# Related to Fermat's Last theorem:The quadratic formula of the equation $X^{n-1} \mp X^{n-2}Y + X^{n-3}Y^2 \mp ... + Y^{n-1} = Z_2^n(nZ_2^n)$ in the cases n = 3, 5 and 7

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**Abstract**

*We give some quadratic formulas ( including Euler' and Dirichlet's formula in [1],[2]) of the equation* $X^{n-1} \mp X^{n-2}Y + X^{n-3}Y^2 \mp ... + Y^{n-1} = Z_2^n(nZ_2^n)$ *in the cases n = 3,5 and 7 for finding a solution in integer.*

The equation $X^{n-1} \mp X^{n-2}Y + X^{n-3}Y^2 \mp ... + Y^{n-1} = Z_2^n$ always has a solution such as :
$X = a(a^{n-1} \mp a^{n-2}b + a^{n-3}b^2 \mp ... + b^{n-1})$
$Y = b(a^{n-1} \mp a^{n-2}b + a^{n-3}b^2 \mp ... + b^{n-1})$
$Z_2 = a^{n-1} \mp a^{n-2}b + a^{n-3}b^2 \mp ... + b^{n-1}$
X,Y and $Z_2$ have a common factor$= a^{n-1} \mp a^{n-2}b + a^{n-3}b^2 \mp ... + b^{n-1}$
Below we consider the case X,Y and $Z_2$ are relative prime and X,Y are odd for n = 3,5 and 7.

## 1 The quadratic formulas of the equation $X^2 - XY + Y^2 = Z_2^3$(related equation $X^3 + Y^3 = Z^3; Z_1 = X + Y$)

1a.

$$x^2 - xy + y^2 = z_2^3 \tag{1}$$

(x + y is not divisible by 3)

Write x = u + v, y = u - v, then:
$x^2 - xy + y^2 = u^2 + 3v^2$
$u = \dfrac{x+y}{2}$ is not divisible by 3,consider the equation $u^2 + 3v^2 = z_2^3$
and $z_2$ can be written as $z_2 = a + 3b$
$z_2^3 = (a + 3b)^3 = a(a - 9b)^2 + 27b(a - b)^2$
select $u_1^2 = a(a - 9b)^2$, $v_1^2 = 9b(a - b)^2$
then a and b are the square,write : $a = c^2$,$b = d^2$
it gives:
$u_1^2 = c^2(c^2 - 9d^2)^2$, $v_1^2 = 9d^2(c^2 - d^2)^2$
then:

$$u_1 = c(c^2 - 9d^2) \tag{2}$$

$$v_1 = 3d(c^2 - d^2) \tag{3}$$

and $z_2 = (c^2 + 3d^2)$
1b.

$$x^2 - xy + y^2 = 3z_2^3 \tag{4}$$

x + y is divisible by 3

$u = \dfrac{x+y}{2}$ is divisible by 3,write $u = 3u'$ then : $u^2 + 3v^2 = 3^2u'^2 + 3v^2$
$x^2 - xy + y^2 = u^2 + 3v^2 = 3^2u'^2 + 3v^2 = 3(3u'^2 + v^2)$ Consider the equation $3(3u'^2 + v^2) = 3z_2^3$
So $(3u'^2 + v^2) = z_2^3$
And by the same way, we obtain:
$v_1' = c(c^2 - 9d^2)$ as $u_1$, $u_1' = 3d(c^2 - d^2)$ as $v_1$and $z_2 = (c^2 + 3d^2)$
x = u + v = 3u' + v, y = u - v = 3u' - v.

If $u, v$ is the one solution, then:

$$u' = \frac{(m^2 - 3n^2)u + 6mnv}{3n^2 + m^2} \tag{5}$$

$$v' = \frac{2mnu + (3n^2 - m^2)v}{3n^2 + m^2} \tag{6}$$

are also solution: $u'^2 + 3v'^2 = u^2 + 3v^2$

## 2 The quadratic formulas of the equation $X^4 - X^3Y + X^2Y^2 - XY^3 + Y^4 = Z_2^5$(related equation $X^5 + Y^5 = Z^5$; $Z_1 = X + Y$)

2a.

$$x^4 - x^3y + x^2y^2 - xy^3 + y^4 = z_2^5 \tag{7}$$

x + y is not divisible by 5

Write x = p + q, y = p - q, then:
$x^4 - x^3y + x^2y^2 - xy^3 + y^4 = p^4 + 10p^2q^2 + 5q^4 = p^4 + 10p^2q^2 + 25q^4 - 20q^4 = (p^2 + 5q^2)^2 - 5(2q^2)^2$

$p = \dfrac{x+y}{2}$ is not divisible by 5, consider the equation:

$(p^2 + 5q^2)^2 - 5(2q^2)^2 = z_2^5$
let $u = p^2 + 5q^2$ and $v = 2q^2$
then $z_2^5 = u^2 - 5v^2$ and $z_2$ can be written as $z_2 = a - 5b$
$z_2^5 = (a - 5b)^5 = a(a^2 + 50ab + 125b^2)^2 - 5^3b(a^2 + 10ab + 5b^2)^2$
select $u_1^2 = a(a^2 + 50ab + 125b^2)^2$, $v_1^2 = 5^2b(a^2 + 10ab + 5b^2)^2$
then a and b are the square,write : $a = c^2$,$b = d^2$
it gives:
$u_1^2 = c^2(c^4 + 50c^2d^2 + 125d^4)^2$
$v_1^2 = 5^2d^2(c^4 + 10c^2d^2 + 5d^4)^2$
then:

$$u_1 = c(c^4 + 50c^2d^2 + 125d^4) \tag{8}$$

$$v_1 = 5d(c^4 + 10c^2d^2 + 5d^4) \tag{9}$$

and $z_2 = c^2 - 5d^2$

2b.

$$x^4 - x^3y + x^2y^2 - xy^3 + y^4 = 5z_2^5 \tag{10}$$

x + y is divisible by 5

$p = \dfrac{x+y}{2}$ is divisible by 5, write $p = 5p'$,so $p^4 + 10p^2q^2 + 5q^4 = (5p)'^4 + 10(5p)'^2q^2 + 5q^4 = 5(125p'^4 + 50p'^2 + q^4)$
$5(125p'^4 + 50p'^2 + q^4) = 5(625p'^4 + 50p'^2 + q^4 - 500p'^4) = 5[(q^2 + 25p'^2)^2 - 5.(10p'^2)^2]$
Consider the equation:$5[(q^2 + 25p'^2)^2 - 5.(10p'^2)^2] = 5z_2^5$
So $(q^2 + 25p'^2)^2 - 5.(10p'^2)^2 = z_2^5$
Let $u = q^2 + 25p'^2, v = 10p'^2$ then $z_2 = c^2 - 5d^2$ and u, v are given by (8);(9).

If $u, v$ is the one solution, then:

$$u' = \frac{(m^2 + 5n^2)u - 10mnv}{5n^2 - m^2} \tag{11}$$

$$v' = \frac{2mnu - (m^2 + 5n^2)v}{5n^2 - m^2} \tag{12}$$

are also solution: $u'^2 - 5v'^2 = u_0^2 - 5v_0^2$
The other one solution* of the equation $u^2 + 5v^2 = (c^2 - 5d^2)^5$ is:

$$u = u_2 = c(c^4 + 10c^2d^2 - 75d^4) \tag{13}$$

$$v = v_2 = d(3c^4 - 10c^2d^2 - 25d^4) \tag{14}$$

( * was not considered by Dirichlet in his proof).

# 3 The quadratic formulas of the equation $X^6 - X^5Y + X^4Y^2 - X^3Y^3 + X^2Y^4 - XY^5 + Y^6 = Z_2^7$ (related equation $X^7 + Y^7 = Z^7; Z_1 = X + Y$)

3a.

$$x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6 = z_2^7 \tag{15}$$

x + y is not divisible by 7

Write x = p + q, y = p - q, then:
$x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6 = p^6 + 21p^4q^2 + 35p^2q^4 + 7q^6$

$= p^6 + 14p^4q^2 + 49p^2q^4 + 7p^4q^2 - 14p^2q^4 + 7q^6$
$= p^2(p^2 + 7q^2)^2 + 7q^2(p^2 - q^2)^2$

$p = \dfrac{x+y}{2}$ is not divisible by 7, consider the equation:$z_2^7 = p^2(p^2 + 7q^2)^2 + 7q^2(p^2 - q^2)^2$ let $u = p(p^2 + 7q^2)$ and $v = q(p^2 - q^2)$
then $z_2^7 = u^2 + 7v^2$ and $z_2$ can be written as $z_2 = a + 7b$
and by the same way as above, we obtain:

$$u_1 = c[c^2(c^2 - 21d^2)^2 - 7d^2(3c^2 - 7d^2)^2 - 14d^2(c^2 - 21d^2)(3c^2 - 7d^2)] \tag{16}$$

$$v_1 = d[2c^2(c^2 - 21d^2)(3c^2 - 7d^2) + c^2(c^2 - 21d^2)^2 - 7d^2(3c^2 - 7d^2)^2] \tag{17}$$

$$u_2 = c[c^2(c^2 - 21d^2)^2 - 7d^2(3c^2 - 7d^2)^2 + 14d^2(c^2 - 21d^2)(3c^2 - 7d^2)] \tag{18}$$

$$v_2 = d[2c^2(c^2 - 21d^2)(3c^2 - 7d^2) - c^2(c^2 - 21d^2)^2 + 7d^2(3c^2 - 7d^2)^2] \tag{19}$$

and $z_2 = c^2 + 7d^2$
3b.

$$x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6 = 7z_2^7 \tag{20}$$

x + y is divisible by 7

$p = \dfrac{x+y}{2}$ is divisible by 7,write p = 7 p', so $p^2(p^2 + 7q^2)^2 + 7q^2(p^2 - q^2)^2 = 7^2p'^2(7^2p'^2 + 7q^2)^2 + 7q^2(7^2p'^2 - q^2)^2$
$= 7[7^3p'^2(7p'^2 + q^2)^2 + q^2(7^2p'^2 - q^2)^2]$
Consider the equation $7[7^3p'^2(7p'^2 + q^2)^2 + q^2(7^2p'^2 - q^2)^2] = 7z_2^7$
So $7^3p'^2(7p'^2 + q^2)^2 + q^2(7^2p'^2 - q^2)^2 = z_2^7$
Let $u = q(7^2p'^2 - q^2)$, $v = 7p'(7p'^2 + q^2)$, then
$z_2^7 = u^2 + 7v^2$,and u , v are given by (16);(17);(18);(19).

If $u, v$ is the one solution, then:

$$u' = \frac{(m^2 - 7n^2)u + 14mnv}{7n^2 + m^2} \tag{21}$$

$$v' = \frac{2mnu + (7n^2 - m^2)v}{7n^2 + m^2} \tag{22}$$

are also solution:$u'^2 + 7v'^2 = u^2 + 7v^2$

Notes:
- Different from case n = 3, for the case n = 5, u and v must satisfy $u = p^2 + 5q^2$ and $v = 2q^2$ ( 2a), $u = q^2 + 25p'^2, v = 10p'^2$ ( 2b),for the case n = 7, u and v must satisfy $u = p(p^2 + 7q^2)$ , $v = q(p^2 - q^2)$ (3a), $u = q(7^2p'^2 - q^2)$, $v = 7p'(7p'^2 + q^2)$ (3b).
- u',v' are integer or not, depend on u,v,m,n.
- For the equation $X^{n-1} + X^{n-2}Y + X^{n-3}Y^2 + ... + Y^{n-1} = Z_2^n(nZ_2^n)$, the algorithm is the same as above.

# References


[1] Quang N V, Euler's proof of Fermat Last's Theorem for n = 3 is incorrect Vixra:1605.0123v3(NT)

[2] Quang N V, Dirichlet's proof of Fermat last's theorem for n = 5 is flawed Vixra:1607.0400v2(NT)



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