# A Quantum Physics Jeopardy Problem of a One-Bound-State Double Dirac Delta Potential with a Centrifugal, Angular-Momentum-Like Tail

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15 April 2018

**Abstract**

Starting from a simple, one-parameter, symmetric wave function, we derive an attractive double Dirac delta potential with a centrifugal, angular-momentum-like tale, which has only one bound state, the energy of which can be set to zero.

**Keywords**: rational wave functions, one-bound-state potentials, double Dirac delta potential, centrifugal tail, zero-energy states, quantum physics jeopardy

## Contents

## 1. Introduction

In [1], Van Heuvelen and Maloney describe a new, reverse type of physics problems in which the answer is given and the question is asked. Such problems, which require from the solver to follow a working-backward approach, are called physics Jeopardy problems (the name was taken from a game show called Jeopardy).

In quantum physics Jeopardy problems, wave functions describing energy eigenstates are given and the respective potentials are asked [2].

Herein, we use a simple, one-parameter, symmetric wave function to derive an attractive, equal-coupling, double Dirac delta potential, which vanishes in the interior region (between the delta functions) and has a centrifugal, angular-momentum-like tail in the exterior region (outside the delta functions). The potential admits only one bound state, the energy of which can be chosen zero.

For a detailed study of the double Dirac delta potential, the reader may refer to [3] and to references therein, while in [4] the reader can find a detailed analysis of the Dirac delta potentials as pedagogical and physical models in quantum physics.

## 2. The wave function

We consider the wave function

$$\psi\left(\tilde{x};s\right)=\frac{A}{\left(\left|\tilde{x}-\tilde{x}_1\right|+\left|\tilde{x}+\tilde{x}_1\right|\right)^s} \qquad (1)$$

where $s>3/2$ [5], $\tilde{x}_1>0$, $A$ the normalization constant, and $\tilde{x}=x/x_0$, with $x$ being the position and $x_0>0$ a length scale.

Since $\tilde{x}_1\neq 0$, the expression $\left|\tilde{x}-\tilde{x}_1\right|+\left|\tilde{x}+\tilde{x}_1\right|$ is strictly positive. Then, the wave function (1) has no singularities, it is everywhere continuous and finite.

Also, we have

$$\psi\left(-\tilde{x};s\right)=\frac{A}{\left(\left|-\tilde{x}-\tilde{x}_1\right|+\left|-\tilde{x}+\tilde{x}_1\right|\right)^s}=\frac{A}{\left(\left|-\left(\tilde{x}+\tilde{x}_1\right)\right|+\left|-\left(\tilde{x}-\tilde{x}_1\right)\right|\right)^s}\overset{|-x|=|x|}{=}\frac{A}{\left(\left|\tilde{x}+\tilde{x}_1\right|+\left|\tilde{x}-\tilde{x}_1\right|\right)^s}$$

Thus

$$\psi\left(-\tilde{x};s\right)=\psi\left(\tilde{x};s\right)$$

That is, $\psi(\tilde{x};s)$ is symmetric (i.e. of even parity).

In the region $|\tilde{x}| > \tilde{x}_1$, we have

i. If $\tilde{x} < -\tilde{x}_1$, then $\tilde{x} + \tilde{x}_1 < 0$ and since $\tilde{x}_1 > 0$, $\tilde{x} - \tilde{x}_1 < \tilde{x} + \tilde{x}_1 < 0$.

Thus $|\tilde{x} - \tilde{x}_1| = -(\tilde{x} - \tilde{x}_1)$ and $|\tilde{x} + \tilde{x}_1| = -(\tilde{x} + \tilde{x}_1)$.

Then (1) becomes

$$\psi(\tilde{x};s) = \frac{A}{\left(-(\tilde{x} - \tilde{x}_1) - (\tilde{x} + \tilde{x}_1)\right)^s} = \frac{A}{(-2\tilde{x})^s} = \frac{A}{2^s(-\tilde{x})^s}$$

That is

$$\psi(\tilde{x};s) = \frac{A}{2^s(-\tilde{x})^s} \tag{2}$$

ii. If $\tilde{x} > \tilde{x}_1$, then $\tilde{x} - \tilde{x}_1 > 0$ and since $\tilde{x}_1 > 0$, $\tilde{x} + \tilde{x}_1 > \tilde{x} - \tilde{x}_1 > 0$.

Thus $|\tilde{x} - \tilde{x}_1| = \tilde{x} - \tilde{x}_1$ and $|\tilde{x} + \tilde{x}_1| = \tilde{x} + \tilde{x}_1$.

Then (1) becomes

$$\psi(\tilde{x};s) = \frac{A}{(\tilde{x} - \tilde{x}_1 + \tilde{x} + \tilde{x}_1)^s} = \frac{A}{(2\tilde{x})^s} = \frac{A}{2^s\tilde{x}^s}$$

That is

$$\psi(\tilde{x};s) = \frac{A}{2^s\tilde{x}^s} \tag{3}$$

Combining (2) and (3), we obtain

$$\psi(\tilde{x};s) = \frac{A}{2^s|\tilde{x}|^s} \tag{4}$$

for $|\tilde{x}| > \tilde{x}_1$.

In the region $|\tilde{x}| \le \tilde{x}_1$, we have

$-\tilde{x}_1 \le \tilde{x} \le \tilde{x}_1$, thus $\tilde{x} - \tilde{x}_1 \le 0$ and $\tilde{x} + \tilde{x}_1 \ge 0$.

Then $|\tilde{x} - \tilde{x}_1| = -(\tilde{x} - \tilde{x}_1)$ and $|\tilde{x} + \tilde{x}_1| = \tilde{x} + \tilde{x}_1$.

Thus (1) becomes

$$\psi(\tilde{x};s) = \frac{A}{\left(-(\tilde{x}-\tilde{x}_1)+\tilde{x}+\tilde{x}_1\right)^s} = \frac{A}{(2\tilde{x}_1)^s}$$

That is

$$\psi(\tilde{x};s) = \frac{A}{(2\tilde{x}_1)^s} \tag{5}$$

**The wave function is constant in the region $|\tilde{x}| \le \tilde{x}_1$.**

Combining (4) and (5), we write the wave function as

$$\psi(\tilde{x};s) = \begin{cases} \dfrac{A}{(2\tilde{x}_1)^s}, & |\tilde{x}| \le \tilde{x}_1 \\ \dfrac{A}{2^s|\tilde{x}|^s}, & |\tilde{x}| > \tilde{x}_1 \end{cases} \tag{6}$$

Since it is continuous, $\psi(\tilde{x};s)$ is Riemann integrable. Also, $\psi(\tilde{x};s)$ is square integrable, as it decays as $1/|\tilde{x}|^s$, with $s > 3/2$.

The normalization constant $A$ can be easily calculated by applying the normalization condition, i.e.

$$\int_{-\infty}^{\infty} dx |\psi(x)|^2 = 1$$

Using that $\tilde{x} = x/x_0$, we obtain $dx = x_0 d\tilde{x}$, and since $x_0 > 0$, the previous integral is written as

$$x_0 \int_{-\infty}^{\infty} d\tilde{x} |\psi(\tilde{x})|^2 = 1$$

or

$$2x_0 \int_0^{\infty} d\tilde{x} |\psi(\tilde{x})|^2 = 1 \tag{7}$$

since $\psi(\tilde{x})$ is symmetric.

Using (6), the integral in the left-hand side of (7) is written as

$$\int_0^\infty d\tilde{x}\left|\psi\left(\tilde{x}\right)\right|^2 = \int_0^{\tilde{x}_1} d\tilde{x}\frac{|A|^2}{\left(2\tilde{x}_1\right)^{2s}} + \int_{\tilde{x}_1}^\infty d\tilde{x}\frac{|A|^2}{2^{2s}\tilde{x}^{2s}} = \frac{|A|^2}{\left(2\tilde{x}_1\right)^{2s}}\tilde{x}_1 + \frac{|A|^2}{2^{2s}}\left(-\frac{1}{2s-1}\frac{1}{\tilde{x}^{2s-1}}\Bigg|_{\tilde{x}_1}^\infty\right) =$$

$$\overset{2s-1>0}{\overbrace{=}} \frac{|A|^2}{\left(2\tilde{x}_1\right)^{2s}}\tilde{x}_1 + \frac{|A|^2}{2^{2s}}\frac{1}{2s-1}\frac{1}{\tilde{x}_1^{2s-1}} = \frac{|A|^2}{2^{2s}\tilde{x}_1^{2s-1}} + \frac{1}{2s-1}\frac{|A|^2}{2^{2s}\tilde{x}_1^{2s-1}} = \left(1+\frac{1}{2s-1}\right)\frac{|A|^2}{2^{2s}\tilde{x}_1^{2s-1}} =$$

$$= \frac{2s|A|^2}{(2s-1)2^{2s}\tilde{x}_1^{2s-1}} = \frac{s|A|^2}{(2s-1)\left(2\tilde{x}_1\right)^{2s-1}}$$

That is

$$\int_0^\infty d\tilde{x}\left|\psi\left(\tilde{x}\right)\right|^2 = \frac{s|A|^2}{(2s-1)\left(2\tilde{x}_1\right)^{2s-1}}$$

Substituting into (7) yields

$$\frac{2x_0 s|A|^2}{(2s-1)\left(2\tilde{x}_1\right)^{2s-1}} = 1 \Rightarrow |A|^2 = \frac{(2s-1)\left(2\tilde{x}_1\right)^{2s-1}}{2x_0 s} = \frac{(2s-1)\left(2\tilde{x}_1\right)^{2s}}{4x_0\tilde{x}_1 s}$$

Thus

$$|A| = \frac{\left(2\tilde{x}_1\right)^s}{2}\sqrt{\frac{2s-1}{x_0\tilde{x}_1 s}}$$

Then, up to a constant phase, the normalization constant is

$$A = \frac{\left(2\tilde{x}_1\right)^s}{2}\sqrt{\frac{2s-1}{x_0\tilde{x}_1 s}} \qquad (8)$$

By means of (8), (6) becomes

$$\psi\left(\tilde{x};s\right) = \begin{cases} \frac{1}{2}\sqrt{\frac{2s-1}{x_0\tilde{x}_1 s}}, & |\tilde{x}| \le \tilde{x}_1 \\ \frac{1}{2}\sqrt{\frac{2s-1}{x_0\tilde{x}_1 s}}\left(\frac{\tilde{x}_1}{|\tilde{x}|}\right)^s, & |\tilde{x}| > \tilde{x}_1 \end{cases} \qquad (9)$$

Using (9), we calculate the wave function derivative, with respect to $\tilde{x}$.

In the region $|\tilde{x}| < \tilde{x}_1$, the wave function is constant, thus its derivative vanishes.

In the region $\tilde{x} < -\tilde{x}_1$, the wave function is, from (9),

$$\psi\left(\tilde{x};s\right)=\frac{1}{2}\sqrt{\frac{2s-1}{x_0\tilde{x}_1 s}}\left(\frac{\tilde{x}_1}{-\tilde{x}}\right)^s$$

Thus

$$\psi'\left(\tilde{x};s\right)=\frac{1}{2}\sqrt{\frac{2s-1}{x_0\tilde{x}_1 s}}\tilde{x}_1^{\,s}\frac{(-s)(-1)}{(-\tilde{x})^{s+1}}=\frac{1}{2}\sqrt{\frac{s^2(2s-1)}{x_0\tilde{x}_1 s}}\frac{\tilde{x}_1^{\,s}}{(-\tilde{x})^{s+1}}=\frac{1}{2}\sqrt{\frac{s(2s-1)}{x_0\tilde{x}_1}}\frac{1}{\tilde{x}_1}\left(\frac{\tilde{x}_1}{-\tilde{x}}\right)^{s+1}=$$
$$=\frac{1}{2}\sqrt{\frac{s(2s-1)}{x_0\tilde{x}_1^{\,3}}}\left(\frac{\tilde{x}_1}{-\tilde{x}}\right)^{s+1}$$

In the region $\tilde{x}>\tilde{x}_1$, the wave function is, from (9),

$$\psi\left(\tilde{x};s\right)=\frac{1}{2}\sqrt{\frac{2s-1}{x_0\tilde{x}_1 s}}\left(\frac{\tilde{x}_1}{\tilde{x}}\right)^s$$

Then

$$\psi'\left(\tilde{x};s\right)=\frac{1}{2}\sqrt{\frac{2s-1}{x_0\tilde{x}_1 s}}\tilde{x}_1^{\,s}\frac{-s}{\tilde{x}^{s+1}}=-\frac{1}{2}\sqrt{\frac{s(2s-1)}{x_0\tilde{x}_1^{\,3}}}\left(\frac{\tilde{x}_1}{\tilde{x}}\right)^{s+1}$$

Thus, the wave function derivative is

$$\psi'\left(\tilde{x};s\right)=\begin{cases} 0,\ |\tilde{x}|<\tilde{x}_1 \\ \frac{1}{2}\sqrt{\frac{s(2s-1)}{x_0\tilde{x}_1^{\,3}}}\left(\frac{\tilde{x}_1}{-\tilde{x}}\right)^{s+1},\ \tilde{x}<-\tilde{x}_1 \\ -\frac{1}{2}\sqrt{\frac{s(2s-1)}{x_0\tilde{x}_1^{\,3}}}\left(\frac{\tilde{x}_1}{\tilde{x}}\right)^{s+1},\ \tilde{x}>\tilde{x}_1 \end{cases} \qquad (10)$$

$\psi'\left(\tilde{x};s\right)$ is odd, as expected, since $\psi\left(\tilde{x};s\right)$ is even.

At $\pm\tilde{x}_1$, $\psi'\left(\tilde{x};s\right)$ is discontinuous, since from (10) we obtain

$$\psi'\left(-\tilde{x}_1^{\,-};s\right)=\frac{1}{2}\sqrt{\frac{s(2s-1)}{x_0\tilde{x}_1^{\,3}}},\ \psi'\left(-\tilde{x}_1^{\,+};s\right)=0$$

and

$$\psi'\left(\tilde{x}_1^{\,-};s\right)=0,\ \psi'\left(\tilde{x}_1^{\,+};s\right)=-\frac{1}{2}\sqrt{\frac{s(2s-1)}{x_0\tilde{x}_1^{\,3}}}$$

Thus, the wave function derivative has at $-\tilde{x}_1$ and $\tilde{x}_1$ finite discontinuities, which are equal, since

$$\psi'\left(-\tilde{x}_1^{\,+};s\right)-\psi'\left(-\tilde{x}_1^{\,-};s\right)=-\frac{1}{2}\sqrt{\frac{s(2s-1)}{x_0\tilde{x}_1^{\,3}}} \tag{11}$$

$$\psi'\left(\tilde{x}_1^{\,+};s\right)-\psi'\left(\tilde{x}_1^{\,-};s\right)=-\frac{1}{2}\sqrt{\frac{s(2s-1)}{x_0\tilde{x}_1^{\,3}}}$$

To summarize, the wave function $\psi(\tilde{x};s)$ is continuous, while its first derivative has equal discontinuities at $\pm\tilde{x}_1$.

## 3. The potential

Since $\psi(\tilde{x};s)$ has no zeros[1], it can be the ground-state wave function of a potential consisting of a term having, at most, finite discontinuities, and of a sum of two Dirac delta functions with one of them acting at $-\tilde{x}_1$ and the other at $\tilde{x}_1$ (see appendix).

1. By zeros, we mean real zeros.

The first term of the potential, which has, at most, finite discontinuities, does not induce discontinuities in the wave function derivative [6, 7]. We'll refer to this term as the regular part of the potential.

A finite discontinuity in the wave function derivative is induced by a Dirac delta potential [7]. Particularly, a delta potential $\lambda\delta(x)$ induces in the wave function derivative a finite discontinuity at zero, which is given by [7]

$$\psi_+'(0)-\psi_-'(0)=\frac{2m\lambda\psi(0)}{\hbar^2} \tag{12}$$

where the prime denotes differentiation with respect to the position $x$.

Since $\tilde{x}=x/x_0$, we have

$$\frac{d}{dx}=\frac{1}{x_0}\frac{d}{d\tilde{x}}$$

and (12) is written as

$$\frac{1}{x_0}\left(\psi_+'(0)-\psi_-'(0)\right)=\frac{2m\lambda\psi(0)}{\hbar^2} \tag{13}$$

where now the prime denotes differentiation with respect to $\tilde{x}$.

Then, from (13), the finite discontinuity in $\psi'(\tilde{x};s)$ at $-\tilde{x}_1$ is induced by a delta potential $\lambda_1\delta(x+x_1)$ and is given by

$$\frac{1}{x_0}\left(\psi'\left(-\tilde{x}_1^{\,+};s\right)-\psi'\left(-\tilde{x}_1^{\,-};s\right)\right)=\frac{2m\lambda_1\psi(-\tilde{x}_1;s)}{\hbar^2}$$

Using (11) and that

$$\psi(-\tilde{x}_1;s)=\frac{1}{2}\sqrt{\frac{2s-1}{x_0\tilde{x}_1 s}},$$

as given by (9), the above discontinuity condition is written as

$$\frac{1}{x_0}\left(-\frac{1}{2}\sqrt{\frac{s(2s-1)}{x_0\tilde{x}_1^{\,3}}}\right)=\frac{2m\lambda_1\frac{1}{2}\sqrt{\frac{2s-1}{x_0\tilde{x}_1 s}}}{\hbar^2}$$

Solving the last equation for the coupling $\lambda_1$ gives, after a little algebra,

$$\lambda_1=-\frac{s\hbar^2}{2mx_0\tilde{x}_1} \tag{14}$$

The delta potential acting at $-\tilde{x}_1$ is thus attractive.

> Let us do a dimensional check on (14).
>
> Since $[\hbar]=PL$, $[x_0]=L$, and $[\tilde{x}_1]=1$ (dimensionless), we have
>
> $$[\lambda_1]=\frac{(PL)^2}{ML}=\frac{P^2}{M}L=EL$$
>
> Then, since $\int_{-\infty}^{\infty}dx\delta(x)=1$, we have $L[\delta(x)]=1$, thus $[\delta(x)]=L^{-1}$, and then $[\lambda_1\delta(x)]=E$, and we are ok.

The wave function is symmetric, thus $\psi(-\tilde{x}_1;s)=\psi(\tilde{x}_1;s)$, and also, the discontinuities in the wave function derivative at $\pm\tilde{x}_1$ are equal. Then, the discontinuity in the wave function derivative at $\tilde{x}_1$ is induced by a delta potential acting at $\tilde{x}_1$, which has the same coupling as the delta potential acting at $-\tilde{x}_1$, i.e. it is induced by the delta potential $\lambda_1\delta(x-x_1)$.

The sum of the two delta potentials is then

$$V_{delta}(x;s)=-\frac{s\hbar^2}{2mx_0\tilde{x}_1}\left(\delta(x+x_1)+\delta(x-x_1)\right) \tag{15}$$

Using that $\delta(ax)=\delta(x)/|a|$, we have, since $x_0>0$,

$$\delta(\tilde{x})=\delta\left(\frac{x}{x_0}\right)=x_0\delta(x)$$

That is

$$\delta(x)=\frac{\delta(\tilde{x})}{x_0}$$

Then

$$\delta(x\pm x_1)=\frac{\delta(\tilde{x}\pm\tilde{x}_1)}{x_0}$$

where $\tilde{x}_1=x_1/x_0$.

Then (15) is written as

$$V_{delta}(\tilde{x};s)=-\frac{s\hbar^2}{2mx_0{}^2\tilde{x}_1}\left(\delta(\tilde{x}+\tilde{x}_1)+\delta(\tilde{x}-\tilde{x}_1)\right) \tag{16}$$

In order to find the regular part of the potential, we'll calculate the second derivative of the wave function.

Using (10), we have

$$\psi''(\tilde{x};s)=\begin{cases} 0, & |\tilde{x}|<\tilde{x}_1 \\ \dfrac{(s+1)}{2}\sqrt{\dfrac{s(2s-1)}{x_0\tilde{x}_1^{\,5}}}\left(\dfrac{\tilde{x}_1}{-\tilde{x}}\right)^{s+2}, & \tilde{x}<-\tilde{x}_1 \\ \dfrac{(s+1)}{2}\sqrt{\dfrac{s(2s-1)}{x_0\tilde{x}_1^{\,5}}}\left(\dfrac{\tilde{x}_1}{\tilde{x}}\right)^{s+2}, & \tilde{x}>\tilde{x}_1 \end{cases}$$

or

$$\psi''(\tilde{x};s)=\begin{cases} 0, & |\tilde{x}|<\tilde{x}_1 \\ \dfrac{(s+1)}{2}\sqrt{\dfrac{s(2s-1)}{x_0\tilde{x}_1^{\,5}}}\left(\dfrac{\tilde{x}_1}{|\tilde{x}|}\right)^{s+2}, & |\tilde{x}|>\tilde{x}_1 \end{cases} \tag{17}$$

Since the wave function describes the ground state of the potential we are looking for, it satisfies the energy eigenvalue equation (in position space)[2], i.e.

$$\psi''(x;s)+\frac{2m}{\hbar^2}\left(E_0-V(x;s)\right)\psi(x;s)=0$$

with $E_0$ being the ground-state energy and $V(x;s)$ the potential.

2. The well-known time-independent Schrödinger equation.

Solving the energy eigenvalue equation for the potential yields

$$V(x;s)=\frac{\hbar^2}{2m}\frac{\psi''(x;s)}{\psi(x;s)}+E_0$$

Using that $\tilde{x}=x/x_0$, we obtain

$$\frac{d^2}{dx^2}=\frac{1}{x_0^{\,2}}\frac{d^2}{d\tilde{x}^2},$$

and thus, in terms of $\tilde{x}$, the potential is written as

$$V(\tilde{x};s)=\frac{\hbar^2}{2mx_0^{\,2}}\frac{\psi''(\tilde{x};s)}{\psi(\tilde{x};s)}+E_0$$

Substituting (9) and (17) into the previous equation, we obtain, after a little algebra, that the regular part of the potential is

$$V_{reg}\left(\tilde{x};s\right)=\begin{cases} E_0, & \left|\tilde{x}\right|<\tilde{x}_1 \\ \dfrac{s\left(s+1\right)\hbar^2}{2m{x_0}^2\tilde{x}^2}+E_0, & \left|\tilde{x}\right|>\tilde{x}_1 \end{cases} \tag{18}$$

From (18), we see that the regular part of the potential has, at $\pm\tilde{x}_1$, finite discontinuities, which are

$$V_{reg}\left(-\tilde{x}_1^{\;+};s\right)-V_{reg}\left(-\tilde{x}_1^{\;-};s\right)=E_0-\left(\frac{s\left(s+1\right)\hbar^2}{2m{x_0}^2\left(-\tilde{x}_1\right)^2}+E_0\right)=-\frac{s\left(s+1\right)\hbar^2}{2m{x_0}^2{\tilde{x}_1}^2}$$

$$V_{reg}\left(\tilde{x}_1^{\;+};s\right)-V_{reg}\left(\tilde{x}_1^{\;-};s\right)=\frac{s\left(s+1\right)\hbar^2}{2m{x_0}^2{\tilde{x}_1}^2}+E_0-E_0=\frac{s\left(s+1\right)\hbar^2}{2m{x_0}^2{\tilde{x}_1}^2}$$

The two discontinuities are thus opposite.

The total potential is the sum of the regular part and the delta functions, i.e.

$$V\left(\tilde{x};s\right)=V_{reg}\left(\tilde{x};s\right)+V_{delta}\left(\tilde{x};s\right) \tag{19}$$

Since $V\left(\pm\infty;s\right)=V_{reg}\left(\pm\infty;s\right)=E_0$, the ground-state energy $E_0$ is the highest bound energy of the potential (for a detailed explanation, see, for instance, [8]).

Thus, the potential (19) has only one bound state, which is also the ground state, of energy $E_0$, which is described by the wave function (9).

Choosing the infinity as reference point and setting $V\left(\pm\infty;s\right)=0$, we obtain

$$E_0=0 \tag{20}$$

i.e. the ground-state energy becomes zero.

The regular part of the potential, given by (18), is then written as

$$V_{reg}\left(\tilde{x};s\right)=\begin{cases} 0, & \left|\tilde{x}\right|<\tilde{x}_1 \\ \dfrac{s\left(s+1\right)\hbar^2}{2m{x_0}^2\tilde{x}^2}, & \left|\tilde{x}\right|>\tilde{x}_1 \end{cases} \tag{21}$$

We've thus ended up at an attractive double Dirac delta potential (of equal couplings) with an angular-momentum-like tail $s\left(s+1\right)\hbar^2/2m{x_0}^2\tilde{x}^2$, which, for every value of $s>3/2$, has only one, zero-energy, bound state.

## 4. Appendix

*For a potential consisting of a part having, at most, finite discontinuities, and of a finite sum of Dirac delta functions, if there exists a nodeless wave function* $\psi_0(x)$ *describing a bound energy eigenstate, then* $\psi_0(x)$ *is the ground-state wave function.*

*Proof*

Let $\hat{a}(x)$ be the position-space operator

$$\hat{a}(x) = \frac{1}{p_0}\left(\hat{p}(x) + i\hbar\frac{\psi_0'(x)}{\psi_0(x)}\right) \tag{22}$$

where $\hat{p}(x) = -i\hbar\, d/dx$ is the momentum operator in position space and $p_0$ is a (positive) momentum scale.

The operator $\hat{a}(x)$ is dimensionless.

The part of the potential which has, at most, finite discontinuities does not induce discontinuities in the wave function derivative [6, 7], while each delta function induces, in the wave function derivative, a finite discontinuity, at the point where the delta function acts [7].

Since $\psi_0(x)$ is nodeless, the function $\psi_0'(x)/\psi_0(x)$ has no singularities, but it has finite discontinuities at the points where the delta functions act.

Since the number of delta functions in the potential is finite, the function $\psi_0'(x)/\psi_0(x)$ has a finite number of finite discontinuities.

Also, since $\psi_0(x)$ is a one-dimensional bound energy eigenfunction, it is real, up to a constant phase [7].

Thus, $\psi_0'(x)/\psi_0(x)$ is a real function, and then the Hermitian conjugate of the operator (22) is given by

$$\hat{a}^\dagger(x) = \frac{1}{p_0}\left(\hat{p}(x) - i\hbar\frac{\psi_0'(x)}{\psi_0(x)}\right) \tag{23}$$

Using (22) and (23), we have

$$\hat{a}^{\dagger}(x)\hat{a}(x)=\frac{1}{p_0^{\ 2}}\left(\hat{p}(x)-i\hbar\frac{\psi_0{}'(x)}{\psi_0(x)}\right)\left(\hat{p}(x)+i\hbar\frac{\psi_0{}'(x)}{\psi_0(x)}\right)=$$

$$=\frac{1}{p_0^{\ 2}}\left(\hat{p}^2(x)+i\hbar\left[\hat{p}(x),\frac{\psi_0{}'(x)}{\psi_0(x)}\right]-\left(i\hbar\frac{\psi_0{}'(x)}{\psi_0(x)}\right)^2\right)=$$

$$=\frac{1}{p_0^{\ 2}}\left(\hat{p}^2(x)+i\hbar\left[\hat{p}(x),\frac{\psi_0{}'(x)}{\psi_0(x)}\right]+\hbar^2\left(\frac{\psi_0{}'(x)}{\psi_0(x)}\right)^2\right)$$

That is

$$\hat{a}^{\dagger}(x)\hat{a}(x)=\frac{1}{p_0^{\ 2}}\left(\hat{p}^2(x)+i\hbar\left[\hat{p}(x),\frac{\psi_0{}'(x)}{\psi_0(x)}\right]+\hbar^2\left(\frac{\psi_0{}'(x)}{\psi_0(x)}\right)^2\right) \qquad (24)$$

If $f(x)$ is an arbitrary function, then $\left[\hat{p}(x),f(x)\right]=-i\hbar f'(x)$, provided that the derivative $f'(x)$ exists. This is easily shown by applying the previous commutator to an arbitrary wave function.

Using the previous commutator, we have

$$\left[\hat{p}(x),\frac{\psi_0{}'(x)}{\psi_0(x)}\right]=-i\hbar\left(\frac{\psi_0{}'(x)}{\psi_0(x)}\right)'=-i\hbar\left(\frac{\psi_0{}''(x)}{\psi_0(x)}-\left(\frac{\psi_0{}'(x)}{\psi_0(x)}\right)^2\right)$$

That is

$$\left[\hat{p}(x),\frac{\psi_0{}'(x)}{\psi_0(x)}\right]=-i\hbar\left(\frac{\psi_0{}''(x)}{\psi_0(x)}-\left(\frac{\psi_0{}'(x)}{\psi_0(x)}\right)^2\right) \qquad (25)$$

Note

At the points where $\psi_0{}'(x)/\psi_0(x)$ has finite discontinuities, i.e. at the points where the delta functions act, the derivative $\left(\psi_0{}'(x)/\psi_0(x)\right)'$ is a delta function.

Similarly, $\psi_0{}''(x)$ is a delta function at the points where the delta functions act.

Therefore, the function in the right-hand side of (25) contains delta functions.

By means of (25), (24) becomes

$$\hat{a}^{\dagger}(x)\hat{a}(x)=\frac{1}{p_0^2}\left(\hat{p}^2(x)-(i\hbar)^2\left(\frac{\psi_0''(x)}{\psi_0(x)}-\left(\frac{\psi_0'(x)}{\psi_0(x)}\right)^2\right)+\hbar^2\left(\frac{\psi_0'(x)}{\psi_0(x)}\right)^2\right)=$$

$$=\frac{1}{p_0^2}\left(\hat{p}^2(x)+\hbar^2\left(\frac{\psi_0''(x)}{\psi_0(x)}-\left(\frac{\psi_0'(x)}{\psi_0(x)}\right)^2\right)+\hbar^2\left(\frac{\psi_0'(x)}{\psi_0(x)}\right)^2\right)=\frac{1}{p_0^2}\left(\hat{p}^2(x)+\hbar^2\frac{\psi_0''(x)}{\psi_0(x)}\right)$$

That is

$$\hat{a}^{\dagger}(x)\hat{a}(x)=\frac{1}{p_0^2}\left(\hat{p}^2(x)+\hbar^2\frac{\psi_0''(x)}{\psi_0(x)}\right) \tag{26}$$

Since $\psi_0(x)$ is an energy eigenfunction, it satisfies the energy eigenvalue equation (in position space), i.e.

$$\psi_0''(x)+\frac{2m}{\hbar^2}\left(E_0-V(x)\right)\psi_0(x)=0$$

where $E_0$ is the energy of the state described by $\psi_0(x)$ and $V(x)$ is the potential. The previous equation gives

$$\hbar^2\frac{\psi_0''(x)}{\psi_0(x)}=2m\left(V(x)-E_0\right)$$

Substituting the previous expression into (26) yields

$$\hat{a}^{\dagger}(x)\hat{a}(x)=\frac{1}{p_0^2}\left(\hat{p}^2(x)+2m\left(V(x)-E_0\right)\right)=\frac{2m}{p_0^2}\left(\frac{\hat{p}^2(x)}{2m}+V(x)-E_0\right)$$

Using that $\frac{\hat{p}^2(x)}{2m}+V(x)$ is the Hamiltonian $\hat{H}(x)$ (in position space), we end up to

$$\hat{a}^{\dagger}(x)\hat{a}(x)=\frac{2m}{p_0^2}\left(\hat{H}(x)-E_0\right) \tag{27}$$

The operator $\hat{a}^{\dagger}(x)\hat{a}(x)$ is Hermitian, since

$$\left(\hat{a}^{\dagger}(x)\hat{a}(x)\right)^{\dagger}=\hat{a}^{\dagger}(x)\left(\hat{a}^{\dagger}(x)\right)^{\dagger}=\hat{a}^{\dagger}(x)\hat{a}(x)$$

Also, using (27), we have

$$\left[\hat{a}^{\dagger}(x)\hat{a}(x),\hat{H}(x)\right]=\left[\frac{2m}{p_0^{\,2}}\left(\hat{H}(x)-E_0\right),\hat{H}(x)\right]=\frac{2m}{p_0^{\,2}}\left[\hat{H}(x)-E_0,\hat{H}(x)\right]=$$

$$=\frac{2m}{p_0^{\,2}}\left(\underbrace{\left[\hat{H}(x),\hat{H}(x)\right]}_{0}-\underbrace{\left[E_0,\hat{H}(x)\right]}_{0}\right)=0$$

That is, the operator $\hat{a}^{\dagger}(x)\hat{a}(x)$ commutes with the Hamiltonian.

Next, we'll prove that $\hat{a}^{\dagger}(x)\hat{a}(x)$ has non-negative eigenvalues.

*Proof*

Let $\psi(x)$ be an eigenfunction of $\hat{a}^{\dagger}(x)\hat{a}(x)$ with eigenvalue $\lambda$.

Since $\hat{a}^{\dagger}(x)\hat{a}(x)$ commutes with the Hamiltonian $\hat{H}(x)$, $\hat{a}^{\dagger}(x)\hat{a}(x)$ and $\hat{H}(x)$ have a common set of eigenfunctions, and thus $\psi(x)$ is an energy eigenfunction.

$\psi(x)$ is continuous, but its derivative $\psi'(x)$ has a finite number of finite discontinuities, at the points where the delta functions of the potential act.

Let us now consider the function

$$\hat{a}(x)\psi(x)=\frac{1}{p_0}\left(\hat{p}(x)+i\hbar\frac{\psi_0'(x)}{\psi_0(x)}\right)\psi(x)\overset{\hat{p}(x)=-i\hbar\frac{d}{dx}}{\widehat{=}}-\frac{i\hbar}{p_0}\left(\psi'(x)-\frac{\psi_0'(x)\psi(x)}{\psi_0(x)}\right)$$

That is

$$\hat{a}(x)\psi(x)=-\frac{i\hbar}{p_0}\left(\psi'(x)-\frac{\psi_0'(x)\psi(x)}{\psi_0(x)}\right)\tag{28}$$

The function $\hat{a}(x)\psi(x)$ has no singularities, since $\psi_0(x)$ is nodeless, but it has a finite number of finite discontinuities.

The wave functions $\psi_0(x)$ and $\psi(x)$ as well as their derivatives $\psi_0'(x)$ and $\psi'(x)$ are everywhere finite, so that the respective probability densities and currents are everywhere finite.

Then, the Riemann integral $\int_{-\infty}^{\infty}dx\left|\hat{a}(x)\psi(x)\right|^2$ exists[3].

3. If a function has a finite number of discontinuities and it is everywhere finite (i.e. if it is bounded), then it is Riemann integrable (see, for instance, [9]).

Using that $|z|^2 = z^* z$, with the asterisk denoting the complex conjugate, we have

$$\int_{-\infty}^{\infty} dx \left|\hat{a}(x)\psi(x)\right|^2 = \int_{-\infty}^{\infty} dx \left(\hat{a}(x)\psi(x)\right)^* \left(\hat{a}(x)\psi(x)\right) \tag{29}$$

Also, by definition[4]

$$\int_{-\infty}^{\infty} dx\psi^*(x)\hat{a}^\dagger(x)\left(\hat{a}(x)\psi(x)\right) = \int_{-\infty}^{\infty} dx\left(\left(\hat{a}^\dagger(x)\right)^\dagger \psi(x)\right)^* \left(\hat{a}(x)\psi(x)\right) \overset{\left(\hat{a}^\dagger(x)\right)^\dagger = \hat{a}(x)}{\overbrace{=}}$$

$$= \int_{-\infty}^{\infty} dx\left(\hat{a}(x)\psi(x)\right)^* \left(\hat{a}(x)\psi(x)\right)$$

That is

$$\int_{-\infty}^{\infty} dx\psi^*(x)\hat{a}^\dagger(x)\left(\hat{a}(x)\psi(x)\right) = \int_{-\infty}^{\infty} dx\left(\hat{a}(x)\psi(x)\right)^* \left(\hat{a}(x)\psi(x)\right) \tag{30}$$

4. In position space, the Hermitian conjugate operator $\hat{O}^\dagger(x)$ of a linear operator $\hat{O}(x)$ is defined by the relation [7, 10]

$$\int_{-\infty}^{\infty} dx\varphi_2^*(x)\hat{O}(x)\varphi_1(x) = \int_{-\infty}^{\infty} dx\left(\hat{O}^\dagger(x)\varphi_2(x)\right)^* \varphi_1(x),$$

where $\varphi_1(x), \varphi_2(x)$ are two arbitrary wave functions (in position space).

Comparing (29) and (30) yields

$$\int_{-\infty}^{\infty} dx\left|\hat{a}(x)\psi(x)\right|^2 = \int_{-\infty}^{\infty} dx\psi^*(x)\hat{a}^\dagger(x)\left(\hat{a}(x)\psi(x)\right) = \int_{-\infty}^{\infty} dx\psi^*(x)\left(\hat{a}^\dagger(x)\hat{a}(x)\psi(x)\right)$$

But $\hat{a}^\dagger(x)\hat{a}(x)\psi(x) = \lambda\psi(x)$, as $\psi(x)$ is an eigenfunction of $\hat{a}^\dagger(x)\hat{a}(x)$ with eigenvalue $\lambda$.

Thus

$$\int_{-\infty}^{\infty} dx\left|\hat{a}(x)\psi(x)\right|^2 = \int_{-\infty}^{\infty} dx\psi^*(x)\lambda\psi(x) = \lambda\int_{-\infty}^{\infty} dx\left|\psi(x)\right|^2$$

That is

$$\int_{-\infty}^{\infty} dx \left|\hat{a}(x)\psi(x)\right|^2 = \lambda \int_{-\infty}^{\infty} dx \left|\psi(x)\right|^2 \qquad (31)$$

In (31), both integrands are non-negative, thus both integrals are also non-negative. Moreover, since $\psi(x)$ is an eigenfunction, it is linearly independent, and thus it cannot be identically zero. Thus, the integral in the right-hand side of (31) is strictly positive and, since the integral in the left-hand side is non-negative, $\lambda$ must be non-negative too.

Therefore, the eigenvalues of $\hat{a}^\dagger(x)\hat{a}(x)$ are non-negative.

Next, using (27) and that $2m/p_0^2 > 0$, we derive that the eigenvalues of $\hat{H}(x) - E_0$ are also non-negative, and thus the eigenvalues of the Hamiltonian, i.e. the energies, are greater than or equal to $E_0$.

Then, since the energy $E_0$ exists, it is the ground-state energy, and the respective wave function $\psi_0(x)$ is the ground-state wave function.

Notes

i. We point out the significance of the existence of the Riemann integral $\int_{-\infty}^{\infty} dx \left|\hat{a}(x)\psi(x)\right|^2$ in the proof.

If the wave function $\psi_0(x)$ had nodes, the function $\hat{a}(x)\psi(x)$, as given by (28), would have singularities and the previous Riemann integral would be ill-defined.

Also, if the discontinuities in the wave function derivative $\psi_0'(x)$ would be infinite (in number or in magnitude), the previous Riemann integral would also be ill-defined.

ii. From (28), we see that $\hat{a}(x)\psi_0(x) = 0$, i.e. the operator $\hat{a}(x)$ kills the ground-state wave function.

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