R with the standard topology, each open interval (a,b) is $b^{\#}$-closed where a ¡ b. All other non-empty intervals are not $b^{\#}$-closed. Let $A = \left(-\frac{1}{n}, \frac{1}{n}\right)$. Then by Example 6.6 of [6], $b^{\#}cl\{0\} = \{0\}$ is not $b^{\#}$-closed. This shows that the set of real numbers R with standard topology is not $b^{\#}$-S_0.*

Theorem 5.4 *Suppose (X,τ) is $b^{\#}$-S_0. Then (X,τ) is $b^{\#}$-T_1 if and only if $\{x\}$ is $b^{\#}$-closed for every $x \in X$.*
Proof. *Suppose (X,τ) is $b^{\#}$-T_1. Then by Theorem 2.3, $b^{\#}cl\{X\} = \{x\}$. Since (X,τ) is $b^{\#}$-S_0, it follows that $\{x\} = b^{\#}cl\{X\}$ is $b^{\#}$-closed. Now Suppose $\{x\}$ is $b^{\#}$-closed for every $x \in X$. Then $b^{\#}cl\{X\} = \{x\}$ for every $x \in X$. Again by Theorem 2.3, (X,τ) is $b^{\#}$-T_1.*

Theorem 5.5 *X is $b^{\#}$-S_0 if and only if $b^{\#}int(A)$ is $b^{\#}$-open for $A \subseteq X$.*
Proof. *Assume X is $b^{\#}$-S_0. Then $b^{\#}cl(A)$ is $b^{\#}$-closed for every subset A of X. Taking complement, $X - b^{\#}cl(A)$ is $b^{\#}$-open that implies $b^{\#}int(X-A)$ is $b^{\#}$-open. This shows $b^{\#}int(A)$ is $b^{\#}$-open for $A \subseteq X$. The proof for the converse is analog.*

Theorem 5.6 *If (X,τ) is $b^{\#}$-R_0 and $b^{\#}$-S_0 then every $b^{\#}$-open set is a union of $b^{\#}$-closed sets.*
Proof. *Suppose (X,τ) is $b^{\#}$-R_0 and $b^{\#}$-S_0. Let U be a $b^{\#}$-open set. Since (X,τ) is $b^{\#}$-R_0, $b^{\#}cl\{X\} \subseteq U$ for every $x \in U$. This implies $U = \bigcup\left\{b^{\#}cl\{X\} : x \in U\right\}$. Since (X,τ) is $b^{\#}$-S_0, $b^{\#}cl\{X\}$ is $b^{\#}$-closed for every $x \in X$. Therefore U is a union of $b^{\#}$-closed sets.*

Proposition 5.7 *If a topological space (X,τ) is $b^{\#}$-T_0, $b^{\#}$-R_0 and $b^{\#}$-S_0 space then it is $b^{\#}$-T_1.*
Proof. *Let x and y be two distinct points of X. Since X is $b^{\#}$-T_0, there exists a $b^{\#}$-open set U*

such that $x \in U$ and $y \notin U$. Since $x \in U$ and $y \notin U$ we have $b^{\#}cl\{X\} \subseteq U$ and $y \notin b^{\#}cl\{X\}$. Take $V = X - b^{\#}cl\{X\}$. This implies $y \in V$ and $x \notin V$. Since X is $b^{\#}$-S_0, $b^{\#}cl\{X\}$ is $b^{\#}$-closed and V is $b^{\#}$-open. Therefore there exist $b^{\#}$-open sets U and V containing x and y respectively, such that $y \notin U$ and $x \notin V$. This implies that X is $b^{\#}$-T_1.

Theorem 5.8 *Let (X, τ) be $b^{\#}$-S_0. Then the following are equivalent.*
(i) (X, τ) is $b^{\#}$-R_0.
(ii) For any $F \in b^{\#}C(X)$, $x \notin F$ there exists $U \in b^{\#}O(X)$ with $F \subseteq U$ and $x \notin U$.
(iii) For $F \in b^{\#}C(X)$, $x \notin F$ the condition $F \cap b^{\#}cl\{x\}$ holds.
(iv) For any $x \neq y$, either $b^{\#}cl\{x\} = b^{\#}cl\{y\}$ or $b^{\#}cl\{x\} \cap b^{\#}cl\{y\} = \phi$.

Proof. *To prove (i) $\Rightarrow$ (ii). Suppose (X, τ) is $b^{\#}$-R_0. Let $F \in b^{\#}C(X)$, $x \notin F$. Therefore $x \in X - F$. Since $X - F$ is $b^{\#}$-open and (X, τ) is $b^{\#}$-R_0, $b^{\#}cl\{x\} \subseteq X - F$. Take $U = X - b^{\#}cl\{x\}$. Since X is $b^{\#}$-S0, $b^{\#}cl\{x\}$ is $b^{\#}$-closed and U is $b^{\#}$-open. Clearly $x \notin U$ and $X - b^{\#}cl\{x\} \supseteq F$. Therefore $F \subseteq U$. Now to prove (ii) $\Rightarrow$ (iii). Suppose $F \in b^{\#}C(X)$, $x \notin F$ there exists $U \in b^{\#}O(X)$ with $F \subseteq U$ and $x \notin U$. Since $U \in b^{\#}O(X)$, $U \cap b^{\#}cl\{x\} = \phi$ and therefore $F \cap b^{\#}cl\{x\} = \phi$. Next to prove (iii) $\Rightarrow$ (iv). Suppose that for any $x \neq y$, $b^{\#}cl\{x\} \neq b^{\#}cl\{y\}$. Then there exists $z \in b^{\#}cl\{x\}$ such that $z \notin b^{\#}cl\{y\}$ and therefore $z \notin b^{\#}cl\{x\}$. Also there exists $V \in b^{\#}O(X)$ such that $y \notin V$, $z \in V$ and $x \in V$. Therefore we have $x \notin b^{\#}cl\{y\}$. By assumption, $b^{\#}cl\{x\} \cap b^{\#}cl\{y\} = \phi$. To prove (iv) $\Rightarrow$ (i). Suppose that for any $x \neq y$, $b^{\#}cl\{x\} \cap b^{\#}cl\{y\} = \phi$. Let $V \in b^{\#}O(X)$ and $x \in V$. For any $x \neq y$, $x \in b^{\#}cl\{y\}$. This implies $b^{\#}cl\{x\} \neq b^{\#}cl\{y\}$. Since $b^{\#}cl\{x\} \cap b^{\#}cl\{y\} = \phi$ for $y \in X - V$, $b^{\#}cl\{x\} \cap \left(\bigcup_{y \in X-V} b^{\#}cl\{y\}\right) = \phi$. Also $V \in b^{\#}O(X)$ and $y \in X - V$ implies $b^{\#}cl\{y\} \subseteq X - V$. Therefore $X - V = \bigcup_{y \in X-V} b^{\#}cl\{y\}$ and $(X - V) \cap (b^{\#}cl\{x\}) = \phi$. This implies $b^{\#}cl\{x\} \subseteq V$ and (X, τ) is $b^{\#}$-R_0.*

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