Queens Puzzle Solution

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Contents

1 Queens Puzzle Definition

Original definition:The aim is to calculate the algorithm allowing the placement of N queens on a chessboard with sides equaling N, with $N \geq 4$ in such way that no queens threaten each other.

2 Queens Puzzle Solution Algorithm

2.1 First Particularization of Conditions

According to the conditions in the successive solution of the puzzle the N parameter will take even (N_{2M}) and uneven (N_{2M+1}) values:

$$N_{2M} = 2 * M, \quad \text{where} \quad M \geq 2, \tag{1}$$

$$N_{2M+1} = 2 * M + 1, \quad \text{where} \quad M \geq 2. \tag{2}$$

Let there be given N_{2M} (1), N_{2M+1} (2), apart from N, and M corresponding to them.

2.2 Conclusion 1

According to the conditions one queen shall be placed in each row of squares, both horizontally and vertically. So the puzzle engages each row of squares, both vertical and horizontal. Therefore, the arrangement of each queen in vertical (x) and horizontal (y) directions for all queens can be presented in the form of two equal sequences of natural numbers:

$$x \in \{x_n \mid n \in \mathbb{N},\ x_n \in \mathbb{N},\ 1 \leq x_n \leq N\}, \tag{3}$$

$$y \in \{y_n \mid n \in \mathbb{N},\ y_n \in \mathbb{N},\ 1 \leq y_n \leq N\}. \tag{4}$$

From expressions (3) and (4) it can be concluded:
Conclusion 1: Since the sequences of natural numbers $\{x_n\}$ and $\{y_n\}$ are the same, all calculation can be made using one variable x_n.

2.3 Formulae for Reference Conditions

The conditions for queens arrangement on the chessboard for sequences $\{x_n\}$ (3) and $\{y_n\}$ (4) can be presented in the following way:

$$x_{n1} + y_{n1} \neq x_{n2} + y_{n2}, \quad \text{where} \quad n1 \in \{n \mid 1 \leq n \leq N\}, \quad n2 \in \{n \mid 1 \leq n \leq N\}; \tag{5}$$

$$x_{n1} - y_{n1} \neq x_{n2} - y_{n2}, \quad \text{where} \quad n1 \in \{n \mid 1 \leq n \leq N\}, \quad n2 \in \{n \mid 1 \leq n \leq N\}. \tag{6}$$

2.4 New Concepts for Queens Arrangement

According to the conditions (2.3) we should introduce new concepts for comparison of arrangement of queens on the chessboard:

$$\Sigma = x + y, \tag{7}$$

$$S = x - y. \tag{8}$$

2.5 Expression for Queens

Taking into account expressions (7), (8) and conditions for queens arrangement on the chessboard (2.3), the following expression will be relevant for each queen from the N-quantity in the condition of the problem:

$$(\Sigma) \wedge x = y + S. \tag{9}$$

2.6 Second Definition of the Puzzle

One of the sequences of the natural numbers, characterizing the arrangement of the queens in the vertical direction, $\{x_n\}$ (3), can be considered as known in advance and moved to the conditions of the puzzle. In this case only the sequence of natural number, characterizing the queens arrangement in the horizontal direction, $\{y_n\}$ (4).
Now the original definition can be specified:
Second definition of the puzzle: The aim is to arrange the sequence of natural numbers $\{y_n\}$ along equal known sequence of natural numbers $\{x_n\}$ in such way that conditions (2.3) will be fulfilled.
(Note to the second definition: The arrangement of one sequence along the other equal one means that if both sequences have matching first members, the last members will also match. If the first member of second sequence corresponds to $(N - x_n)$-member of first sequence, then the last member of second sequence will correspond to $(N - (x_n + 1)))$-member of first sequence, where $1 \leq x_n \leq N - 2$.)

2.7 Second Particularization of Conditions

Due to the ambiguity of conditions (2.3) it is necessary to divide sequences $\{x_n\}$ (3) and $\{y_n\}$ (4) into equal subsets for puzzle solution. To eliminate the ambiguity of conditions (2.3), the division of sequences $\{x_n\}$ and $\{y_n\}$ into subsets shall be performed according to different rules.

2.7.1 Vertical Division of Chessboard

We divide the sequence of natural numbers $\{x_n\}$ (3) into sequence of natural numbers up to M and sequence of natural numbers from $(M + 1)$ to N:

$$x \in \{x_{\leq M} \mid x_{\leq M} \in \mathbb{N},\ 1 \leq x_{\leq M} \leq M\}, \tag{10}$$

$$x \in \{x_{>M} \mid x_{>M} \in \mathbb{N},\ M+1 \leq x_{>M} \leq N\}. \tag{11}$$

2.7.2 Horizontal Division of Chessboard

We divide the sequence of natural numbers $\{y_n\}$ (4) into sequence of even numbers $\{y_{2x}\}$ and sequence of uneven numbers $\{y_{2x-1}\}$:

$$y \in \{2 * x_n \mid n \in \mathbb{N}, 1 \leq x_n \leq M\}, \tag{12}$$

$$y \in \{2 * x_n - 1 \mid n \in \mathbb{N}, \quad \text{for } N_{2M} : 1 \leq x_n \leq M; \quad \text{for } N_{2M+1} : 1 \leq x_n \leq M+1\}. \tag{13}$$

2.8 Special Feature of New Sequences

According to the quantity of members for N_{2M} (1), as well as for N_{2M+1} (2) sequence $\{x_{\leq M}\}$ (10) is equal to sequence $\{y_{2x}\}$ (12), and sequence $\{x_{>M}\}$ (11) is equal to $\{y_{2x-1}\}$ (13). Therefore the queens will be placed in the lower half of the chessboard on the squares even in horizontal direction. In the upper half of the chessboard the queens will be placed on the squares uneven in horizontal direction.

2.9 Conclusion 2

From expression (9) it follows that only S can take both negative and positive values and also take value $S = 0$. Even though according to (8), N-number of queens can have a number of values $S_n < 0$ equal to $(N - 1)$. Conditions (2.3) limit this number in such way, so we can conclude the following:
Conclusion 2: For N number of queens on the chessboard with sides equaling N, the number of values $S_n < 0$ shall be equal or approximately equal to the number of values $S_n \geq 0$ (8).
Taking into account Conclusion 2, value of first member and sequence step of even numbers $\{y_{2x}\}$, all values

$S_n < 0$ or absolute majority of values $S_n < 0$ shall be placed in expressions for queens (9) which comply with known sequence of natural numbers up to M $\{x_{\leq M}\}$ (10) or on lower half of the chessboard. In this case the expressions for queens, conforming to known sequence of natural numbers from $(M+1)$ to N $\{x_{>M}\}$ (11) or in upper half of the chessboard, will mostly have values $S_n \geq 0$.
In accordance with the fact that the solution of the puzzle begins with small values ($M \geq 2$) (2.1), only two variants of arrangement of sequence $\{y_{2x}\}$ (12) along the sequence $\{x_{\leq M}\}$ are suitable for puzzle solution: first variant - all expression for queens have $S_n < 0$; second variant - expression for one queen has $S_n \geq 0$.

2.10 First Variant

This variant in expression of queens (9), placed in lower half of the chessboard, i.e. including sequence $\{x_{\leq M}\}$ (10), only has values $S_n < 0$ (8).
To place all values $S_n < 0$ in expressions for queens (9), including $\{x_{\leq M}\}$ (10) and to comply with conditions (2.3), the sequence of even numbers $\{y_{2x}\}$ (12) shall be placed along equal sequence of natural numbers $\{x_{\leq M}\}$ in such way that first members of both sequences are matching.
We provide the sequence of arrangement of queens in the form of expression (9) in the lower half of the chessboard conforming to sequence $\{x_{\leq M}\}$. This expression will be the first variant for puzzle solution:

$$(3 * x_n) \wedge x_n = 2 * x_n + (-x_n), \quad \text{where} \quad 1 \leq n \leq M, \quad 1 \leq x_n \leq M. \tag{14}$$

Expression (14) has the following regularity for all queens in this sequence:

$$\Sigma_n = 3 * x_n, \quad \text{where} \;\; 1 \leq x_n \leq M. \tag{15}$$

Due to the fact that this algorithm the sequence $\{y_{2x}\}$ is placed along sequence $\{x_{\leq M}\}$, condition (5) is sufficient to provide that in case of one expression for queens (9) in puzzle solution all Σ_n belong to one of sets given below:

$$\Sigma_n \in \{3 * x_n \mid x_n \geq 1\}, \tag{16}$$

$$\Sigma_n \in \{3 * x_n + 1 \mid x_n \geq 1\}, \tag{17}$$

$$\Sigma_n \in \{3 * x_n + 2 \mid x_n \geq 1\}. \tag{18}$$

In case of two or three expressions for queens in one puzzle solutions, for condition (5) it is sufficient that Σ_n of those two or three expressions for queens belong to different sets (16), (17) or (18).

2.11 Second Variant

This variant in expressions for queens (9) placed in lower half of chessboard, i.e. including the sequence $\{x_{\leq M}\}$ (10), apart from values $S_n < 0$ (8) also has one value $S_n \geq 0$.
To comply with Conclusion 2 and conditions (2.3), the sequence of even numbers $\{y_{2x}\}$ (12) shall be placed along the equal sequence of natural numbers $\{x_{\leq M}\}$ (10) in such way that first member of sequence $\{y_{2x}\}$ is matching with last member of sequence $\{x_{\leq M}\}$.
We provide the sequence of arrangement of queens in the lower half of the chessboard, conforming to sequence $\{x_{\leq M}\}$, now in two expressions which will constitute the second variant for puzzle solution:

$$\begin{aligned} &1.\ (3 * x_n + 2) \wedge x_n = (2 * x_n + 2) + (-x_n - 2), \quad \text{where} \quad 1 \leq n \leq M-1, \quad 1 \leq x_n \leq M-1; \\ &2.\ (M+2) \wedge M = 2 + (M-2). \end{aligned} \tag{19}$$

2.12 Conclusion 3

In (19) the first expression has noticeable regularity for expressions with values $S_n < 0$:

$$\Sigma_n = 3 * x_n + 2, \quad \text{where} \quad 1 \leq n \leq M - 1, \quad 1 \leq x_n \leq M - 1. \tag{20}$$

Therefore due to the fact that $(M + 2) < (3 * x_n + 2)$ with $x_n = M - 1$, in order to conform to condition (5) the following inequation shall be complied with:

$$M + 2 \neq 3 * x_n + 2, \quad \text{where} \quad x_n \geq 1. \tag{21}$$

Let's express from expression (21) M:

$$M \neq 3 * x_n, \quad \text{where} \quad x_n \geq 1. \tag{22}$$

From expression (22) we conclude the following:
Conclusion 3: The expressions (19) will constitute the second variant for puzzle solution for even and uneven N (N_{2M} and N_{2M+1}) at $M = 3 * x_n + 1$ or $M = 3 * x_n - 1$, where $x_n \geq 1$.

2.13 Conclusion 4

The ambiguity of conditions (2.3), Conclusion 3 and necessity of compliance of integrity of sequences $\{x_n\}$ and $\{y_n\}$ in expressions for queens conclude to the following:
Conclusion 4: For final elimination of ambiguity of conditions (2.3) during the problem solution the natural sequence $N \geq 4$ shall be divided not only into sequences of even (N_{2M}) and uneven (N_{2M+1}) numbers, but also according to the following values M of this N:

$$M \in \{3 * x_n \mid x_n \geq 1\}, \tag{23}$$

$$M \in \{3 * x_n + 1 \mid x_n \geq 1\}, \tag{24}$$

$$M \in \{3 * x_n - 1 \mid x_n \geq 1\}. \tag{25}$$

2.14 Final Definition

Because of the small number of variants of expression (14) and (19) (provided the compliance of expressions (19) to Conclusion 3) can be moved to the condition of the puzzle. Namely, it is necessary to give the following specification:
Final definition: In case of given in (14) or (19) expressions for queens in the lower half of the chessboard up to $x = M$, to arrange the sequence of uneven numbers $\{y_{2x-1}\}$ (13) in the upper half of the board along the sequence of natural numbers from $M + 1$ to N ($\{x_{>M}\}$) in such way that conditions (2.3) are complied with.

2.15 First Main Solution

No we can start with puzzle solution according to Conclusion 4. Let's analyze the sequence $\{x_{>M}\}$ (11) under the condition of first variant (14) for even N_{2M} (1).
In first variant of (14) all values of $S < 0$ are collected. To provide that the expressions for queens (9) in the upper half of the chessboard, i.e. with sequence $\{x_{>M}\}$, have only $S \geq 0$, the matching of first members of sequences $\{x_{>M}\}$ and $\{y_{2x-1}\}$ (13) (13) is necessary.
We provide the sequence of arrangement of queens in the upper half of the chessboard in the form of the expression for queens (9), conforming to the sequence $\{x_{>M}\}$:

$$\begin{gathered}(3 * x_n - 2 * M - 1) \wedge x_n = (2 * x_n - 2 * M - 1) + (2 * M - x_n + 1), \\ \text{where} \quad M + 1 \leq n \leq 2 * M, \quad M + 1 \leq x_n \leq 2 * M.\end{gathered} \tag{26}$$

Due to the possibility of matching Σ_n of last members of expression for queens (14) with Σ_n of first member of expression for queens (26) in (14) $\Sigma_n = 3 * x_n$, where $1 \le x_n \le M$, the following inequation shall be complied with to satisfy condition (5):

$$3 * (M + 1) - 2 * M - 1 \ne 3 * x_n, \quad \text{where} \quad 1 \le x_n \le M \tag{27}$$

Let's express from (27) the condition for M:

$$M \ne 3 * x_n - 2 = 3 * (x_n - 1) + 1, \quad \text{where} \quad x_n \ge 2. \tag{28}$$

Therefore, $M \in \{3 * x_n \mid x_n \ge 1\}$ or $M \in \{3 * x_n - 1 \mid x_n \ge 1\}$ (see Conclusion 4)
During the addition of expressions (26) and (14) we will receive the expressions for queens for whole chessboard:

$$\begin{aligned}
&1.\ (3 * x_n) \wedge x_n = 2 * x_n + (-x_n), \quad \text{where} \quad 1 \le n \le M, \quad 1 \le x_n \le M; \\
&2.\ (3 * x_n - 2 * M - 1) \wedge x_n = (2 * x_n - 2 * M - 1) + (2 * M - x_n + 1), \\
&\qquad \text{where} \quad M + 1 \le n \le 2 * M, \quad M + 1 \le x_n \le 2 * M.
\end{aligned} \tag{29}$$

Removing from the expressions for queens (29) Σ_n and S_n we will receive **First main solution** of puzzle for even N_{2M} (1) with $M \in \{3 * x_n \mid x_n \ge 1\}$ or $M \in \{3 * x_n - 1 \mid x_n \ge 1\}$ where x_n means the value of queens in vertical direction and y_n - value of queens in horizontal direction:

$$\begin{aligned}
&1.\ y_n = 2 * x_n, \quad \text{where} \quad 1 \le n \le M, \quad 1 \le x_n \le M; \\
&2.\ y_n = 2 * x_n - 2 * M - 1, \quad \text{where} \quad M + 1 \le n \le 2 * M, \quad M + 1 \le x_n \le 2 * M.
\end{aligned} \tag{30}$$

2.16 Second Main Solution

Now we will analyze the sequence $\{x_{>M}\}$ (11) under the condition of first variant (14) for uneven N_{2M+1} (2). Due to the condition only $S \ge 0$ in expressions for queens the matching of first members of sequence $\{x_{>M}\}$ and $\{y_{2x-1}\}$ (13) is necessary.
Let's provide the sequence of arrangement of queens in the upper half of the chessboard in the form of the expression for queens (9), corresponding to the sequence $\{x_{>M}\}$:

$$\begin{aligned}
&(3 * x_n - 2 * M - 1) \wedge x_n = (2 * x_n - 2 * M - 1) + (2 * M - x_n + 1), \\
&\quad \text{where} \quad M + 1 \le n \le 2 * M + 1, \quad M + 1 \le x_n \le 2 * M + 1.
\end{aligned} \tag{31}$$

The expression (31) is different from (26) only in addition of last number with $x = 2*M+1$ and $y = 2*M+1$. It has $S = 0$. Also the condition (28) is correct for expression (31).
During the addition of expressions (31) and (14) we will receive the expressions for queens for the whole chessboard:

$$\begin{aligned}
&1.\ (3 * x_n) \wedge x_n = 2 * x_n + (-x_n), \quad \text{where} \quad 1 \le n \le M, \quad 1 \le x_n \le M; \\
&2.\ (3 * x_n - 2 * M - 1) \wedge x_n = (2 * x_n - 2 * M - 1) + (2 * M - x_n + 1), \\
&\qquad \text{where} \quad M + 1 \le n \le 2 * M + 1, \quad M + 1 \le x_n \le 2 * M + 1.
\end{aligned} \tag{32}$$

Removing from expressions for queens (32) Σ_n and S_n, we will receive **Second main solution** of puzzle for uneven N_{2M+1} (2) with $M \in \{3 * x_n \mid x_n \ge 1\}$ or $M \in \{3 * x_n - 1 \mid x_n \ge 1\}$, where x_n means the value for queens in vertical direction and y_n - value of queens in horizontal direction:

$$\begin{aligned}
&1.\ y_n = 2 * x_n \quad \text{where} \quad 1 \le n \le M, \quad 1 \le x_n \le M; \\
&2.\ y_n = 2 * x_n - 2 * M - 1, \quad \text{where} \quad M + 1 \le n \le 2 * M + 1, \quad M + 1 \le x_n \le 2 * M + 1.
\end{aligned} \tag{33}$$

2.17 Third Main Solution

Due to the condition (28) and Conclusion 3 for even N_{2M} (1) and uneven N_{2M+1} (2) with $M \in \{3 * x_n + 1 \mid x_n \geq 1\}$ the second variant (19) will constitute the condition of puzzle solution.
Let's consider the expressions for queens (9) with sequence $\{x_{>M}\}$ (11) under the condition (19) for even N_{2M}.
Due to the fact that (19) has one expression for queens with positive $S_n = M - 2$, and negative $S_n \leq -3$, it is necessary that expressions for queens with sequence $\{x_{>M}\}$ have values of negative $0 > S_n > -3$. For maximum $y_n = 2*M-1$ there are only two variants of mutual arrangement of sequences $\{x_{>M}\}$ and $\{y_{2x+1}\}$ (13).

Let's provide the sequence of arrangement of queens in the upper half of the chessboard in the form of expressions for queens (9), corresponding to sequence $\{x_{>M}\}$ with one negative value $S_n = -1$:

$$\begin{aligned} &1.\ (3 * x_n - 2 * M + 3) \wedge x_n = (2 * x_n - 2 * M + 3) + (2 * M - x_n - 3), \\ &\qquad \text{where} \quad M + 1 \leq n \leq 2 * M - 2, \quad M + 1 \leq x_n \leq 2 * M - 2; \\ &2.\ (3 * x_n - 2 * M - 5) \wedge x_n = (2 * x_n - 2 * M - 5) + (2 * M - x_n + 5), \\ &\qquad \text{where} \quad 2 * M - 1 \leq n \leq 2 * M, \quad 2 * M - 1 \leq x_n \leq 2 * M. \end{aligned} \tag{34}$$

The smallest Σ_n of first and second expression (34), during the replacement of x_n to M, will be bigger than $\Sigma_n = M + 2$ of the last expression (19):

$$3 * x_n - 2 * M + 3 = 3 * M + 3 - 2 * M + 3 = M + 6 > M + 2, \quad \text{where} \quad M \geq 4; \tag{35}$$

$$3 * x_n - 2 * M - 5 = 3 * M - 3 - 2 * M - 5 = 4 * M - 8 > M + 2, \quad \text{where} \quad M \geq 4. \tag{36}$$

Here we will identify x_n of expressions (19) as x_{n1}. Due to the fact that first expression (19) has $\Sigma_n = 3*x_{n1}+2$, we will find to which from the sets (16), (17) or (18) Σ_n of first and second expressions (34) belongs to, by substituting for this purpose the value $M = 3 * x_{n1} + 1$, where $x_{n1} \geq 1$, in expressions Σ_n from condition of (2.17):

$$3 * x_n - 2 * M + 3 = 3 * (x_n - 2 * x_{n1}) + 1, \quad \text{where} \quad M + 1 \leq x_n \leq 2 * M - 2, \quad x_{n1} \geq 1; \tag{37}$$

$$3 * x_n - 2 * M - 5 = 3 * (x_n - 2 * x_{n1} - 3) + 2, \quad \text{where} \quad 2 * M - 1 \leq x_n \leq 2, \quad x_{n1} \geq 1. \tag{38}$$

Σ_n of first expression (19) and Σ_n (38) belongs to one set (18). Σ_n (37) belong to set (17). Let's compare the smallest Σ_n (38) and biggest Σ_n of first expression (19), representing them through M:

$$3 * x_n - 2 * M - 5 = 4 * (M - 2) < 3 * x_{n1} + 2 = 3 * (M - 1) + 2, \quad \text{where} \quad 4 \leq M \leq 7. \tag{39}$$

Σ_n (38) at $M < 8$ will always match with Σ_n of first expression (19). The condition (5) is not complied with and expressions (34) can not constitute the puzzle solution.

Let's provide the sequence of arrangement of queens in the upper half of the chessboard in the form of expressions for queens (9), corresponding to the sequence $\{x_{>M}\}$ with two negative values $S_n \geq -2$:

$$\begin{aligned} &1.\ (3 * x_n - 2 * M + 5) \wedge x_n = (2 * x_n - 2 * M + 5) + (2 * M - x_n - 5), \\ &\qquad \text{where} \quad M + 1 \leq n \leq 2 * M - 3, \quad M + 1 \leq x_n \leq 2 * M - 3; \\ &2.\ (3 * x_n - 2 * M - 3) \wedge x_n = (2 * x_n - 2 * M - 3) + (2 * M - x_n + 3), \\ &\qquad \text{where} \quad 2 * M - 2 \leq n \leq 2 * M, \quad 2 * M - 2 \leq x_n \leq 2 * M. \end{aligned} \tag{40}$$

The smallest Σ_n for first and second expression (40) during replacement of x_n with M will be bigger than $\Sigma_n = M + 2$ of the last expression (19):

$$3 * x_n - 2 * M + 5 = 3 * M + 3 - 2 * M + 5 = M + 8 > M + 2, \quad \text{where} \quad M \geq 4; \tag{41}$$

$$3 * x_n - 2 * M - 3 = 3 * M - 3 - 2 * M - 3 = 4 * M - 9 > M + 2, \quad \text{where} \quad M \geq 4. \tag{42}$$

Here we will identify x_n of expressions (19) as x_{n1}. Due to the fact that first expression (19) has $\Sigma_n = 3*x_{n1}+2$, we will find to which from the sets (16), (17) or (18) Σ_n of first and second expressions (40) belongs to, by substituting for this purpose the value $M = 3 * x_{n1} + 1$, where $x_{n1} \geq 1$, in expressions Σ_n from condition of (2.17):

$$3 * x_n - 2 * M + 5 = 3 * (x_n - 2 * x_{n1} + 1), \quad \text{where} \quad M + 1 \leq x_n \leq 2 * M - 3, \quad x_{n1} \geq 1; \tag{43}$$

$$3 * x_n - 2 * M - 3 = 3 * (x_n - 2 * x_{n1} - 2) + 1, \quad \text{where} \quad 2 * M - 2 \leq x_n \leq 2 * M, \quad x_{n1} \geq 1. \tag{44}$$

Σ_n of first expression (19) belong to set (18), Σ_n (43) belong to set (16), and Σ_n (44) belong to set (17). I.e. all conditions for puzzle solution are complied with.
During the addition of expressions (40) and (19) we will receive the expressions for queens for the whole chessboard:

$$\begin{aligned}
&1.\ (3 * x_n + 2) \wedge x_n = (2 * x_n + 2) + (-x_n - 2), \quad \text{where} \quad 1 \leq n \leq M - 1, \quad 1 \leq x_n \leq M - 1;\\
&2.\ (M + 2) \wedge M = 2 + (M - 2);\\
&3.\ (3 * x_n - 2 * M + 5) \wedge x_n = (2 * x_n - 2 * M + 5) + (2 * M - x_n - 5),\\
&\qquad \text{where} \quad M + 1 \leq n \leq 2 * M - 3, \quad M + 1 \leq x_n \leq 2 * M - 3;\\
&4.\ (3 * x_n - 2 * M - 3) \wedge x_n = (2 * x_n - 2 * M - 3) + (2 * M - x_n + 3),\\
&\qquad \text{where} \quad 2 * M - 2 \leq n \leq 2 * M, \quad 2 * M - 2 \leq x_n \leq 2 * M.
\end{aligned} \tag{45}$$

Removing from expressions for queens (45) Σ_n and S_n, we will receive **Third main solution** of puzzle for even N_{2M} (1) with $M \in \{3 * x_n + 1 \mid x_n \geq 1\}$, where x_n means the value for queens in vertical direction and y_n - value of queens in horizontal direction:

$$\begin{aligned}
&1.\ y_n = 2 * x_n + 2, \quad \text{where} \quad 1 \leq n \leq M - 1, \quad 1 \leq x_n \leq M - 1;\\
&2.\ y_n = 2; \quad \text{where} \quad x_n = M;\\
&3.\ y_n = 2 * x_n - 2 * M + 5, \quad \text{where} \quad M + 1 \leq n \leq 2 * M - 3, \quad M + 1 \leq x_n \leq 2 * M - 3;\\
&4.\ y_n = 2 * x_n - 2 * M - 3, \quad \text{where} \quad 2 * M - 2 \leq n \leq 2 * M, \quad 2 * M - 2 \leq x_n \leq 2 * M.
\end{aligned} \tag{46}$$

2.18 Fourth Main Solution

Let's analyze the expressions for queens (9) with sequence $\{x_{>M}\}$ (11) under the condition (19) for uneven N_{2M+1} (2) with $M \in \{3 * x_n + 1 \mid x_n \geq 1\}$.
In this case the calculation is the same as in (2.17). It is necessary that expressions for queens with sequence $\{x_{>M}\}$ has the values of negative $0 > S_n > -3$. For maximum $y_n = 2 * M + 1$ there are only two variants of mutual arrangement of subsequences $\{x_{>M}\}$ and $\{y_{2x+1}\}$ (13).

Let's provide the sequence of arrangement of queens in the upper half of the chessboard in the form of expressions for queens (9), corresponding to the sequence $\{x_{>M}\}$ with one negative value $S_n = -1$:

$$\begin{aligned}
&1.\ (3 * x_n - 2 * M + 1) \wedge x_n = (2 * x_n - 2 * M + 1) + (2 * M - x_n - 1),\\
&\qquad \text{where} \quad M + 1 \leq n \leq 2 * M, \quad M + 1 \leq x_n \leq 2 * M;\\
&2.\ (2 * M + 2) \wedge 2 * M + 1 = 1 + 2 * M.
\end{aligned} \tag{47}$$

The smallest Σ_n of first expression (47) during the replacement of x_n with M and Σ_n of second expression (47) will be bigger than $\Sigma_n = M + 2$ of the last expression (19):

$$3 * x_n - 2 * M + 1 = 3 * M + 3 - 2 * M + 1 = M + 4 > M + 2; \tag{48}$$

$$2 * M + 2 > M + 2. \tag{49}$$

Here we will identify x_n of expressions (19) as x_{n1}. Due to the fact that first expression (19) has $\Sigma_n = 3*x_{n1}+2$, we will find to which from the sets (16), (17) or (18) Σ_n of first and second expressions (47) belongs to, by substituting for this purpose the value $M = 3 * x_{n1} + 1$, where $x_{n1} \geq 1$, in expressions Σ_n from condition of (2.18):

$$3 * x_n - 2 * M + 1 = 3 * (x_n - 2 * x_{n1} - 1) + 2, \quad \text{where} \quad M + 1 \leq x_n \leq 2 * M, \quad x_{n1} \geq 1; \tag{50}$$

$$2 * M + 2 = 3 * (2 * x_{n1} + 1) + 1, \quad \text{where} \quad x_{n1} \geq 1. \tag{51}$$

Σ_n (50) and Σ_n of first expression (19) belong to one set (18). Σ_n (51) belong to set (17). Let's compare the smallest Σ_n (50) and bigger Σ_n of first expression (19), representing it through M:

$$3 * x_n - 2 * M + 1 = M + 4 < 3 * x_{n1} + 2 = 3 * (M - 1) + 2, \quad \text{where} \quad M \geq 4. \tag{52}$$

Σ_n (50) will always match with Σ_n of first expression (19). The condition (5) is not complied with and expressions for queens (47) can not constitute the puzzle solution.

Let's provide the sequence of arrangement of queens in the upper half of the chessboard in the form of expressions for queens (9), corresponding to sequences $\{x_{>M}\}$ with two negative values $S_n \geq -2$:

$$\begin{aligned}
&1.\ (3 * x_n - 2 * M + 3) \wedge x_n = (2 * x_n - 2 * M + 3) + (2 * M - x_n - 3), \\
&\qquad \text{where} \quad M + 1 \leq n \leq 2 * M - 1, \quad M + 1 \leq x_n \leq 2 * M - 1; \\
&2.\ (3 * x_n - 2 * M - 7) \wedge x_n = (2 * x_n - 2 * M - 7) + (2 * M - x_n + 7), \\
&\qquad \text{where} \quad 2 * M \leq n \leq 2 * M + 1, \quad 2 * M \leq x_n \leq 2 * M + 1.
\end{aligned} \tag{53}$$

The smallest Σ_n of first and second expression (53) during the replacement of x_n with M will be bigger than $\Sigma_n = M + 2$ of the last expression (19):

$$3 * x_n - 2 * M + 3 = 3 * M + 3 - 2 * M + 3 = M + 6 > M + 2; \tag{54}$$

$$3 * x_n - 2 * M - 7 = 3 * 2 * M - 3 - 2 * M - 7 = 4 * M - 7 > M + 2, \quad \text{where} \quad M \geq 4. \tag{55}$$

Here we will identify x_n of expressions (19) as x_{n1}. Due to the fact that first expression (19) has $\Sigma_n = 3*x_{n1}+2$, we will find to which from the sets (16), (17) or (18) Σ_n of first and second expressions (53) belongs to, by substituting for this purpose the value $M = 3 * x_{n1} + 1$, where $x_{n1} \geq 1$, in expressions Σ_n from condition of (2.18):

$$3 * x_n - 2 * M + 3 = 3 * (x_n - 2 * x_{n1}) + 1, \quad \text{where} \quad M + 1 \leq x_n \leq 2 * M - 1, \quad x_{n1} \geq 1; \tag{56}$$

$$3 * x_n - 2 * M - 7 = 3 * (x_n - 2 * x_{n1} - 3), \quad \text{where} \quad 2 * M \leq x_n \leq 2 * M + 1, \quad x_{n1} \geq 1. \tag{57}$$

Σ_n of first expression (19) belong to set (18), Σ_n (56) belong to set (17), and Σ_n (57) belong to set (16). I.e. all conditions of puzzle solution are complied with.

During the addition of expressions (53) and (19) we will receive the expressions for queens for the whole chessboard:

$$\begin{aligned}
&1.\ (3 * x_n + 2) \wedge x_n = (2 * x_n + 2) + (-x_n - 2), \quad \text{where} \quad 1 \leq n \leq M - 1, \quad 1 \leq x_n \leq M - 1; \\
&2.\ (M + 2) \wedge M = 2 + (M - 2); \\
&3.\ (3 * x_n - 2 * M + 3) \wedge x_n = (2 * x_n - 2 * M + 3) + (2 * M - x_n - 3), \\
&\qquad \text{where} \quad M + 1 \leq n \leq 2 * M - 1, \quad M + 1 \leq x_n \leq 2 * M - 1; \\
&4.\ (3 * x_n - 2 * M - 7) \wedge x_n = (2 * x_n - 2 * M - 7) + (2 * M - x_n + 7), \\
&\qquad \text{where} \quad 2 * M \leq n \leq 2 * M + 1, \quad 2 * M \leq x_n \leq 2 * M + 1.
\end{aligned} \tag{58}$$

Removing from expressions for queens (58) Σ_n and S_n, we will receive **Fourth main solution** of puzzle for uneven N_{2M+1} (2) with $M \in \{3 * x_n + 1 \mid x_n \geq 1\}$, where x_n means the value for queens in vertical direction and y_n - value of queens in horizontal direction:

$$\begin{aligned}
&1.\ y_n = 2 * x_n + 2, \quad \text{where} \quad 1 \leq n \leq M - 1, \quad 1 \leq x_n \leq M - 1;\\
&2.\ y_n = 2; \quad \text{where} \quad x_n = M;\\
&3.\ y_n = 2 * x_n - 2 * M + 3, \quad \text{where} \quad M + 1 \leq n \leq 2 * M - 1, \quad M + 1 \leq x_n \leq 2 * M - 1;\\
&4.\ y_n = 2 * x_n - 2 * M - 7, \quad \text{where} \quad 2 * M \leq n \leq 2 * M + 1, \quad 2 * M \leq x_n \leq 2 * M + 1.
\end{aligned} \tag{59}$$

2.19 Main Solution for All $N \geq 4$

N-number of queens on the chessboard with sides equaling to N, where $N \geq 4$, is arranged with the help of **Fourth main solutions**:
1) For $N = 2 * M$ with $M \in \{3 * x_n \mid x_n \geq 1\}$ or $M \in \{3 * x_n - 1 \mid x_n \geq 1\}$, where x_n means the value of queens in vertical direction, y_n - value of queens in horizontal direction:

$$\begin{aligned}
&1.\ y_n = 2 * x_n, \quad \text{where} \quad 1 \leq n \leq M, \quad 1 \leq x_n \leq M;\\
&2.\ y_n = 2 * x_n - 2 * M - 1, \quad \text{where} \quad M + 1 \leq n \leq 2 * M, \quad M + 1 \leq x_n \leq 2 * M.
\end{aligned} \tag{60}$$

2) For $N = 2 * M + 1$ with $M \in \{3 * x_n \mid x_n \geq 1\}$ or $M \in \{3 * x_n - 1 \mid x_n \geq 1\}$, where x_n means the value of queens in vertical direction, y_n - value of queens in horizontal direction:

$$\begin{aligned}
&1.\ y_n = 2 * x_n \quad \text{where} \quad 1 \leq n \leq M, \quad 1 \leq x_n \leq M;\\
&2.\ y_n = 2 * x_n - 2 * M - 1, \quad \text{where} \quad M + 1 \leq n \leq 2 * M + 1, \quad M + 1 \leq x_n \leq 2 * M + 1.
\end{aligned} \tag{61}$$

3) For $N = 2 * M$ with $M \in \{3 * x_n + 1 \mid x_n \geq 1\}$, where x_n means the value of queens in vertical direction, y_n - value of queens in horizontal direction:

$$\begin{aligned}
&1.\ y_n = 2 * x_n + 2, \quad \text{where} \quad 1 \leq n \leq M - 1, \quad 1 \leq x_n \leq M - 1;\\
&2.\ y_n = 2; \quad \text{where} \quad x_n = M;\\
&3.\ y_n = 2 * x_n - 2 * M + 5, \quad \text{where} \quad M + 1 \leq n \leq 2 * M - 3, \quad M + 1 \leq x_n \leq 2 * M - 3;\\
&4.\ y_n = 2 * x_n - 2 * M - 3, \quad \text{where} \quad 2 * M - 2 \leq n \leq 2 * M, \quad 2 * M - 2 \leq x_n \leq 2 * M.
\end{aligned} \tag{62}$$

4) For $N = 2 * M + 1$ with $M \in \{3 * x_n + 1 \mid x_n \geq 1\}$, where x_n means the value of queens in vertical direction, y_n - value of queens in horizontal direction:

$$\begin{aligned}
&1.\ y_n = 2 * x_n + 2, \quad \text{where} \quad 1 \leq n \leq M - 1, \quad 1 \leq x_n \leq M - 1;\\
&2.\ y_n = 2; \quad \text{where} \quad x_n = M;\\
&3.\ y_n = 2 * x_n - 2 * M + 3, \quad \text{where} \quad M + 1 \leq n \leq 2 * M - 1, \quad M + 1 \leq x_n \leq 2 * M - 1;\\
&4.\ y_n = 2 * x_n - 2 * M - 7, \quad \text{where} \quad 2 * M \leq n \leq 2 * M + 1, \quad 2 * M \leq x_n \leq 2 * M + 1.
\end{aligned} \tag{63}$$

Therefore if the solution of **Queens Puzzle** for sequence of natural numbers $N \geq 4$, the main solution consists in successive application of the following algorithm:

1.	**First Main Solution** (60)
2.	**Second Main Solution** (61)
3.	**First Main Solution** (60)
4.	**Second Main Solution** (61)
5.	**Third Main Solution** (62)
6.	**Fourth Main Solution** (63)

2.20 Additional and Derived Solutions

The set of **Four main solutions** is the easiest and the most exhaustive algorithm for **Queens puzzle solution**. However, there are the set of additional solutions, duplicating the easiest main solutions or having additional conditions.

2.20.1 First Additional Solution

In (2.12) in Conclusion 3 it is specified that the second variant (19) can be the condition for puzzle solution for N with $M = 3 * x_n - 1$, where $x_n \geq 1$. But for such M more suitable **First** (30) and **Second** (33) **main solutions** are correct too. Using the calculation for **Third** (46) and **Fourth** (59) main solutions given in (2.17) and (2.18) it can be calculated that **Third main solution** of puzzle (46) for even N_{2M} (1) with $M \in \{3 * x_n + 1 \mid x_n \geq 1\}$ also constitutes **First additional solution** for puzzle for even N_{2M} with $M \in \{3 * x_n - 1 \mid x_n \geq 1\}$.

2.20.2 Second Additional Solution

In (2.17) it was specified that due to the condition (28) for even N_{2M} (1) with $M \in \{3*x_n+1 \mid x_n \geq 1\}$ only second variant (19) can be the condition of puzzle solution.
But if we add the condition $M \in \{2 * x_{n1} \mid x_{n1} \geq 2\}$ to the condition $M \in \{3 * x_n + 1 \mid x_n \geq 1\}$ then we can return to the first variant (14). Let's return to **First main solution** of puzzle (30).
Firstly we will calculate the additional condition for this solution:

$$M = 2 * x_{n1} = 3 * x_{n2} + 1 = 3 * (2x_n - 1) + 1, \quad \text{where} \quad x_n \geq 1, \quad x_{n2} = 2 * x_n - 1. \tag{64}$$

The sequence of uneven numbers $\{y_{2x-1}\}$ (13), the first member of which matches with the first member of sequence of natural numbers $\{x_{>M}\}$ (11), we will divide into pairs on the whole length. We will perform the exchange of $y_{(2x-1)1}$ and $y_{(2x-1)2}$ inside each pair.
Let's provide the expressions for queens (9) in the upper half of the chessboard, conforming to sequence $\{x_{>M}\}$:

$$\begin{aligned}
&1.\ (3 * x_n - 2 * M + 1) \wedge x_n = (2 * x_n - 2 * M + 1) + (2 * M - x_n - 1),\\
&\qquad \text{where} \quad n \in \{M + (2 * x_{n3} - 1) \mid 1 \leq x_{n3} \leq (M/2)\},\\
&\qquad\qquad x_n \in \{M + (2 * x_{n3} - 1) \mid 1 \leq x_{n3} \leq (M/2)\};\\
&2.\ (3 * x_n - 2 * M + 3) \wedge x_n = (2 * x_n - 2 * M - 3) + (2 * M - x_n + 3),\\
&\qquad \text{where} \quad n \in \{M + (2 * x_{n3}) \mid 1 \leq x_{n3} \leq (M/2)\},\\
&\qquad\qquad x_n \in \{M + (2 * x_{n3}) \mid 1 \leq x_{n3} \leq (M/2)\}.
\end{aligned} \tag{65}$$

Values of Σ_n in (65) match with values in previously performed calculations. According to (50), Σ_n of the first expression (65) belong to set (18). According to (37), Σ_n of the second expression (65) belong to set (17). Σ_n (14) belong to set (16). I.e. all conditions for puzzle solution are complied with.
During the addition of expressions (65) and (14) we will receive the expressions for queens for the whole chessboard:

$$\begin{aligned}
&1.\ (3 * x_n) \wedge x_n = 2 * x_n + (-x_n), \quad \text{where} \quad 1 \leq n \leq M, \quad 1 \leq x_n \leq M;\\
&2.\ (3 * x_n - 2 * M + 1) \wedge x_n = (2 * x_n - 2 * M + 1) + (2 * M - x_n - 1),\\
&\qquad \text{where} \quad n \in \{M + (2 * x_{n3} - 1) \mid 1 \leq x_{n3} \leq (M/2)\},\\
&\qquad\qquad x_n \in \{M + (2 * x_{n3} - 1) \mid 1 \leq x_{n3} \leq (M/2)\};\\
&3.\ (3 * x_n - 2 * M + 3) \wedge x_n = (2 * x_n - 2 * M - 3) + (2 * M - x_n + 3),\\
&\qquad \text{where} \quad n \in \{M + (2 * x_{n3}) \mid 1 \leq x_{n3} \leq (M/2)\},\\
&\qquad\qquad x_n \in \{M + (2 * x_{n3}) \mid 1 \leq x_{n3} \leq (M/2)\}.
\end{aligned} \tag{66}$$

Removing from expressions for queens (66) Σ_n and S_n, we will receive **Second additional solution** of the puzzle for even N_{2M} (1) with $M \in \{3 * (2 * x_n - 1) + 1 \mid x_n \geq 1\}$, where x_n means the value of queens in vertical direction, y_n - number of queens in horizontal direction:

$$\begin{aligned}
&1.\ y_n = 2 * x_n, \quad \text{where} \quad 1 \leq n \leq M, \quad 1 \leq x_n \leq M; \\
&2.\ y_n = 2 * x_n - 2 * M + 1, \\
&\qquad \text{where} \quad n \in \{M + (2 * x_{n3} - 1) \mid 1 \leq x_{n3} \leq (M/2)\}, \\
&\qquad\qquad x_n \in \{M + (2 * x_{n3} - 1) \mid 1 \leq x_{n3} \leq (M/2)\}; \\
&3.\ y_n = 2 * x_n - 2 * M - 3, \\
&\qquad \text{where} \quad n \in \{M + (2 * x_{n3}) \mid 1 \leq x_{n3} \leq (M/2)\}, \\
&\qquad\qquad x_n \in \{M + (2 * x_{n3}) \mid 1 \leq x_{n3} \leq (M/2)\}.
\end{aligned} \tag{67}$$

Under these conditions for uneven N_{2M+1} (2) the puzzle does not have a solution due to the condition given in (2.16) only $S \geq 0$ in expressions for queens corresponding to sequence $\{x_{>M}\}$ (11).

2.20.3 Third Additional Solution

In (2.17) it was specified that the sequence of arrangement of queens in the upper half of the chessboard in the form of expressions for queens (9), corresponding to sequence $\{x_{>M}\}$ with one negative value $S_n = -1$, can not constitute the solution of puzzle due to the fact that $\Sigma_n(38)$ at $M < 8$ will match with Σ_n of first expression (19). Therefore, in case of $M \geq 8$ the conditions for puzzle solution will be complied with.
During the addition of expressions (34) and (19) we will receive the expressions for queens for the whole chessboard:

$$\begin{aligned}
&1.\ (3 * x_n + 2) \wedge x_n = (2 * x_n + 2) + (-x_n - 2), \quad \text{where} \quad 1 \leq n \leq M - 1, \quad 1 \leq x_n \leq M - 1; \\
&2.\ (M + 2) \wedge M = 2 + (M - 2); \\
&3.\ (3 * x_n - 2 * M + 3) \wedge x_n = (2 * x_n - 2 * M + 3) + (2 * M - x_n - 3), \\
&\qquad \text{where} \quad M + 1 \leq n \leq 2 * M - 2, \quad M + 1 \leq x_n \leq 2 * M - 2; \\
&4.\ (3 * x_n - 2 * M - 5) \wedge x_n = (2 * x_n - 2 * M - 5) + (2 * M - x_n + 5), \\
&\qquad \text{where} \quad 2 * M - 1 \leq n \leq 2 * M, \quad 2 * M - 1 \leq x_n \leq 2 * M.
\end{aligned} \tag{68}$$

Removing from expressions for queens (68) Σ_n and S_n, we will receive **Third additional solution** for even N_{2M} (1) with $M \in \{3 * x_n + 1 \mid x_n \geq 3\}$, where x_n the value of queens in vertical direction and y_n - values of queens in horizontal direction:

$$\begin{aligned}
&1.\ y_n = 2 * x_n + 2, \quad \text{where} \quad 1 \leq n \leq M - 1, \quad 1 \leq x_n \leq M - 1; \\
&2.\ y_n = 2; \quad \text{where} \quad x_n = M; \\
&3.\ y_n = 2 * x_n - 2 * M + 3, \quad \text{where} \quad M + 1 \leq n \leq 2 * M - 2, \quad M + 1 \leq x_n \leq 2 * M - 2; \\
&4.\ y_n = 2 * x_n - 2 * M - 5, \quad \text{where} \quad 2 * M - 1 \leq n \leq 2 * M, \quad 2 * M - 1 \leq x_n \leq 2 * M.
\end{aligned} \tag{69}$$

2.20.4 Derived Solutions

Solutions are named like this because of the fact that they are derived from main and additional puzzle solutions. They are based on the fact that according to specific x_n for adjacent uneven y_n of one of sets Σ_n (16), (17) or (18), the following expression is correct:

$$y_{n1} = y_{n2} \pm 6. \tag{70}$$

Therefore, in the complete expressions for queens of main and additional solutions you can choose such y_n, the difference among which will equal 6, and exchange them. Herewith the additional calculations shall be

made in order to provide that each expression for queens (9) has a unique set Σ_n, x_n, y_n and S_n. Thus, for expressions of (45) the **derived solution** will be successful in case of exchange of $y_n = 9$ and $y_n = 3$. For expressions of (58) the **derived solution** will be successful in case of exchange of $y_n = 7$ and $y_n = 1$. More difficult exchanges with three and more y_n, effecting the expressions for queens, Σ_n of which will belong to other set of (16), (17) and (18) as a result of exchange, if such Σ_n will not equal to any Σ_n from this set in expression for queens both for lower and upper halves of the chessboard. With increasing of N the number of possible exchanges of y_n is also rising.
But it requires additional calculations at already complete main and additional solutions and adds nothing new to **Algorithm of Queens puzzle solution**.

Publications: http://samlib.ru/editors/b/bezymjannyj_a/w8.shtml