Question 447 : Some trigonometric formulas

for the argument $\frac{2n\pi}{13}, n=1,2,3,4,5,6$.

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abstract

This note presents continued radicals and trigonometric formulas for the argument $2n\pi/13, n=1,2,3,4,5,6$.

1. Introduction. Some Continued Radicals

Let $A=\frac{13+\sqrt{13}}{6}, B=\frac{26+5\sqrt{13}}{27}$. then

$$2\cos\frac{4\pi}{13}=-\frac{1+\sqrt{13}}{6}+\sqrt[3]{B+A\sqrt[3]{B+A\sqrt[3]{B+\ldots}}}=$$

$$=-\frac{1+\sqrt{13}}{6}+\sqrt[3]{\frac{26+5\sqrt{13}}{27}+\frac{13+\sqrt{13}}{6}\sqrt[3]{\frac{26+5\sqrt{13}}{27}+\ldots}} \quad (1)$$

$$2\cos\frac{10\pi}{13}=-\frac{1+\sqrt{13}}{6}-\left\{\frac{B}{A}+\frac{1}{A}\left(\frac{B}{A}+\frac{1}{A}\left(\frac{B}{A}+\ldots\right)^3\right)^3\right\}=$$

$$=-\frac{1+\sqrt{13}}{6}-\left\{\frac{7+\sqrt{13}}{18}+\frac{6}{13+\sqrt{13}}\left(\frac{7+\sqrt{13}}{18}+\frac{6}{13+\sqrt{13}}\left(\frac{7+\sqrt{13}}{18}+\ldots\right)^3\right)^3\right\} \quad (2)$$

$$2\cos\frac{12\pi}{13}=-\frac{1+\sqrt{13}}{6}-\sqrt{A-\frac{B}{\sqrt{A-\frac{B}{\sqrt{A-\ldots}}}}}=$$

$$=-\frac{1+\sqrt{13}}{6}-\sqrt{\frac{13+\sqrt{13}}{6}-\frac{\left(26+5\sqrt{13}\right)/27}{\sqrt{\frac{13+\sqrt{13}}{6}-\frac{\left(26+5\sqrt{13}\right)/27}{\sqrt{\frac{13+\sqrt{13}}{6}-\ldots}}}}} \quad (3)$$

Let $C=\frac{13-\sqrt{13}}{6}, D=\frac{26-5\sqrt{13}}{27}$. then

$$2\cos\frac{2\pi}{13}=\frac{\sqrt{13}-1}{6}+\sqrt[3]{D+C\sqrt[3]{D+C\sqrt[3]{D+...}}}=$$
$$=\frac{\sqrt{13}-1}{6}+\sqrt[3]{\frac{26-5\sqrt{13}}{27}+\frac{13-\sqrt{13}}{6}\sqrt[3]{\frac{26-5\sqrt{13}}{27}+...}} \tag{4}$$

$$2\cos\frac{6\pi}{13}=\frac{\sqrt{13}-1}{6}-\left\{\frac{D}{C}+\frac{1}{C}\left(\frac{D}{C}+\frac{1}{C}\left(\frac{D}{C}+...\right)^3\right)^3\right\}=$$
$$=\frac{\sqrt{13}-1}{6}-\left\{\frac{7-\sqrt{13}}{18}+\frac{6}{13-\sqrt{13}}\left(\frac{7-\sqrt{13}}{18}+\frac{6}{13-\sqrt{13}}\left(\frac{7-\sqrt{13}}{18}+...\right)^3\right)^3\right\} \tag{5}$$

$$2\cos\frac{8\pi}{13}=\frac{\sqrt{13}-1}{6}-\sqrt{C-\frac{D}{\sqrt{C-\frac{D}{\sqrt{C-...}}}}}=$$
$$=\frac{\sqrt{13}-1}{6}-\sqrt{\frac{13-\sqrt{13}}{6}-\frac{\left(26-5\sqrt{13}\right)/27}{\sqrt{\frac{13-\sqrt{13}}{6}-\frac{\left(26-5\sqrt{13}\right)/27}{\sqrt{\frac{13-\sqrt{13}}{6}-...}}}}} \tag{6}$$

2. π as sum of arctangents

$$\pi=4\tan^{-1}\left(\frac{27-5\sqrt{13}}{13+5\sqrt{13}}\right)+4\tan^{-1}\left(\cos\frac{2\pi}{13}\right)+4\tan^{-1}\left(\cos\frac{6\pi}{13}\right)+4\tan^{-1}\left(\cos\frac{8\pi}{13}\right) \tag{7}$$

$$\pi=4\tan^{-1}\left(\left(2\cos\frac{2\pi}{13}\right)^{-1}\right)-4\tan^{-1}\left(2\cos\frac{6\pi}{13}\right)$$
$$-4\tan^{-1}\left(2\cos\frac{8\pi}{13}\right)-4\tan^{-1}\left(\frac{4}{\sqrt{13}}-1\right) \tag{8}$$

$$\pi = 2\tan^{-1}\left(\frac{20}{7+5\sqrt{13}}\right) - 2\tan^{-1}\left(\cos\frac{4\pi}{13}\right) - 2\tan^{-1}\left(\cos\frac{10\pi}{13}\right) - 2\tan^{-1}\left(\cos\frac{12\pi}{13}\right) \quad (9)$$

$$\begin{aligned}\pi = 4\tan^{-1}\left(\frac{\sqrt{13}}{4+\sqrt{13}}\right) - 4\tan^{-1}\left(\left(2\cos\frac{4\pi}{13}\right)^{-1}\right) \\ - 4\tan^{-1}\left(\left(2\cos\frac{10\pi}{13}\right)^{-1}\right) - 4\tan^{-1}\left(\left(2\cos\frac{12\pi}{13}\right)^{-1}\right)\end{aligned} \quad (10)$$

3. Trigonometric Identities

$$\cos\frac{2\pi}{13} + \cos\frac{6\pi}{13} + \cos\frac{8\pi}{13} = \frac{\sqrt{13}-1}{4} \quad (11)$$

$$\cos\frac{4\pi}{13} + \cos\frac{10\pi}{13} + \cos\frac{12\pi}{13} = -\frac{\sqrt{13}+1}{4} \quad (12)$$

$$\sum_{n=1}^{6}\cos\frac{2n\pi}{13} = -\frac{1}{2} \quad (13)$$

$$\left(2\cos\frac{2\pi}{13} - \frac{\sqrt{13}-1}{6}\right)\left(2\cos\frac{6\pi}{13} - \frac{\sqrt{13}-1}{6}\right)\left(2\cos\frac{8\pi}{13} - \frac{\sqrt{13}-1}{6}\right) = \frac{26-5\sqrt{13}}{27} \quad (14)$$

$$\left(\frac{\sqrt{13}+1}{6} + 2\cos\frac{4\pi}{13}\right)\left(\frac{\sqrt{13}+1}{6} + 2\cos\frac{10\pi}{13}\right)\left(\frac{\sqrt{13}+1}{6} + 2\cos\frac{12\pi}{13}\right) = \frac{26+5\sqrt{13}}{27} \quad (15)$$

$$\cos\frac{2\pi}{13}\cos\frac{6\pi}{13}\cos\frac{8\pi}{13} = -\frac{\sqrt{13}-3}{16} \quad (16)$$

$$\cos\frac{4\pi}{13}\cos\frac{10\pi}{13}\cos\frac{12\pi}{13} = \frac{\sqrt{13}+3}{16} \quad (17)$$

4. Recurrences

$$x_{n+6} = 5x_{n+5} - 5x_{n+4} - 6x_{n+3} + 7x_{n+2} + 2x_{n+1} - x_n \quad , n \in \mathbb{N} \quad (18)$$

$$x_n = \{1, 5, 20, 69, 222, 682, 2035, \ldots\} \quad (19)$$

$$\lim_{n\to\infty}\left(\frac{x_{n+1}}{x_n} - 1\right) = 2\cos\frac{2\pi}{13} \quad (20)$$

$$x_{n+6} = 6x_{n+5} + 11x_{n+4} - 6x_{n+3} - 15x_{n+2} - 7x_{n+1} - x_n \quad , n \in \mathbb{N} \tag{21}$$

$$x_n = \{1, 6, 47, 342, 2518, 18491, 135844, \ldots\} \tag{22}$$

$$\lim_{n\to\infty}\left(\frac{x_n}{x_{n+1}} + 1\right) = 2\cos\frac{4\pi}{13} \tag{23}$$

$$x_{n+6} = 3x_{n+5} + 6x_{n+4} - 4x_{n+3} - 5x_{n+2} + x_{n+1} + x_n \quad , n \in \mathbb{N} \tag{24}$$

$$x_n = \{1, 3, 15, 59, 250, 1030, 4283, \ldots\} \tag{25}$$

$$\lim_{n\to\infty}\frac{x_n}{x_{n+1}} = 2\cos\frac{6\pi}{13} \tag{26}$$

$$x_{n+6} = 2x_{n+5} + 7x_{n+4} - 6x_{n+3} - 5x_{n+2} + 5x_{n+1} - x_n \quad , n \in \mathbb{N} \tag{27}$$

$$x_n = \{1, 2, 11, 30, 120, 379, 1372, \ldots\} \tag{28}$$

$$\lim_{n\to\infty}\left(\frac{x_n}{x_{n+1}} - 1\right) = 2\cos\frac{8\pi}{13} \tag{29}$$

$$x_{n+6} = 336x_{n+5} - 84x_{n+4} - 1216x_{n+3} + 1360x_{n+2} - 512x_{n+1} + 64x_n \quad , n \in \mathbb{N} \tag{30}$$

$$x_n = \{1, 336, 112812, 37875392, 12716248288, 4269341168896, 1433384564668544, \ldots\} \tag{31}$$

$$\lim_{n\to\infty}\left(\frac{x_n}{x_{n+1}} - \frac{3}{2}\right) = 2\cos\frac{10\pi}{13} \tag{32}$$

$$x_{n+6} = 21x_{n+5} - 70x_{n+4} + 84x_{n+3} - 45x_{n+2} + 11x_{n+1} - x_n \quad , n \in \mathbb{N} \tag{33}$$

$$x_n = \{1, 21, 371, 6405, 110254, 1897214, 32645269, \ldots\} \tag{34}$$

$$\lim_{n\to\infty}\left(\frac{x_n}{x_{n+1}} - 2\right) = 2\cos\frac{12\pi}{13} \tag{35}$$

References

1. B.C. Berndt, Ramanujan's Notebooks,Part IV,Springer-Verlag,New York, 1994.
2. S. Ramanujan, Collected Papers, Chelsea, New York,1962.