On expanding a function into raw moment series (i.e. expansion based on raw moment matching)

Andrej Liptaj*

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Abstract

I focus in this text on the construction of functions f_j with the delta property $C_i(f_j) = \delta_{i,j}$, where C_i are operators which associate to a function its i-th raw moment. A formal method for their construction is found, however results are divergent, from what a non-existence of such functions is conjectured. This also prevents an elegant series expansion with order-by-order moment matching. For a finite interval some partial results are presented: a method of expansion into raw moment series for finite number of moments and a "non-delta" method based on computing Legendre-expansion coefficients from moments (an already known method [1]). As by-product some coefficients formulas are found: coefficients for expanding a Hermite function into the Taylor series and coefficients for expanding into the Taylor series an element of a Fourier series (i.e. common formula for sine and cosine) thus formally merging the two (sine and cosine) Fourier sub-series into one.

1 Introduction

An approximation is often based on matching an infinite set of numbers $\{c_i\}$ which characterizes the function to be expanded (I will denote it g) by some different, well chosen function form. Let $\{C_i\}$ be a set of functionals

$$c_i = C_i(g).$$

If C_i is linear then the most efficient way of constructing an expansion is:

- Find a set of functions $\{f_j\}$ having the delta property with respect to $\{C_i\}$

$$\delta_{i,j} = C_i(f_j)$$

- Construct the approximation A_g as sum

$$A_g = \sum_i c_i f_i \equiv \sum_i C_i(g) f_i.$$

The approximation A_g has the same characterization as g:

$$\begin{aligned} C_i(A_g) &= C_i\left(\sum_j c_j f_j\right) \\ &= \sum_j c_j C_i(f_j) \\ &= \sum_j c_j \delta_{i,j} \\ &= c_i \\ &= C_i(g) \end{aligned}$$

*Institute of Physics, Bratislava, Slovak Academy of Sciences, andrej.liptaj@savba.sk
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In this text the characteristic numbers are raw moments

$$c_n(g) \equiv m_n(g) \equiv \int_a^b x^n g(x)\, dx.$$

It seems this is an interesting and active field of mathematics with many people involved in the so-called "moment problem". Here a notice should be made: in this text g can be an arbitrary function with appropriate properties (integrable, etc...) which is too loose for statistics. A statistical probability needs to be always positive and this is not guaranteed for g. Therefore the link with statistics in this text might be limited.

2 Orthonormal function approach

2.1 Interval (-1,1) and finite intervals

A finite interval in the moment computation can be seen as equivalent to a moment computed over the whole real axis for a function which is non-zero only on some limited interval. Then the recipe for constructing functions with the delta property is as follows. Starting with

$$\int_a^b x^n f_m(x)\, dx = \delta_{n,m}$$

one expands both, x^n and $f_m(x)$ into series of (some) orthonormal functions $\{P_k\}$ (in the L^2 metrics).

$$\begin{aligned} \delta_{n,m} &= \int_a^b \left[\sum_{k=0}^{\infty} a_{n,k} P_k(x)\right] \left[\sum_{l=0}^{\infty} b_{m,l} P_l(x)\right] dx \qquad (1)\\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_{n,k} b_{m,l} \int_a^b P_k(x) P_l(x)\, dx \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_{n,k} b_{m,l} \delta_{k,l} \\ &= \sum_{k=0}^{\infty} a_{n,k} b_{m,k} \end{aligned}$$

What does the last line mean? In the matrix notation it can be rewritten as

$$1 = a.b^T$$

or, in other words, if $\{a_k\}$ are the coefficients which express how x^n is built from $P_k(x)$, then the coefficients $\{b_n\}$ are their inverse (up to a transpose), i.e. they encode how $P_k(x)$ is expanded into x^n. This is usually a know information for many common cases, it is the power expansion of the orthonormal system $\{P_k\}$. The approximation of g can be written as

$$g(x) = \sum_{n=0}^{\infty} m_n f_n$$

with

$$f_n = \sum_{l=0}^{\infty} w_{l,n} P_l(x)$$

where $w_{l,n}$ are such that

$$P_n(x) = \sum_{k=0}^{\infty} w_{n,k} x^k$$

One can notice an interesting property which I would call "fractal composition": the same coefficients (with exchanged indices) are used twice in the summation

$$\begin{aligned} f_n &= \sum_{l=0}^{\infty} w_{l,n} \sum_{k=0}^{\infty} w_{l,k} x^k \\ &= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} w_{l,n} w_{l,k} x^k \end{aligned}$$

Now the bad news: it seems such summations do not converge.

2.2 Using Legendre polynomials

The best known orthonormal system on the interval $(-1,1)$ is the system of normalized Legendre polynomials

$$\mathcal{L}_n(x) = \sqrt{\frac{2n+1}{2}} P_n(x)$$

where P_n are Legendre polynomials

$$P_n(x) = \sum_{k=0}^{n} q_{n,k} x^k \tag{2}$$

with

$$q_{n,k} = 2^n \binom{n}{k} \binom{\frac{n+k-1}{2}}{n}.$$

Summation in (2) implies that also the "inverse" summation in the first term of (1) is finite. The coefficients $w_{n,k}$ take the form

$$w_{n,k} = \sqrt{\frac{2n+1}{2}} 2^n \binom{n}{k} \binom{\frac{n+k-1}{2}}{n}.$$

One might construct functions with the delta property f_n progressively, by defining the number of terms in the series

$$f_n^N = \sum_{k=0}^{N} w_{k,n} \mathcal{L}_k(x).$$

It might be easy to show (presumably at $x=1$, where the value of all Legendre polynomials is 1) that the limit $N \to \infty$ does not exist, the function values tend to infinity (in many points). The "small good news" is that f_n^N keeps the delta property for $m \le N$, i.e.

$$\int_{-1}^{1} x^m f_n^N(x)\, dx = \delta_{n,m} \text{ for } m \le N.$$

All f_n^{10} functions are shown in Fig. (1). The behavior of the f_2^N functions with increasing N is shown in a quasi-logarithmic plot $\mathrm{sig}(f_2^N) \times \ln\left(1 + \left|f_2^N\right|\right)$ in Fig. (2). It seems the functions $f_n^N(x)$ gain more and more variations as N increases and therefore one may assume that even a shape-limit for these functions does not exist.

2.3 Moment expansion in situation where order of approximation is known in advance

If one desires to match N raw moments $\{m_i\}_{i=0}^{N-1}$ of a function g, $m_i = \int_{-1}^{1} x^i g(x)\, dx$, the recipe is as follows:

- Construct functions f_n^{N-1}

  $$f_n^{N-1} = \sum_{k=0}^{N-1} w_{k,n} \mathcal{L}_k(x),$$

  where

  $$w_{k,n} = \sqrt{\frac{2k+1}{2}} 2^k \binom{k}{n} \binom{\frac{k+n-1}{2}}{k} \tag{3}$$

  and

  $$\begin{aligned}\mathcal{L}_k(x) &= \sum_{m=0}^{\infty} w_{k,m} x^m \\ &= \sqrt{\frac{2k+1}{2}} P_k(x)\end{aligned}$$

  with $P_k(x)$ being the Legendre polynomials.

- Construct the approximation $A_g(x)$

  $$A_g^{N-1}(x) = \sum_{j=0}^{N-1} m_j f_j^{N-1}$$

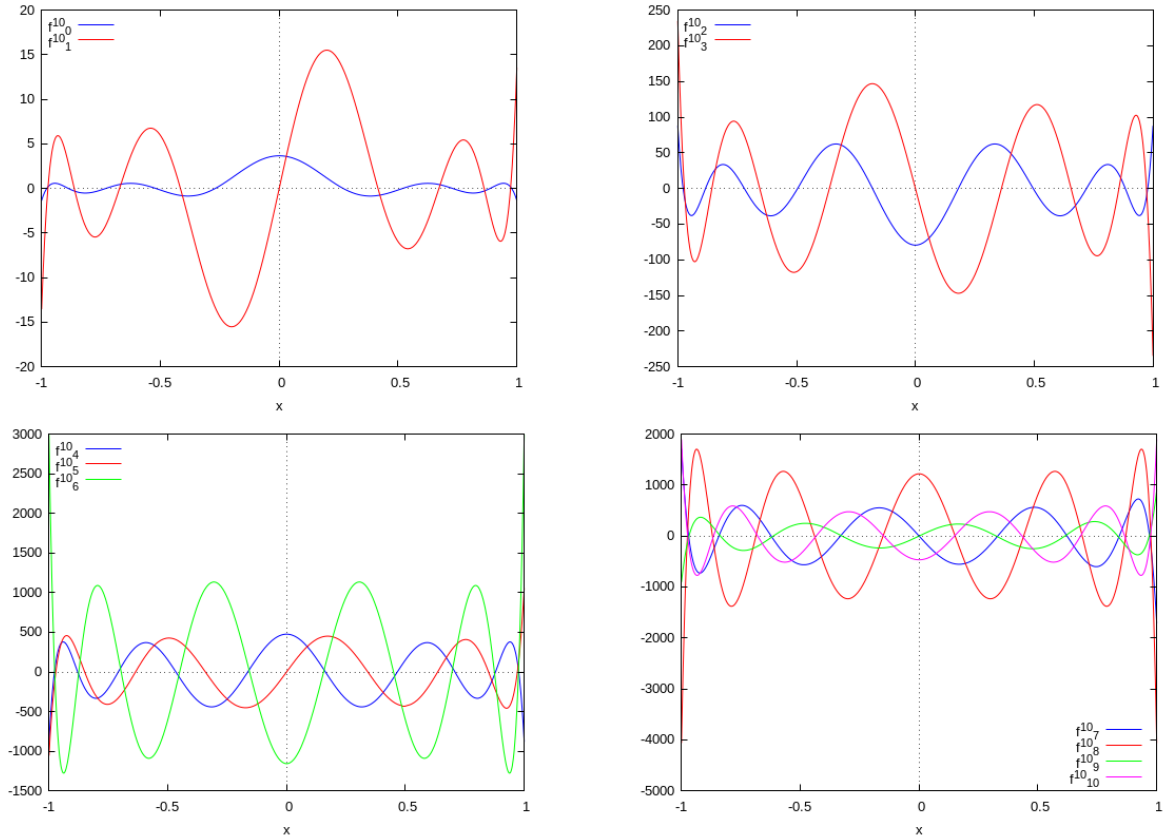

Figure 1: Plots of the f_n^{10} functions, $n = 0, \ldots, 10$.

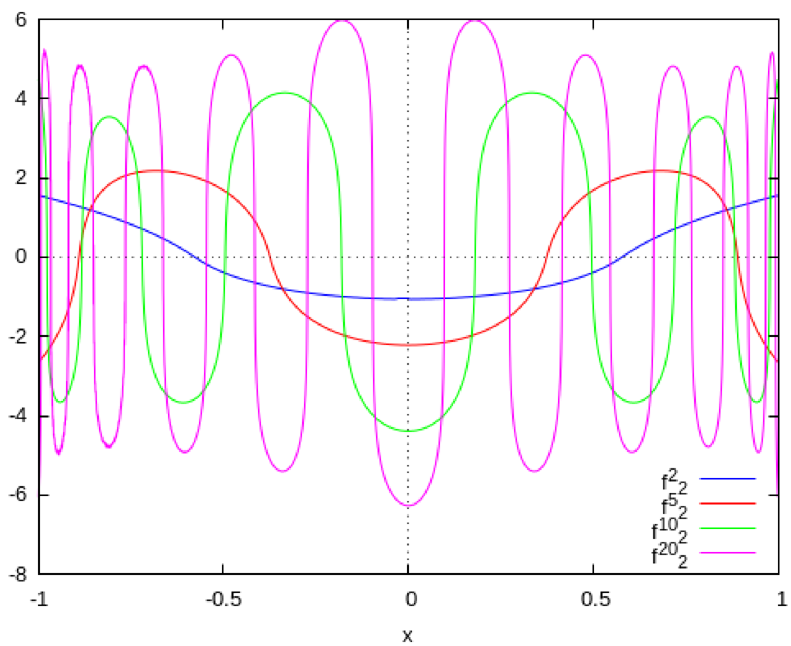

Figure 2: Behavior of the functions f_2^2, f_2^5, f_2^{10} and f_2^{20} shown in a quasi-logarithmic plot. Lines represent $\mathrm{sig}(f_2^N) \ln\left(1 + \left|f_2^N\right|\right)$.

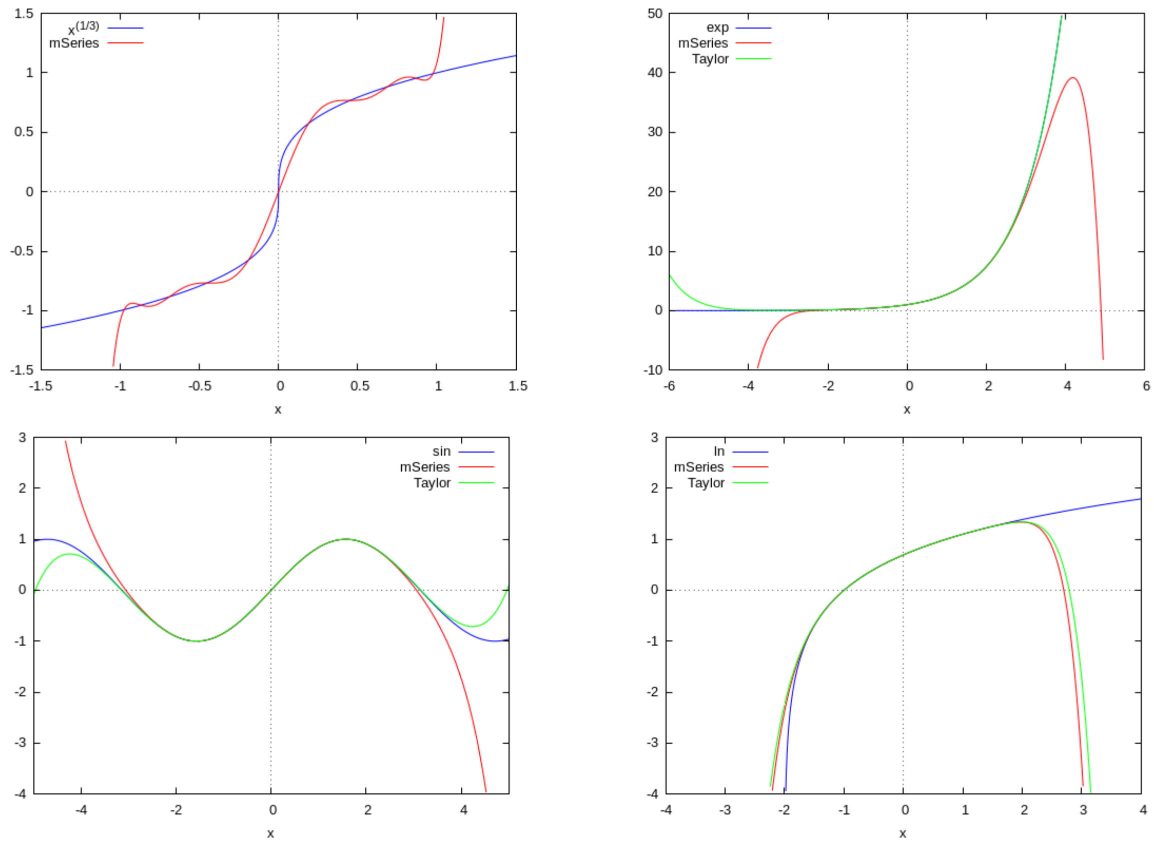

Figure 3: Functions $\sqrt[3]{x}$, $\exp(x)$, $\sin(x)$ and $\ln(x+2)$ approximated by moment and Taylor series with 11 terms.

Approximations of four functions[1] $\sqrt[3]{x}$, $\exp(x)$, $\sin(x)$ and $\ln(x+2)$ with 11 moments matched, together with Taylor series of the identical length (value and 10 derivatives matched) are shown in Fig. (3). For any finite polynomial both, moment expansion (based on Legendre polynomials) and Taylor series, give an exact approximation. Therefore I omit the x^2 function which I usually include in my other texts and replace it with $\sqrt[3]{x}$. Graphs are shown on a grater interval than $(-1,1)$, it is assumed that for many functions the approximation may hold also in some neighborhood of this interval. The method is integral-based and therefore there are hopes for convergence also in cases of non-analytic functions (e.g. $\sqrt[3]{x}$).

2.4 Non-delta approach

Numerical results suggest that for an arbitrary high (but fixed) N a valid approximation A_g^N can be constructed for many common functions. It seems the "infinite" functions $f_n \equiv f_n^\infty$ can be combined so as to give finite results. Maybe one could define some finite linear combinations of these functions

$$\mathcal{F}_n = \sum_u c_{n,u} f_u$$

[1] In my texts I usually chose $\ln(x+1)$, which is however inappropriate in the current settings: all higher moments could be computed but not the zero-order one.

and use those in expansion. Clearly, functions $\mathcal{F}_n$ loose the delta property

$$\begin{aligned} C_i\left(\mathcal{F}_n\right) &= C_i\left(\sum_u c_{n,u} f_u\right) \\ &= \sum_u c_{n,u} C_i\left(f_u\right) \\ &= \sum_u c_{n,u} \delta_{i,u} \\ &= c_{n,i} \\ &= i\text{-th moment of } \mathcal{F}_n \end{aligned}$$

Question of moment matching arises. What is the i-th moment of a series of $\mathcal{F}_n$ functions?

$$\begin{aligned} C_i\left(\sum_{n=0}^{\infty} \lambda_n \mathcal{F}_n\right) &= \sum_{n=0}^{\infty} \lambda_n C_i\left(\mathcal{F}_n\right) \\ &= \sum_{n=0}^{\infty} \lambda_n c_{n,i} \end{aligned}$$

One wishes to match this moment to a fixed moment m_i given in advance

$$m_i = \sum_{n=0}^{\infty} c_{n,i} \lambda_n.$$

This should be seen as a condition on λ_n, coefficients $c_{n,i}$ are interpreted as fixed. In matrix notation

$$m = c^T \lambda$$

so

$$\lambda = \left(c^{-1}\right)^T m.$$

Let me summarize the recipe:

- One has functions $\mathcal{F}_n$ defined as linear combinations of (infinite) functions f_u via mixing matrix $c_{n,u}$.
- One has moments m_i of a function g to be approximated.
- One computes coefficients λ_n as

$$\lambda_n = \sum_{i=0}^{\infty} \left(c^{-1}\right)_{i,n} m_i$$

- The moment expansion of g can then be written as

$$A_g\left(x\right) = \sum_{n=0}^{\infty} \lambda_n \mathcal{F}_n, \tag{4}$$

  where one needs to keep in mind that this approximation is not "order-by-order" but holds in the limit of large number of terms.

2.5 Example of a non-delta approach

A clever person might invent its own, original version of the mixing $c_{n,u}$. I will try the most obvious one where $\mathcal{F}_n$ functions are scaled Legendre polynomials. One has

$$f_n = \sum_{k=0}^{\infty} w_{k,n} \mathcal{L}_k\left(x\right)$$

or, in the matrix language

$$f = w^T \mathcal{L},$$

where the coefficient w are given by (3). So

$$\mathcal{L} = \left(w^{-1}\right)^T f$$

or

$$\mathcal{L}_n = \sum_{u=0}^{\infty} w^{-1}_{u,n} f_u$$

leading to one concrete realization of the mixing

$$c_{n,u} \equiv w^{-1}_{u,n}$$

Let me recall the computation of λ

$$\begin{aligned}\lambda_n &= \sum_{i=0}^{\infty} \left(c^{-1}\right)_{i,n} m_i \\ &= \sum_{i=0}^{\infty} w_{n,i} m_i\end{aligned}$$

This is probably not a very surprising result: we are in the standard "Legendre expansion", with moments related to the expansion coefficients in a straightforward way as shown by the following computation

$$\begin{aligned}a_n &= \int_{-1}^{1} g\left(x\right) \mathcal{L}_n\left(x\right) dx \\ &= \int_{-1}^{1} dx\, g\left(x\right) \sum_i w_{n,i} x^i \\ &= \sum_i w_{n,i} \int_{-1}^{1} g\left(x\right) x^i dx \\ &= \sum_i w_{n,i} m_i\end{aligned}$$

Well, a different choice of the mixing $c_{n,u}$ is still an open possibility. The important question for using this approach in analytic formulas for moment expansion is: *Am I able to compute (analytically) the sum $\sum_i w_{n,i} m_i$?*

This kind of expansion is already know, see [1].

2.6 Using Fourier series

I was maybe wrong claiming the Legendre polynomials as being the most famous orthonormal system on $(-1, 1)$. It might actually be the system of trigonometric functions as appearing in the Fourier series

$$\frac{1}{\sqrt{2}},\ \sin\left(n\pi x\right),\ \cos\left(n\pi x\right),\ n = 1, 2, 3, \ldots$$

The "problem" with this system is that it is usually written as two series, sine and cosine. To get the knowledge of the $w_{k,n}$ coefficients one needs to have a general expression giving the dependence of the k-th function on x^m. With some work one can succeed and merge the two series into one:

$$t_k\left(x\right) = \begin{cases} \frac{1}{\sqrt{2}} & \text{for } k = 0 \\ \sin\left(\frac{k+1}{2}\pi x\right) & \text{for } k = 1, 3, 5, \ldots \\ \cos\left(\frac{k}{2}\pi x\right) & \text{for } k = 2, 4, 6, \ldots \end{cases}.$$

Here is the general power expansion for t_k:

$$t_k\left(x\right) = \sum_{n=0}^{\infty} w_{k,n} x^n,$$

where

$$w_{k,n} = \begin{cases} \frac{1}{\sqrt{2}} \delta_{k,n} & \text{for } k = 0 \\ \mathrm{Re}\left[\frac{\left(\sqrt{\frac{1}{2}}+i\sqrt{\frac{1}{2}}\right)^{2n+(-1)^k-1} \times \left(\pi\left\lfloor\frac{k+1}{2}\right\rfloor\right)^n}{n!}\right] & \text{for } k > 0 \end{cases} \tag{5}$$

with $\lfloor\ \rfloor$ denoting the floor function.

The "general theory" tells us that we should build the functions f_n^N as

$$f_n^N(x) = \sum_{k=0}^{N} w_{k,n} t_k(x).$$

Numerical computations lead to a deception: functions f_n^N does not seem to have the delta property for any fixed N (and m), neither seem to approach this property with increasing N. For some reason the construction of "delta" functions fails, presumably it is associated with some inappropriate order changing of integral and infinite summation which appears in formulas following the expression (1). Momentum matching approximation based on the Fourier series seems to be an impasse.

The non-delta approach for Fourier series follows from

$$\begin{aligned} a_k &= \int_{-1}^{1} g(x)\, t_k(x) \\ &= \int_{-1}^{1} g(x) \sum_{n=0}^{\infty} w_{k,n} x^n \\ &= \sum_{n=0}^{\infty} w_{k,n} \int_{-1}^{1} g(x)\, x^n \\ &= \sum_{n=0}^{\infty} w_{k,n} m_n \end{aligned}$$

with $w_{k,n}$ given by (5). This approach also fails (numerical observation).

2.7 Finite interval (a,b)

On a general interval (a,b) one needs to scale everything properly. System of orthonormal functions on such interval is written as

$$\mathcal{O}_n^{(a,b)}(x) = \frac{2}{b-a}\mathcal{O}_n\left(\frac{2}{b-a}x - \frac{a+b}{b-a}\right),$$

where $\{\mathcal{O}_n\}$ represents an orthonormal system on $(-1,1)$. One needs to determine the coefficients $w_{n,q}^{(a,b)}$. A brute-force computation gives

$$\begin{aligned} \mathcal{O}_n^{(a,b)}(x) &= \frac{2}{b-a}\mathcal{O}_n\left(\frac{2}{b-a}x - \frac{a+b}{b-a}\right) \\ &= \frac{2}{b-a}\sum_{k=0}^{\infty} w_{n,k}\left(\frac{2}{b-a}x + \frac{a+b}{a-b}\right)^k \\ &= \frac{2}{b-a}\sum_{k=0}^{\infty} w_{n,k}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{2}{b-a}\right)^{k-j}\left(\frac{a+b}{a-b}\right)^j x^{k-j} \\ & \quad k-j=q, \quad j=k-q \\ &= \frac{2}{b-a}\sum_{k=0}^{\infty}\sum_{q=0}^{k} w_{n,k}\binom{k}{k-q}\left(\frac{2}{b-a}\right)^q\left(\frac{a+b}{a-b}\right)^{k-q} x^q \end{aligned}$$

If I assume the infinite sum can be re-arranged[2], then

$$\mathcal{O}_n^{(a,b)}(x) = \frac{2}{b-a}\sum_{q=0}^{\infty}\sum_{k=q}^{\infty} w_{n,k}\binom{k}{k-q}\left(\frac{2}{b-a}\right)^q\left(\frac{a+b}{a-b}\right)^{k-q} x^q$$
$$= \sum_{q=0}^{\infty}\left[\sum_{k=q}^{\infty} w_{n,k}\binom{k}{k-q}\left(\frac{2}{b-a}\right)^{q+1}\left(\frac{a+b}{b-a}\right)^{k-q}\right] x^q$$
$$= \sum_{q=0}^{\infty} w_{n,q}^{(a,b)} x^q$$

So one has

$$w_{n,q}^{(a,b)} = \sum_{k=q}^{\infty} w_{n,k}\binom{k}{k-q}\left(\frac{2}{b-a}\right)^{q+1}\left(\frac{a+b}{a-b}\right)^{k-q}$$

I did no numerical computation for a general interval (a,b), I have no numerical results supporting or dismissing the validity of the last formula.

With a system of orthonormal functions and with coefficients $w_{n,q}^{(a,b)}$, one can proceed in a way identical to the $(-1,1)$ case.

2.8 Infinite interval $(-\infty,\infty)$: unsuccessful attempt

One well know system of orthonormal function on the interval $(-\infty,\infty)$ are Hermite functions

$$\psi_n(x) = \left(2^n n!\sqrt{\pi}\right)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} H_n(x),$$

where H_n are Hermite polynomials

$$H_n = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}.$$

The tricky thing is to find the coefficients $w_{n,k}$ appearing in power expansion of the Hermite functions

$$\psi_n(x) = \sum_{k=0}^{\infty} w_{n,k}x^k.$$

It is a technical task and the result is

$$w_{n,k} = \left(2^n n!\sqrt{\pi}\right)^{-\frac{1}{2}}\frac{K_{n,k}}{\left(2\left\lfloor\frac{k}{2}\right\rfloor\right)!!} \tag{6}$$
$$K_{n,k} = \sum_{j=k-2\left\lfloor\frac{k}{2}\right\rfloor}^{k-\left\lfloor\frac{k}{2}\right\rfloor}(-1)^{\left\lfloor j-\frac{k}{2}\right\rfloor}\omega_{k,j}h_{n,2\left(j+\left\lfloor\frac{k}{2}\right\rfloor\right)-k}$$
$$\omega_{k,j} = \frac{\left(2\left\lfloor\frac{k}{2}\right\rfloor\right)!!}{\left[2\left(k-j-\left\lfloor\frac{k}{2}\right\rfloor\right)\right]!!}$$
$$h_{n,k} = \begin{cases}0 & \text{for } k>n\\ H_{n,k} & \text{else}\end{cases}$$
$$H_{n,k} = \begin{cases}0 & \text{if } n \text{ and } k \text{ have a different parity}\\ \frac{n!(-1)^{\frac{(n-k)}{2}}2^k}{k!\left(\frac{n-k}{2}\right)!} & \text{if } n \text{ and } k \text{ have the same parity}\end{cases}$$

Now one can proceed mechanically, functions with the delta property should be built as

$$f_k(x) = \sum_{n=0}^{\infty} w_{n,k}\psi_n.$$

[2] In an infinite triangle with elements indexed by k, q, $q<k$ rows were summed. Now summing the same triangle by columns.

Numerical computation show that this construction does not work. The cut-off functions $f_k^N(x)$ seem not to exhibit any delta property and seem not to approach the delta property with increasing N.

I also tested the "non-delta" approach on the $(-\infty, \infty)$ interval using the standard normal distribution[3]

$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

The method is identical to the non-delta approach on a finite interval (4) where I chose ψ_n as functions in the approximation series

$$A_g(x) = \sum_{n=0}^{\infty} \lambda_n \psi_n.$$

Coefficients λ_n are given by

$$\lambda_n = \sum_{i=0}^{\infty} w_{n,i} m_i,$$

with m_i moments to be matched and $w_{n,i}$ given by (6). Attempt leads to a failure: approximation seems to diverge.

3 Conclusion, Summary

Ideas and results presented in this text are rather disappointing because it seems that functions with the delta property (with respect to the moment expansion) cannot be found. Some practical recipes are proposed for finite interval: Given a sequence of numbers $\{m_i\}$ (interpreted as moments) one can build approximation in two ways:

- One fixes the number of moments to match to a finite number $\{m_i\}_{i<N}$, then constructs cut-off functions with the delta property and builds the series as described in section (2.3).
- One computes (if clever and lucky) Legendre-expansion coefficients from moments and performs standard expansion into the orthogonal basis of the Legendre polynomials, see sections (2.4,2.5).

Besides, I constructed two possibly interesting coefficients formulas (5) and (6) which seem not to be common in the literature. The first one allows to formally merge sine and cosine series of a Fourier series into one series, the second one represents the Taylor expansion of the Hermite functions.

References

[1] G. Talenti, "Recovering a function from a finite number of moments", Inverse Problems 3(3), (1987) 501.

[3] One of rare common functions with all moments defined and easily computable on the whole real axis.