A formulation of vacuum Euclidean relativity on a general class of metric spaces.

Johan Noldus*

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Abstract

A general geometric apparatus extending the standard calculus on Riemannian manifolds in a coordinate independent way is developed. It is, moreover, obvious that all quantities involved give rise to the correct limits in such case when convergence is subtle enough. We finish by giving the correct equivalent of the Einstein tensor which approximately is covariantly conserved on a certain deformation scale.

1 Introduction.

Discrete Lorentzian or Riemannian structures often emerge in some approaches to quantum gravity, ranging from Lorentzian simplicial spacetimes, causal sets to abstract Lorentz spaces as studied by this author in the past. The problem is then how to look for generalizations of the Torsion, Riemann, Ricci and Einstein tensor having no notion of differentiability at ones disposal. We shall adress these questions in this paper and provide for a unique geometric setting for metric spaces avoiding some nonlocal subtleties associated to the Lorentzian distance treatment due to compactness of metric balls. First of all, we start by generalizing the notion of differentiability in a coordinate independent way, one which is actually valid for general topological spaces and not only for metric ones which is important to keep in mind when treating the Lorentzian case. Next, we discuss the natural absence of a notion of torsion in the general metric case; something which will only be recuperated when comparing with a differentiable Riemannian space close enough to the metric one. The reason is evident, without a compass, we have no rotation and therefore no twist or Torsion; general metric spaces are not provided with such which explains the matter. Nevertheless, there is a method to extract the "Levi-Civita connection" and it goes very subtle by noticing that the *usual* first Bianchi identity holds and that the "Riemann tensor" has all usual symmetries which is equivalent to the vanishing of torsion in differential geometry. Given that the absence of a linear structure denies one the right of a Jacobi identity, which explains why no second Bianchi identity for the generalized connection needs to hold, the "Levi Civita connection" is

*email: johan.noldus@gmail.com, Relativity group, departement of mathematical analysis, University of Gent, Belgium.

extracted by means of an optimization criterium that the second Bianchi identity needs to hold as closely as possible. This then defines Torsion for a general metric compatible connection.

2 Topological differentials.

We construct an intrinsic notion of differentiability to be used in the next section to define the notion of torsion in a metric space without rulers and compasses, but merely bi-functions indicating distances between points. In other words, we define a fundamental *geometrical* notion of transport: take two topological spaces X, Y with reflexive and symmetric relations[1] $R \subset X \times X, T \subset Y \times Y$ respectively which are both topologically open. Then

$$\nabla_X : \{(x,y,z) : y,z \in R(x,\cdot)\} \to R : (x,y,z) \to \nabla_{(x,y)}(x,z) \in R(y,\cdot)$$

is called the transported relation regarding (x,z) over (x,y) from x to y and as such it indicates a preffered path or geodesic at least locally. ∇_X should obey a division property in order to recuperate the usual linearity properties at small scales if possible. Weakly put, there exists an open environment $\mathcal{Q} \subset R$ with $\pi_1(\mathcal{Q}) = X$ such that for all $(x,w) \in \mathcal{Q}$ there exists an open $\mathcal{W}$ around w such that $\mathcal{O} \subset R(x,\cdot) \subset X$ and

$$T_{\mathcal{O}} : \{(x,y,z) : y,z \in \mathcal{O}\} \to X : (x,y,z) \to \pi_2(\nabla_{(x,z)}(x,y))$$

is continuous and surjective on $\mathcal{W}$. Here, π_j is the projection on the j-th factor. A stronger version would demand that $D \subset \mathcal{Q}$ where D is the diagonal. Given that a topological space has no orientation, it is impossible to define a rotation; therefore, a general definition for the *commutator* between two elements in $R(x,\cdot)$ would read as

$$[(x,w),(x,v)] = ((v,x) \circ P(\nabla_{(x,v)}(x,w)))\sharp(\nabla_{(x,w)}(x,v) \circ (x,w))$$

waarbij $(w,z)\sharp(x,v) = \left(\nabla_{(w,v)}(w,z)\right) \circ (x,v)$ and $P(x,w) = (w,x)$. Note that, in a sense, our connection is "torsion free" which is logical given that the standard definition of a commutator depends upon the algebraic and topological structure of $\mathbb{R}^n$. Since the latter does not exist in the general case, we have no choice but to define it by means of the connection. We call a function $F : X \to Y$ topologically differentiable in a surrounding of $x \in X$, with respect to a continuous scaling functions $h, g : R, T \to \mathbb{R}^+$ satisfying $g(x,x) = 0 = h(x',x')$ and $h(x',y'), g(x,y) > 0$ for $x \neq y, x' \neq y'$, in case for any open $\mathcal{V} \subset T(F(x),\cdot)$ there exists an open neighborhood $\mathcal{O} \subset R(x,\cdot)$ as well as a bi-continuous mapping $DF_w(w,v) : (w,v) \in \mathcal{O}^2 \to \mathcal{V}^2$, $v, w \in \mathcal{O}$ defined by $(F(v), F(w)) = DF_v(v,w)$ implying

$$DF_x\left(\left(\nabla_{(x,w)}(x,y)\right) \circ (x,w)\right) = DF_w(\nabla_{(x,w)}(x,y)) \circ DF_x(x,w)$$

and satisfying the demand

$$\frac{g(P(DF_x(x,v))\sharp DF_x(x,w))}{h(v,w)} \leq C$$

for some constant $C > 0$. We have used both arguments v, w in contrast to the standard definition with $w = x$ in order to ensure "differentiability" of ∇ as well as the linearity of DF_x. Here, the composition $\circ$ on R (T)

[1] In order to make transport symmetrical and to have a notion of zero transport.

is defined as $(w, z) \circ (x, w) = (x, z)$ and is therefore a kind of sum. This constitutes one way of approaching the matter which is still too close to the old fashioned metric approach to my liking; in general, one introduces the notation

$$(x, v) \oplus (x, w) = \nabla_{(x,w)}(x, v) \circ (x, w)$$

and

$$(x, w) \ominus (x, v) = P(x, v)\sharp(x, w).$$

Therefore,

$$DF_x((x, v) \oplus (x, w)) = DF_w(\nabla_{(x,w)}(x, v)) \circ DF_x(x, w)$$

meaning the ∇ operation shifts trough F. Now, one may look at $DF_x(x, v)$ and utter further that it satisfies

$$g(DF_x(x, v) \ominus (\nabla_{DF_w(w,x)} DF_w(\nabla_{(x,w)}(x, v)))) \leq Ch(x, v)h(x, w)$$

where C is some constant. This definition differs from the previous but captures better the linearity. A connection is called metric constant if and only if

$$d(\nabla_{(xy)}(xz)) = d((xz))$$

which does not imply that $\nabla_{(x,w)}(x, \cdot) : v \in R(x, .) \to X : v \to \pi_2(\nabla_{(x,w)}(x, v))$ is differentiable with respect to the scaling function d since

$$d(P(\pi_2(\nabla_{(x,w)}(x, v)), \pi_2(\nabla_{(x,w)}(x, z)))\sharp(\pi_2(\nabla_{(x,w)}(x, v)), \pi_2(\nabla_{(x,w)}(x, y))))$$

cannot be written in terms of original $\nabla_{(x,w)}(x, q)$. Given that we have a notion of addition on R, it is possible to define a functional ω_X to be a ∇-symmetric continuous function on the displacements (x, y) satisfying

$$\omega_X((y, z) \circ (x, y)) = \omega_X((x, y)) + \omega((y, z))$$

and

$$\omega_X((x, y)) = -\omega_X((y, x)), \omega(\nabla_X(x, y, z)) = \omega_X((x, z)).$$

For general spaces X, a curve is a one dimensional object without "holes"; more concretely $\gamma \subset X$ is a curve if and only if there exists a homeomorphism ψ from γ to a subset $A \subset \mathbb{R}$ such that for $r < s \in A$ the displacement $(\psi^{-1}(r), \psi^{-1}(s))$ is irreducible in the limit for s to r in A. A displacement is irreducible if and only if

$$\lim_{s \to > r} \left| \frac{\omega_X(D\psi_r^{-1}(r, s))}{s - r} \right| < C(\psi^{-1}, \omega_X)$$

for any continuous functional ω_X. This is logical because the definition of an irreducible hole requires some notion of displacement which is thightly associated to the definition of a differential.

We therefore obtain a notion of forward differential equations

$$\frac{d}{ds} F(\psi^{-1}(s)) = g(\psi^{-1}(s))$$

for all continuous functions $F, g : X \to \mathbb{C}$ where $\frac{d}{ds}$ stands for

$$\lim_{r \to > s} \frac{f(r) - f(s)}{r - s}$$

and everything is assumed to be ψ independent. This signifies that the last expression remains invariant under order preserving diffeomorphisms $\phi : A \to A$ in the sense that the equation is invariant under the substitution $\psi \to \psi \circ \phi$.

3 Riemannian geometry.

Consider a (locally) compact path metric space (X, d), where the path metric property signifies that for any $x, y \in X$ holds that there exists a $z \in X$ such that

$$d(x, z) = d(y, z) = \frac{d(x, y)}{2}.$$

The latter is equivalent to stating that there exists a curve, called a geodesic, $\gamma : [0, 1] \to X$ which minimizes the length functional L for paths with endpoints x, y and, moreover, $L(\gamma) = d(x, y)$. The latter is defined by

$$L(\gamma) = \sup_{0=t_0<t_1\ldots<t_n=1, n>0} \sum_{j=0}^{n-1} d(\gamma(t_j), \gamma(t_{j+1}))$$

and γ can be parametrized in arc-length parametrization by means of the Radon Nikodym derivative. We shall henceforth define all important quantities as well as derive some results.

- Consider a point $x \in X$ and take a sequence of points y_n, z_n placed on two half geodesics emanating from x converging in the limit for n to infinty towards x. In case the limit

  $$\lim_{n\to\infty} \frac{d(x, y_n)^2 + d(x, z_n)^2 - d(y_n, z_n)^2}{2d(x, y_n)d(x, z_n)}$$

  exists, we define the angle $\theta_x(y, z)$ between both geodesics by equating the latter expression to $\cos(\theta_x(y, z))$.

- Alexandrov curvature: in flat Euclidean geometry, the midpoint r of a line segment $[ab]$ satisfies

  $$\vec{xr} = \frac{1}{2}(\vec{xa} + \vec{xb})$$

  for any x. Hence, one arrives at

  $$d(x, r)^2 = \frac{1}{4}(d(x, a)^2 + d(x, b)^2 + 2d(x, a)d(x, b)\cos(\theta_x(a, b))).$$

  Considering sequences y_n, z_n as previously and defining r_n as a midpoint of the geodesic segment $[y_n z_n]$ consider the sequence

  $$R_n(y, z) = \frac{-d(x, y_n)^2 - d(x, z_n)^2 - 2d(x, y_n)d(x, z_n)\cos(\theta_x(y_n, z_n)) + 4d(x, r_n)^2}{d(x, y_n)^2 d(x, z_n)^2 \sin^2(\theta_x(y_n, z_n))}$$

  or alternatively

  $$R_n(y, z) = \frac{-2d(x, y_n)^2 - 2d(x, z_n)^2 + d(y_n, z_n)^2 + 4d(x, r_n)^2}{d(x, y_n)^2 d(x, z_n)^2 \sin^2(\theta_x(y_n, z_n))}$$

  both quantities having dimension of m^{-2}. For differentiable, metric compatible and torsionless theories, one has in general that

  $$d^2(x+w, x+v) = g_x(v-w, v-w) + \gamma(N)g_x(R_x(v, w)v, w) + \delta(N)g_x(C_x(v, w)v, w) +$$

  $$\kappa(N)R_x(g_x(v, v)g_x(w, w) - g(v, w)^2) + \text{higher order terms}$$

  where R_x is the Ricci scalar and C_x the Weyl tensor. A term of the form $\zeta(N)(R_x(v, v)g_x(w, w) + R_x(w, w)g_x(v, v))$ is forbidden given that it does not vanish for $v = w$. However, it is easily argued from the geodesic equation that the coefficients must be dimension independent and that therefore $\delta(N) = \kappa(N) = 0$ whereas $\gamma(N) = \gamma$ some constant. Logically, the expression

$$-2d(x, y_n)^2 - 2d(x, z_n)^2 + d(y_n, z_n)^2 + 4d(x, r_n)^2$$

has therefore the structure

$$\widetilde{\gamma(N)} g_x(R_x(v, w)v, w) + \widetilde{\delta}(N) g_x(C_x(v, w)v, w) + \widetilde{\kappa}(N) R_x(g_x(v, v)g_x(w, w) - g(v, w)^2) + \text{higher order terms}$$

given that it vanishes in the flat limit and the midpoint

$$r_n = \frac{v_n + w_n}{2} + \text{third order corrections vanishing when } v_n = w_n.$$

Moreover, we have that $\widetilde{\delta}(N) = \widetilde{\kappa}(N) = 0$ and that $\widetilde{\gamma}(N)$ is N independent, given that the midpoint depends upon the Riemann tensor only given that no metric contractions take place. In order to substract the Riemann tensorial part, one might take midpoints of midpoints and points, and so on, in order to obtain distinct coefficients.

- In case $R < 0$, then we call the space Alexandrov hyperbolic; in the other case it is Alexandrov spheric and otherwise flat.

- We now define a volume form akin to a determinant; given that we do not consider n-beins, we have no notion of orientation. Hence, the only natural candidate is symmetric in the arguments instead of antisymmetric. Consider therefore

$$V_{n,x}(a_1, \ldots, a_n)^2 = \left(\prod_{i=1}^{n} d(x, a_i) \right)^2 \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^{n} \cos(\theta(i, \sigma(i))).$$

One verifies that

$$V_4(a, b, a, b)^2 = d(x, a)^4 d(x, b)^4 \left(2 - 4\cos(\theta_x(a, b))^2 + 2\cos(\theta_x(a, b))^4 \right) = 2V_2(a, b)^2$$

and therefore the odd terms in $\cos(\theta)$ vanish which is logical given that the expression must be invariant under $\theta \to \pi - \theta$.

- We now arrive to the notion of Riemann curvature; consider a point x as well as points a_m, b_n, c_n, d_n on four geodesics emanating from x. Denote by $\widehat{ab}$ the midpoint between a and b. Note first that we can define a quantity, which is not a tensor in an appropriate limit but which has the symmetries of the metric, torsionless Riemann tensor and reduces in the "sectional case" to the Alexandrov curvature. It is,

$$T_n(a, c, b, d) = 4 \frac{d(x, a_n)d(x, b_n) - d(x, a_n)d(x, d_n) - d(x, c_n)d(x, b_n) + d(x, c_n)d(x, d_n)}{V_x(a_n, b_n, c_n, d_n)}$$

$$+4 \frac{2d(x, \widehat{a_n b_n})d(x, \widehat{c_n d_n}) - 2d(x, \widehat{a_n d_n})d(x, \widehat{b_n c_n})}{V_x(a_n, b_n, c_n, d_n)}$$

$$-2 \frac{d(a_n, d_n)d(b_n, c_n) - d(a_n, b_n)d(c_n, d_n)}{V_x(a_n, b_n, c_n, d_n)}.$$

This results in

$$T_n(a,c,a,c) = 2\frac{d(x,a_n)^2 - 2d(x,a_n)d(x,c_n) + d(x,c_n)^2}{V_x(a_n,c_n)^2}$$

$$+\frac{4d(x,a_n)d(x,c_n) - 4d(x,\widehat{a_n c_n})^2 - d(a_n,c_n)^2}{V_x(a_n,c_n)^2} = R_n(a,c)$$

which is the correct expression and an easy calculation shows that

$$T_n(a,c,b,d) + T_n(c,b,a,d) + T_n(b,a,c,d) = 0.$$

Before we proceed, it is good to understand the following result.

- Given $\alpha = (12), \beta = (34)$ and $\tau = (13)(24)$, then α, β, τ generate a subgroup K of the permutation group with following relations $\alpha^2 = \beta^2 = \tau^2 = 1$ and $\alpha \circ \beta = \beta \circ \alpha, \alpha \circ \tau = \tau \circ \beta, \beta \circ \tau = \tau \circ \alpha$. These relations have meaning in physics where α, β could indicate rotation of a spinorial point particle around the z-axis for 360 degrees, whereas τ could indicate spatial rotation around the midpoint of both particles with respect to the z-axis for 180 degrees. K has in total six elements. Suppose we are given a function

$$H(a_i; i = 1 \ldots 4) = \sum_{\sigma \in K} \mathrm{Sign}(\sigma) F(a_{\sigma(i)})$$

whereby F remains invariant with respect to a particular odd permutation $\rho \in S_4$ which is not in K. Then, $H(a_i; i = 1 \ldots 4)$ obeys the first Bianchi identity

$$\sum_{\sigma \in S_3} \mathrm{Sign}(\sigma) H(a_0, a_{\sigma(i)}) = 0.$$

Proof: let $\rho \in S_3$, then with $G(a_0, a_i) = \sum_{\sigma \in S_3} \mathrm{Sign}(\sigma) H(a_0, a_{\sigma(i)})$ we have that

$$G(a_0, a_{\rho(i)}) = \sum_{\sigma \in S_3} \mathrm{Sign}(\sigma) H(a_0, a_{\sigma(\rho(i))}).$$

The group S_3 is generated by $r = (123), t = (23)$ where

$$(123)(23) = (12), (23)(123) = (13)$$

and $(123)^3 = e$ the identity element. Every element is therefore of the form $r^j t^p$ with $j = 0 \ldots 2; p = 0, 1$; hence

$$G(a_0, a_i) = \sum_{\sigma \in S_3} \mathrm{Sign}(\sigma) \sum_{\kappa \in K} \mathrm{Sign}(\kappa) F(a_{\sigma(\kappa(i))})$$

which is further reduced to

$$2 \sum_{j=0}^{2} \sum_{\kappa \in K} \mathrm{Sign}(\kappa) F(a_{r^j(\kappa(i))}).$$

Group theoretically, $r\kappa r^2 \in K_+$ for $\kappa \in K_+$ even and $r\kappa r \in K_-$ for $\kappa \in K_-$ odd. Hence, the latter sum reduces to two times

$$\sum_{\kappa \in K_+} \mathrm{Sign}(\kappa) F(a_{\kappa(i)}) + \sum_{\kappa \in K_-} \mathrm{Sign}(\kappa) F(a_{\kappa(i)})$$

$$+\sum_{\kappa\in K_+}\mathrm{Sign}(\kappa)F(a_{\kappa(r(i))})+\sum_{\kappa\in K_-}\mathrm{Sign}(\kappa)F(a_{\kappa(r^2(i))})+$$
$$\sum_{\kappa\in K_+}\mathrm{Sign}(\kappa)F(a_{\kappa(r^2(i))})+\sum_{\kappa\in K_-}\mathrm{Sign}(\kappa)F(a_{\kappa(r(i))}).$$

Hence, we have that

$$G(a_0,a_i)=2\sum_{j=0}^{2}\sum_{\kappa\in K}\mathrm{Sign}(\kappa)F(a_{\kappa(r^j(i))})=2\sum_{\rho\in S_4}\mathrm{Sign}(\rho)F(a_{\rho(i)})$$

which proves that the averaging procedures commute. Note that $(13)=(12)(23)(12)$. We now use the symmetry σ' of F; in case F has (14) or (24) we conjugate it by means of an element in K to (23) meaning $\sigma'=s(23)s^{-1}$ with $s\in K$; hence, a small calculation using the above commutation property of symmetrization reveals that

$$G(a_0,a_{(23)(i)})=G(a_0,a_i)=-G(a_0,a_i)=0.$$

Now, we have learned so far that the metric easily allows for such symmetric terms each with a distinct symmetry of course to recuperate the full Riemann tensor.

- We now return to our analysis of the Riemann tensor. Actually, our choice is not a very good one exclusively due to the presence of non-quadratic metric terms of the kind $d(b_n,a_n)$ in the last term of the nominator which gives it a nondifferentiable character near the origin. Although this edifice is easily repaired, we shall consider the following, far more general, case

$$T_x^{\epsilon,\alpha,\beta,\kappa,\delta,\lambda,\gamma,\mu,\zeta,\rho,\nu,\sigma,\psi,\phi,\pi}(a,b,c,d)=\frac{1}{\epsilon^4}\left(\alpha(d(\widehat{bd},\widehat{ac})^2-d(\widehat{bc},\widehat{ad})^2)+\beta(d(x,\widehat{\widehat{ac}\widehat{bd}})^2-d(x,\widehat{\widehat{ad}\widehat{bc}})^2)\right)$$
$$+\frac{1}{\epsilon^4}\left(\kappa(d(a,c)^2-d(a,d)^2-d(b,c)^2+d(b,d)^2)+\delta(d(a,\widehat{bd})^2-d(a,\widehat{bc})^2-d(b,\widehat{ad})^2+d(b,\widehat{ac})^2)\right)$$
$$+\frac{1}{\epsilon^4}\left(\lambda(d(x,\widehat{ac})^2-d(x,\widehat{ad})^2-d(x,\widehat{bc})^2+d(x,\widehat{bd})^2)+\gamma(d(x,\widehat{ac})d(x,\widehat{bd})-d(x,\widehat{ad})d(x,\widehat{bc}))\right)$$
$$+\frac{\mu}{\epsilon^4}(d(x,a)d(x,c)-d(x,a)d(x,d)-d(x,b)d(x,c)+d(x,b)d(x,d))$$
$$+\frac{\rho}{\epsilon^4}(d(x,\widehat{ad})d(x,\widehat{bc})-d(x,\widehat{ac})d(x,\widehat{bd}))$$
$$+\frac{\zeta}{\epsilon^4}(d(x,a)d(x,\widehat{ac})-d(x,a)d(x,\widehat{ad})-d(x,b)d(x,\widehat{bc})+d(x,b)d(x,\widehat{bd})+d(x,c)d(x,\widehat{ac}))$$
$$+\frac{\zeta}{\epsilon^4}(-d(x,c)d(x,\widehat{cb})-d(x,d)d(x,\widehat{da})+d(x,d)d(x,\widehat{db}))$$
$$+\frac{\nu}{\epsilon^4}(d(x,\widehat{a\widehat{ac}})d(x,\widehat{bc})-d(x,\widehat{b\widehat{bc}})d(x,\widehat{ac})-d(x,\widehat{a\widehat{ad}})d(x,\widehat{bd})+d(x,\widehat{b\widehat{bd}})d(x,\widehat{ad}))$$
$$+\frac{\nu}{\epsilon^4}(d(x,\widehat{c\widehat{ac}})d(x,\widehat{da})-d(x,\widehat{d\widehat{ad}})d(x,\widehat{ac})-d(x,\widehat{c\widehat{bc}})d(x,\widehat{bd})+d(x,\widehat{d\widehat{bd}})d(x,\widehat{bc}))$$
$$+\frac{\sigma}{\epsilon^4}(d(x,a)d(x,\widehat{d\widehat{cb}})-d(x,a)d(x,\widehat{c\widehat{db}})-d(x,b)d(x,\widehat{d\widehat{ca}})+d(x,b)d(x,\widehat{c\widehat{da}}))$$
$$+\frac{\sigma}{\epsilon^4}(d(x,c)d(x,\widehat{b\widehat{ad}})-d(x,c)d(x,\widehat{a\widehat{db}})-d(x,d)d(x,\widehat{b\widehat{ca}})+d(x,d)d(x,\widehat{a\widehat{bc}}))$$
$$+\frac{\psi}{\epsilon^4}(d(x,a)d(x,\widehat{d\widehat{ab}})-d(x,a)d(x,\widehat{c\widehat{ab}})-d(x,b)d(x,\widehat{d\widehat{ba}})+d(x,b)d(x,\widehat{d\widehat{db}}))$$
$$+\frac{\psi}{\epsilon^4}(d(x,c)d(x,\widehat{b\widehat{cd}})-d(x,c)d(x,\widehat{a\widehat{cd}})-d(x,d)d(x,\widehat{d\widehat{cd}})+d(x,d)d(x,\widehat{b\widehat{db}}))$$

$$+\frac{\phi}{\epsilon^4}(d(x,a)d(x,\widehat{b\widehat{c}b})-d(x,a)d(x,\widehat{b\widehat{d}b})-d(x,b)d(x,\widehat{a\widehat{c}a})+d(x,b)d(x,\widehat{a\widehat{d}a}))$$
$$+\frac{\phi}{\epsilon^4}(d(x,c)d(x,\widehat{d\widehat{a}d})-d(x,c)d(x,\widehat{d\widehat{b}d})-d(x,d)d(x,\widehat{c\widehat{c}a})+d(x,d)d(x,\widehat{c\widehat{c}b}))$$
$$+\frac{\pi}{\epsilon^4}(d(x,a)d(x,\widehat{a\widehat{c}b})-d(x,a)d(x,\widehat{a\widehat{d}b})-d(x,b)d(x,\widehat{b\widehat{c}a})+d(x,b)d(x,\widehat{b\widehat{d}a}))$$
$$+\frac{\pi}{\epsilon^4}(d(x,c)d(x,\widehat{c\widehat{a}d})-d(x,c)d(x,\widehat{c\widehat{b}d})-d(x,d)d(x,\widehat{d\widehat{c}a})+d(x,d)d(x,\widehat{d\widehat{c}b})).$$

The reader notices that all terms up to the ρ term satisfy our criterium for the first Bianchi identity and that the subsequent terms do not. Explicit computation reveals that, indeed, all those terms are first Bianchi violating. "Ricci-Alexandrov" contraction provides one with

$$T_x^{\epsilon,\alpha,\beta,\kappa,\delta,\lambda,\gamma,\mu,\zeta,\rho,\nu,\sigma,\psi,\phi,\pi}(a,b,a,b)=\frac{1}{\epsilon^4}\left((2\delta-2\kappa+\alpha)\,d(a,b)^2+(\lambda+\mu+2\zeta)(d(x,a)^2+d(x,b)^2)\right)$$
$$+\frac{1}{\epsilon^4}\left(-(2\lambda+\gamma-\rho)d(x,\widehat{ab})^2+(\gamma-2\mu-\rho-4\phi)d(x,a)d(x,b)-\delta(d(a,\widehat{ab})+d(b,\widehat{ab}))\right)$$
$$+\frac{2((\nu-\sigma)-\zeta)}{\epsilon^4}\left(d(x,a)d(x,\widehat{ab})+d(x,b)d(x,\widehat{ab})\right)+$$
$$\frac{2((\sigma-\nu)+\phi-(\pi-\psi))}{\epsilon^4}\left(d(x,a)d(x,\widehat{b\widehat{a}b})+d(x,b)d(x,\widehat{a\widehat{a}b})\right)$$
$$+\frac{2(\pi-\psi)}{\epsilon^4}\left(d(x,a)d(x,\widehat{a\widehat{a}b})+d(x,b)d(x,\widehat{b\widehat{a}b})\right)$$

Therefore, $-2\lambda-\gamma+\rho=4,\lambda+\mu+2\zeta=-2,\zeta-\nu+\sigma=0$ and

$$2\delta-2\kappa+\alpha=1,\gamma-2\mu-\rho-4\phi=0,\delta=0=\pi-\psi=\zeta-\phi$$

which leaves six free parameters of in total of fourteen parameters given that the system has no degeneracy. α,β,ν however appear to be redundant given that the total number of letters in the d arguments is bigger than four which leaves for three free parameters. In differential geometry, we have two ambiguities up to second order with the symmetries of the Riemann tensor, which produce a vanishing sectional term: they are given by $R\sqrt{g}\epsilon_{\mu_1\mu_2\mu_3\mu_4},\sqrt{g}\epsilon_{\mu_1\mu_2\mu_3\mu_4}$ which suggests another parameter should be fixed.

The last three coefficients of our "Ricci contraction" given by

$$(\pi-\psi),(\sigma+\phi)-(\pi-\psi),-(\sigma+\phi)$$

reveal a cycle structure in the sense that they sum up to zero. This is a kind of vanishing of the edges condition over a triangle given by

$$0,(\pi-\psi),(\sigma+\phi).$$

It suggests that some related area quantity vanishes which is precisely the case due to the dissapearance of the associated contribution to the Alexandrov curvature by means of "Ricci contraction" and it merely creates a diagonal term

$$(d(x,a)-d(x,b))^2$$

by means of ϕ which is what it was supposed to do. We are therefore left with $\psi,\phi,\mu,\sigma,\alpha,\beta$ and moreover

$$\delta=0,\pi=\psi,\zeta=\phi,\nu=\zeta+\sigma,\lambda=-2-\mu-2\zeta,\gamma=2\mu+\rho+4\phi,\kappa=-\frac{1}{2}(1-\alpha).$$

There is, moreover, something really embarrasing about the μ, γ, ρ terms which is that they represent simple coordinates and are in a form pure gauge, under rescalings of the vectors $(xa), (xb), \ldots$ to be absorbed into a gauge piece of the other tensors; therefore $\mu = \gamma = \rho = \alpha = \beta = \nu = \delta = 0$ and we are left with the tensor

$$T_x^{\epsilon,\alpha,\beta,\kappa,\delta,\lambda,\gamma,\mu,\zeta,\rho,\nu,\sigma,\psi,\phi,\pi}(a,b,c,d) = T_x^{\epsilon,0,0,-\frac{1}{2},0,-2(1+\zeta),4\zeta,0,\zeta,0,0,-\zeta,\pi,\zeta,\pi}(a,b,c,d)$$

and the reader verifies that the sum of all coefficients equals $-\frac{5}{2} + 3\zeta + 2\pi$ which provides for another cocycle identity for four volumes in case

$$3\zeta + 2\pi = \frac{5}{2}.$$

Obviously, this case is forbidden as it would remove the Riemann tensor from our curvature expression; therefore, we have some "critical value" left with vanishing curvature corresponding to a combination of our previous two volume forms (which is well known to be true in four dimensions). The right value clearly corresponds to $\zeta = 0$ as there should be no zero'th order contributions left whereas π represents a kind of scale freedom which is not present in continuum theories. It must depend upon the fine grained details of our geometry as well as ϵ and we shall fix it in the upcoming paragraph.

- We shall now work towards the Einstein tensor and show that it is God given upon a constant. Define the "metric"

$$g_x^{\epsilon}(a,b) = \frac{d(x,a)d(x,b)\cos(\theta_x(a,b))}{\epsilon^2}$$

and consider the unique inverse $g_x^{\epsilon}(\widehat{a}, \widehat{b})$ on a general locally compact path metric space with duality relation $\widehat{a}(b) = \delta(a,b)$ where we shall shortly define the equation

$$\int_V d\mu_d(a) f(a) \delta(a,b) = f(b)$$

with $d\mu_d$ the Hausdorff measure defined by the metric d and V contains a as well as b. The Hausdorff measure is defined by stacking balls of radius r with balls of smaller radius and by asserting that

$$\mathrm{Vol}(x,r) = \alpha(x) r^{d_H(x)}$$

in the limit for r to zero and $d_H(x)$ is the so called Hausdorff dimension of x. Both $\alpha(x), d_H(x)$ are assumed to be continuous functions almost everywhere and they are partially determined by means of

$$\sum_{\text{Small balls stacking big ball } B(x_i,r_i) \in B(y,r)} \alpha_i r_i^{d_{H,i}} = \alpha_y r^{d_{H,y}}$$

in the limit for r to zero. The reader verifies that this fixes d_H and α up to a constant. The definition now is that

$$\int_{B(x,\epsilon)} d\mu_d(b) g_x^{\epsilon}(\widehat{a}, \widehat{b}) g_x^{\epsilon}(b,c) = \delta(a,c).$$

The construction of a δ function is totally obvious and the existence of a unique inverse follows from the fact that $g_x^{\epsilon}(b,c)$ defines a

Toeplitz operator with vanishing kernel, $g^{\epsilon}_x(\widehat{a},\widehat{b})$ is then the standard Green's function with respect to the Hausdorff measure. As a proof, just remark that the functions $g^{\epsilon}_x(b,c)$ separate the points meaning that the functions

$$g^{\epsilon}_x(b,\cdot) : B(x,\epsilon) \to \mathbb{R} : c \to g^{\epsilon}_x(b,c)$$

constitute a basis with respect to the ordinary Hilbert space scalar product.

- Before we define contractions with the metric tensor, we must be a bit careful given that the expression

$$\int_{B(x,\epsilon)} \int_{B(x,\epsilon)} d\mu_d(b) d\mu_d(a) g^{\epsilon}_x(\widehat{a},\widehat{b}) g^{\epsilon}_x(b,a)$$

is ill defined which suggests a point splitting and limiting procedure providing for the correct answer. In either, we take

$$\int_{B(x,\epsilon)} \int_{B(x,\epsilon)} d\mu_d(b) d\mu_d(a) \int_{B(a,\delta)} \int_{B(b,\delta)} d\mu_d(d) d\mu_d(c) g^{\epsilon}_x(\widehat{a},\widehat{b}) g^{\epsilon}_x(c,d) \sim$$

$$\alpha^2(x)\delta^{2d_{H,x}}(1 + \text{epsilon, delta corrections})$$

which suggests to take the δ derivative and divide this expression through

$$2\alpha^2(x)\delta^{2d_{H,x}-1}$$

when taking the limit $\delta \to 0$. In that vein, define the "Alexandrov Ricci tensor" on a scale ϵ by means of

$$T^{\epsilon,\pi}_x(a,b) := \lim_{\delta\to 0} \frac{1}{2\alpha^2(x)\delta^{2d_{H,x}-1}} \frac{d}{d\delta} \int_{B(x,\epsilon)} d\mu_d(k) \int_{B(x,\epsilon)} d\mu_d(l) \int_{B(k,\delta)} d\mu_d(r) \int_{B(l,\delta)} d\mu_d(s) g^{\epsilon}_x(\widehat{k},\widehat{l})$$

$$T_x^{\epsilon,0,0,-\frac{1}{2},0,-2,0,0,0,0,0,0,\pi,0,\pi}(r,a,s,b)$$

and a Ricci scalar $T^{\epsilon,\pi}_x$ by means of a similar contraction. The Einstein tensor is then defined as

$$G^{\epsilon,\pi}_x(a,b) = T^{\epsilon,\pi}_x(a,b) - \frac{1}{2} g^{\epsilon}_x(a,b) T^{\epsilon,\pi}_x.$$

Now, we come to the crucial part which should extract the real Riemann tensor and remove any further discrete ambiguity; the trace of the Einstein tensor, $G^{\epsilon,\pi}_x$ must satisfy

$$G^{\epsilon,\pi}_x = -\frac{d_{H,x}-2}{2} T^{\epsilon,\pi}_x$$

exactly which turns π into a function of x and ϵ. The reader might argue that the presence of Bianchi violating terms due to the π coefficient are a consequence of non-smoothness and should vanish in the appropriate limit.

- Define g^{ϵ} geodesics as curves γ which minimize the length fiunctional

$$L(\gamma) = \sup_{0=t_0,\ldots t_n=1, n>0} \sum_{j=0}^{n-1} \sqrt{g^{\epsilon}_{\gamma(t_j)}(\gamma(t_{j+1}),\gamma(t_{j+1}))}\, \epsilon.$$

It is clear that this definition coincides with the usual case.

- We now come to the definition of the Levi Civita transporter using results from the previous section. We call a transporter $\nabla^{\epsilon,\delta}_{(x,a)}$ metric compatible if and only if

$$\nabla^{\epsilon,\delta}_{(x,a)}(g^{\epsilon}_{x}(b,c)) = g^{\epsilon}_{a}(\pi_2(\nabla^{\epsilon,\delta}_{(x,a)}(x,b)), \pi_2(\nabla^{\epsilon,\delta}_{(x,a)}(x,c))) = g^{\epsilon}_{x}(b,c)$$

for all x and $a, b, c \in B(x, \epsilon)$ with $0 \leq \theta_x(a,c), \theta_x(a,b) < \frac{\pi-\delta}{2}$. Clearly, all the ∇-ϵ dependency hides in the condition that a, b, c are sufficiently close to x and δ serves to allow for conal singularities. Obviously, $d(x,a) = d(\nabla^{\epsilon,\delta}_{(x,b)}(x,a))$ for all x, a, b for which $\nabla^{\epsilon,\delta}$ has been defined. On a conal space, plenty of $\nabla^{\epsilon,\delta}$'s exist whereas 2δ exceeds the deficiency angle in case of a spherical cone.

- A ∇ geodesic is an autoparallel curve in arclength parametrization: that is

$$\nabla_{(\gamma(t),\gamma(t+\epsilon))}(\gamma(t), \gamma(t+\epsilon)) \circ (\gamma(t), \gamma(t+\epsilon)) = (\gamma(t), \gamma(t+2\epsilon)).$$

Show, by means of an example, that not every geodesic is a ∇-geodesic for a given ∇ such as occurs in spaces with hyperbolic conal singularities. Reversely, not every ∇-geodesic is a geodesic in the sense that

$$d(\gamma(t), \gamma(t+2\delta)) < 2\delta$$

what often happens globally for Riemannian manifolds or in case of spherical conal singularities where plenty of ∇ geodesics exist which are not geodesics. Obviously, every geodesic is a ∇-geodesic for *some* ∇; the very fact that every ∇-geodesic should always be a geodesic locally imposes the condition that

$$\theta_y(x, \pi_2(\nabla_{(x,y)}(x,z))) = \pi - \theta_x(y,z)$$

for $\theta_x(y,z) < \pi - \delta$ which is a forwards-backwards condition related to differentiability. Moreover, the demand that every geodesic is a ∇ geodesic for *any* ∇ imposes that no hyperbolic conical singularities may occur. Therefore, coincidence of both notions suggests the condition that space is locally $\mathbb{R}^n$.

- Define the scalar product

$$\langle T(\cdot)|g(\cdot)\rangle_{x,\epsilon} = \int_{B(x,\epsilon} \ldots \int_{B(x,\epsilon} d\mu_d(a_1)\ldots d\mu_d(a_k) d\mu_d(b_1)\ldots d\mu_d(b_k) \overline{T(a_i)} g(b_i) g^{\epsilon}_{x}(\kappa_i(a_i), \kappa_i(b_i))$$

where κ_i equals the identity or duality operation. Then, this scalar product is not necessarily positive definite which it should be for spaces which are positively Alexandrov curved and might not be for those which are negatively Alexandrov curved. In any case, any connection which minimizes the Bianchi tensor

$$\sum_{\sigma \in S_3} \mathrm{Sign}(\sigma) \nabla_{a_{\sigma(1)}} R_x(a_{\sigma(2)}, a_{\sigma(3)}, b, c)$$

an expression having precisely as many degrees of freedom as the Torsion tensor does in four dimensions, due to Hodge duality, with respect to this scalar product, is a candidate Levi-Civita connection. In more than four dimensions, vanishing of the Bianchi tensor involves many more degrees of freedom as the Torsion tensor provides for but it is nevertheless equivalent to it; the system is reducible. In two dimensions, there is no second Bianci identity whereas in three dimensions one is left with nine Torsion degrees of freedom whereas the Bianchi tensor only produces three of them which presents an unfortunate exception to our way of thinking about torsion.

- A better way to define the Levi-Civita connection is by demanding that some appropriate expression should match the Riemann tensor by using the above definition for the Lie-Bracket given a metric compatible connection. Given that

$$[(x,v),(x,w)] = (\nabla_{(x,v)}(x,w) \circ (x,v)) \ominus (\nabla_{(x,w)}(x,v) \circ (x,w))$$

is the natural definition for the Lie-Bracket, zero torsion

$$T^{\epsilon,\delta}_x(a,b) \in R(x,\cdot)$$

with $a,b \in B(x,\epsilon)$, is defined by stating that $R_x(a,b,c)$ is given by

$$\left(\nabla^{\epsilon,\delta}_{(x,a)}(\nabla^{\epsilon,\delta}_{(x,b)}(x,c) \circ (x,b)) \circ (x,a) \ominus \nabla^{\epsilon,\delta}_{(x,b)}(\nabla^{\epsilon,\delta}_{(x,a)}(x,c) \circ (x,a)) \circ (x,b)\right) \ominus$$

$$\nabla^{\epsilon,\delta}_{[(x,a),(x,b)]}(x,c) \circ [(x,a),(x,b)]$$

and that

$$\hat{d}(R_x(a,b,c)) = \int_{B(x,\epsilon)} d\mu_d(k) g^{\epsilon}_x(\hat{d},\hat{k}) R_x(a,b,c,k)$$

for a metric compatible connection $\nabla^{\epsilon,\delta}$. This demand might be somewhat strong and minimization of an associated quadratic identity somewhat desirable. Such connection is called Levi-Civita and torsion of another associated metric compatible connection $\widetilde{\nabla}^{\epsilon,\delta}$ is defined by

$$T^{\epsilon,\delta}_x = \widetilde{\nabla}^{\epsilon,\delta} \ominus \nabla^{\epsilon,\delta}.$$

This does not imply that

$$\sum_{\sigma \in S_3} \mathrm{Sign}(\sigma) \nabla_{a_{\sigma(1)}} R_x(a_{\sigma(2)}, a_{\sigma(3)}, b, c) = 0$$

is satisfied.

4 Simplicial refinements.

Simplicial metric spaces are simple examples of generalizations of Riemannian manifolds and the metric structure is fully characterized by distances $d(v_0 v_1)$ on the edges $(v_0 v_1)$. Given that one has more structure than usual, it is possible to get more close to the manifold language which is what we shall develop partially in this concluding section. We first start by defining the operators $x_w(v_0 \ldots v_i) = (w v_0 \ldots v_i)$ en $\partial_w(w v_0 \ldots v_i) = (v_0 \ldots v_i)$ in case none of the v_j equals w. In case this would be true, $\partial_w(w) = \mathbf{1}$, $x_w \mathbf{1} = (w)$ where $\mathbf{1} = ()$ equals the empty simplex. From this follows that $(x_w)^2 = 0$ as well as $(\partial_w)^2 = 0$ giving rise to a natural point Grasmann algebraic structure. One notices that $\partial = \sum_{w \in S} \partial_w$ which shows that ∂_w is the correct partial differential operator associated to the Hodge boundary operator ∂ giving rise to a natural theory of k forms. The empty simplex constitutes the identity element regarding the cross product $*$ defined by

$$(v_0 \ldots v_i) * (w_0 \ldots w_j) = (v_0 \ldots v_i w_0 \ldots w_j).$$

One simply verifies indeed that $x_w x_v = -x_v x_w$ and likewise for the operators ∂_v, ∂_w as well as

$$\partial_v x_w + x_w \partial_v = \delta(v, w)$$

giving rise to the usual Heisenberg duality where ∂_v would be the associated momentum operator. One verifies that x_w, ∂_v satisfy the fermionic Leibniz rule with respect to the $*$ product and that $\mathbf{1}$ is bosonic with respect to the action of x_w regarding $*$. Bosonic operators are then formed by considering even simplicial structures; the line segment provides one with

$$\partial_{(vw)} = \partial_w \partial_v$$

which obeys

$$\partial_{(vw)}(yz) = \delta(v, y)\delta(w, z) - \delta(v, z)\delta(w, y)$$

providing one with an oriented derivative. The simplex algebra is generated by polynomials constituting of monomials which are free products of $(v_0 \ldots v_j)$ for all $j : 0 \ldots n$; mind, the formal product does *not* equal the cross product implying that $\mathbf{1}$ is no longer equal to unity. Since on general metric spaces, bi-relations are merely characterized by means of a metric d the function algebra is limited to monomials in $(v_0 v_1)$ given that higher simplices do not provide for independent higher invariants. Assuming, furthermore, that $\mathbf{1}$ is bosonic with respect to the action of x_w, also for the free product, taking into account that ∂_v, x_w are both fermionic operators, one arrives at

$$\partial_v((w)Q) = \partial_v((x_w \mathbf{1})Q) = \partial_v x_w(\mathbf{1}Q) - \partial_v(\mathbf{1}x_w Q) = (k+1)\delta(v, w)\mathbf{1}Q - x_w(\mathbf{1}\partial_v Q) - \partial_v(\mathbf{1}x_w Q)$$

which reduces to

$$(k+1)\delta(v, w)\mathbf{1}Q - (x_w)\partial_v Q - \mathbf{1}x_w \partial_v Q - \mathbf{1}\partial_v x_w Q = \delta(v, w)\mathbf{1}Q - (x_w)\partial_v Q$$

where k is the degree of the monomial Q, which means the number of factors. This follows immediately from the Leibniz rule for bosonic operators

$$x_w \partial_v + \partial_v x_w = \delta(v, w).$$

Henceforth, akin to the $*$ product, the even simplex variables behave bosonic whereas the odd ones behave fermionic. Indeed,

$$\partial_v((wz)Q) = \partial_v((x_w(z))Q) = \partial_v(x_w((z)Q) + ((z)x_w Q)) = -x_w \partial_v((z)Q) - (z)(\partial_v x_w Q)$$

what reduces to

$$= x_w((z)\partial_v Q) - (z)(\partial_v x_w Q) = (wz)\partial_v Q.$$

The reason for introducing the formal product as a supplementary structure over the $*$ product resides in the fact that the latter allows only for linear function in the edge variables and the standard operations on real numbers would have to be recuperated in a rather different fashion by means of infinite pulverisation (excluding diagonal terms) instead of direct comparison with the simplicial line segments.

Standard derivatives are defined by means of an infinitesimal line segment $(x - |\epsilon|, x + |\epsilon|)$ where $f(v + \epsilon, v - \epsilon)$ has been defined by means of $f(x)$. This is logical because the $v \pm \epsilon$ are fermionic and independent whereas the segments $(v-\epsilon, v+\epsilon) \sim x$ are bosonic. Note that formal products of the kind $(v-\epsilon)(v+\epsilon)$ may be further derived and that

$$\partial_x f(x) = \mathbf{L}\left[\partial_{(v-\epsilon,v+\epsilon)} f(v-\epsilon, v+\epsilon)\right]$$

whereby $\mathbf{L}$ only retains monomials depending upon the line segments. To under stand this, consider $(vw)^2$ whose (vw) derivative equals

$$2(vw) - 2(v)(w).$$

In order to define the standard bosonic multiplication operator on line segments (vw), we define

$$\widehat{x}_{(vw)} Q := x_{(vw)} x_{\mathbf{1}} Q$$

where Q is a free polynomial in the line segments (r, s) and $x_{(vw)}$ is a bosonic Leibniz operator defined by means of

$$x_{(vw)}(v_0 \dots v_j) = (vwv_0 \dots v_j).$$

By definition, one has that

$$x_{(vw)}(rs) = 0$$

if and only if r or s equals v, w as well as

$$(x_{(vw)} + x_{(rs)})((vw) + (rs)) = 2(vwrs)$$

which vanishes identically unless (r, s) equals the opposite side of a four simplex which is never possible for geodesics amongst other curves. For geodesics

$$\gamma(v_0 v_i) := (v_0 v_1) + (v_1 v_2) + \dots (v_{i-1} v_i)$$

we have that

$$x_{\gamma(v_0 v_i)} := \sum_{j=1}^{i} x_{(v_{j-1} v_j)}$$

and therefore

$$x_{\gamma(v_0 v_i)} \gamma(v_0, v_i) = 0.$$

Next, we define derivatives

$$\partial_{\gamma(v_0, v_i)} := \sum_{j=1}^{i} \partial_{(v_{j-1} v_j)}$$

and consider the operator

$$\widehat{\partial}_{\gamma(v_0,v_i)} = \mathbf{L} \circ \partial_{\gamma(v_0,v_i)}$$

which satisfies

$$\widehat{\partial}_{\gamma(v_0,v_i)} \widehat{x}_{\gamma(v_0,v_i)} - \widehat{x}_{\gamma(v_0,v_i)} \widehat{\partial}_{\gamma(v_0,v_i)} = 1$$

on the function space of monomials Q of the form $(\gamma(v_0, v_i))^k$ where $k > 0$.

5 The Lorentzian case.

The question now is how to generalize the above setting to spaces equipped with a Lorentz distance. That is, we consider spaces (X, d) with a compact topology such that $d : X \times X \to \mathbb{R}^+$ is continuous and satisfies

- $d(x, y) \geq 0$ and $d(x, x) = 0$
- $d(x, y) > 0$ implies that $d(y, x) = 0$
- $d(x, y) > 0$ and $d(y, z) > 0$ implies that $d(x, z) > 0$.

As is well known, this defines a chronology relation $y \in I^+(x)$ if and only if $d(x, y) > 0$ where $I^+(x)$ is the set of all events lying in the chronological future of x. Likewise, one has the chronological past $I^-(x)$ containing all y such that $d(y, x) > 0$. Now, we assume the following regarding the partial order $\prec$ defined by $x \prec y$ if and only if $d(x, y) > 0$. That is, for any open $\mathcal{O}$ around x one has points y, z such that $y \prec x \prec z$ and $I^-(z) \cap I^+(y) \equiv A(y, z) \subset \mathcal{O}$. The sets $A(x, y)$ called the Alexandrov sets clearly define the basis for a topology and what we are saying is that the Alexandrov topology must coincide with the space topology. Looking back at the construction of the Riemann tensor, taking into account that only the κ, λ terms do not vanish, one needs either that $a, b \in I^-(c) \cap I^-(d) \cap I^+(x)$ where timelike geodesics are defined by means of a maximization instead of minimization procedure. The other way around is $c, d \in I^-(a) \cap I^-(b) \cap I^+(x)$ or two similar options with a, b, c, d to the past of x. It is well known that a Lorentzian geometry does not provide for compact neighborhoods but that one can nevertheless try to define a Hausdorff measure identically than before for Alexandrov neighborhoods. However, such measure would be direction independent which is the case for manifolds where the metric exhibits Lorentz invariance but not true for piecewise linear manifolds with conal singularities where the result may be direction dependent. Henceforth, it is much better to choose an additional Riemannian metric d and define the Lorentzian metric tensor $g^{\pm}_{\epsilon}(a, b)$ on the pair of points $(a, b) \in I^{\pm}(x)$ for which holds that $d(a, b) > 0$ or $d(b, a) > 0$ so that hyperbolic angles, replacing sine and cosine by sinh and cosh, and so forth are well defined. Calling these regions $Z^{\pm}(x)$, we may define the inverse $g^{\pm,\epsilon}(\hat{a}, \hat{b})$ by means of integration over $(B(x, \epsilon) \times B(x, \epsilon)) \cap Z^{\pm}(x)$ respectively. Herefrom, it is obvious to define the remaining contractions and Ricci tensor and scalars. It is obviuous that for general spacetimes and Riemannian metrics, the limit of ϵ to zero is independent of the choice of the latter.

6 Conclusions.

We have made first steps with developing geometry for general metric spaces as well as a natural gravitational theory defined upon it. It would be interesting to generalize this to the setting of Lorentz spaces endowed with a supplementary Riemannian notion of locality provided by a particular class of metrics.