On The Proving Method of Fermat's Last Theorem

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Abstract: In this paper the author gives an elementary mathematics method to solve ***Fermat's Last Theorem*** (FLT), in which let this equation become an one unknown number equation, in order to solve this equation the author invented a method called "Order reducing method for equations", where the second order root compares to one order root, and with some necessary techniques the author successfully proved when $x^{n-1}+y^{n-1}- z^{n-1} <= x^{n-2}+y^{n-2}- z^{n-2}$ there are no positive solutions for this equation, and also proves with the increasing of x there are still no positive integer solutions for this equation when $x^{n-1}+y^{n-1}- z^{n-1} <= x^{n-2}+y^{n-2}- z^{n-2}$ is not satisfied.

1. Some Relevant Theorems

There are some theorems for proving or need to be known. ***All symbols in this paper represent positive integers unless they are stated to be not.***

Theorem 1.1. In the equation of

$$\begin{cases} x^n + y^n = z^n \\ \gcd(x, y, z) = 1 \\ n > 2 \end{cases} \tag{1-1}$$

x, y, z meet

$x \neq y$,

$x + y > z$,

and if

$x > y$

then

$z > x > y$.

Proof: Let

$x = y$,

we have

$$2x^n = z^n$$

and

$$\sqrt[n]{2}x = z$$

where $\sqrt[n]{2}$ is not an integer and x, z are all positive integers, so $x \neq y$.

Since

$$(x + y)^n = x^n + C_n^1 x^{n-1} y + ... + C_n^{n-1} x y^{n-1} + y^n > z^n ,$$

so we get

$$x+y>z.$$

Since

$$x^n+y^n=z^n,$$

so we have

$$z^n>x^n, z^n>y^n$$

and get

$$z>x>y$$

when

$$x>y.$$

Theorem 1.2. In the equation of (1-1), x, y, z meet

$$\gcd(x,y)=\gcd(y,z)=\gcd(x,z)=1.$$

Proof: Since $x^n+y^n=z^n$, if $\gcd(x,y)>1$ then we have $\left(x_1^{\,n}+y^n{}_1\right)\times\left[\gcd(x,y)\right]^n=z^n$ which causes $\gcd(x,y,z)>1$ since the left side contains the factor of $\left[\gcd(x,y)\right]^n$ then the right side must also contains this factor but contradicts against (1-1) in which $\gcd(x,y,z)=1$, so we have $\gcd(x,y)=1$. Using the same way we have $\gcd(x,z)=\gcd(y,z)=1$.

Theorem 1.3. If there is no positive integer solution for

$$x^p+y^p=z^p$$

when $p>2$ is a prime number then there is also no positive integer solution for

$$\left(x^p\right)^k+\left(y^p\right)^k=\left(z^p\right)^k.$$

Proof: Since $x^p+y^p=z^p$ has no positive integer solution, so there still no positive integer solution for

$$\left(x^k\right)^p+\left(y^k\right)^p=\left(z^k\right)^p$$

which means there is also no positive integer solution for

$$\left(x^p\right)^k+\left(y^p\right)^k=\left(z^p\right)^k.$$

So we only need to prove there is no positive integer solution for equation (1-1) when n is a prime number.

Theorem 1.4. There are no positive integer solutions for equation (1-1) when x or y is a prime number .

Proof: When x is a prime number, since

$$x^n = z^n - y^n = (z-y)\left(z^{n-1} + z^{n-2}y + \ldots + zy^{n-2} + y^{n-1}\right),$$

so we have

$$\gcd(z-y, x) = x\,,$$

which means

$$z - y \geq x\,,$$

we have

$$x + y \leq z\,,$$

that contradicts against **Theorem 1.1** in which $x + y > z$, so it is with y , which means there are no positive integer solutions for equation (1-1) when x or y is a prime number .

Theorem 1.5. There are no positive integer solutions for equation (1-1) when z is a prime number .

Proof: When z is a prime number, from **Theorem 1.3** we only consider the case of $n > 2$ is a prime number, since

$$x^n + y^n = z^n = (x+y)\left(x^{n-1} + \ldots + y^{n-1}\right),$$

so we have

$$\gcd(x+y, z) = z\,,$$

from **Theorem 1.1** we know $x + y > z$, so we get

$$x + y \geq 2z\,,$$

that contradicts against **Theorem 1.1** in which $z > x > y \Rightarrow x + y < 2z$, which means there are no positive integer solutions for equation (1-1) when z is a prime number .

Theorem 1.6. There are no positive integer solutions for

$$1^n + y^n = z^n\,.$$

Proof: Since

$$1 = z^n - y^n = (z-y)\left(z^{n-1} + z^{n-2}y + \ldots + zy^{n-2} + y^{n-1}\right)$$

where

$$\begin{cases} z - y = 1 \\ \left(z^{n-1} + z^{n-2}y + \ldots + zy^{n-2} + y^{n-1}\right) = 1 \end{cases}$$

that causes z, y to be non positive integers, so there are no positive integer solutions for

$$1^n + y^n = z^n\,.$$

Theorem 1.7. There are no positive integer solutions for

$$2^n + y^n = z^n .$$

Proof: Since

$$2^n = z^n - y^n = (z-y)\left(z^{n-1} + z^{n-2}y + \ldots + zy^{n-2} + y^{n-1}\right),$$

if

$$\begin{cases} z - y = 1 \\ z^{n-1} + z^{n-2}y + \ldots + zy^{n-2} + y^{n-1} = 2^n \end{cases}$$

then taking the least value for $y = 2, z = 3$, we have

$$3^{n-1} + 2\times 3^{n-2} + \ldots + 2^{n-1} > 2^n$$

when $n > 2$ that is impossible. If

$$\begin{cases} z - y = 2^i \\ z^{n-1} + z^{n-2}y + \ldots + zy^{n-2} + y^{n-1} = 2^j \\ i + j = n \\ i \geq 1 \end{cases}$$

then $z > 2$ and taking the least value of $y = 2, z = 3$, we get

$$3^{n-1} + 2\times 3^{n-2} + \ldots + 2^{n-1} > 2^j$$

with $n > 2$ that is also impossible, so there are no positive integer solutions for

$$2^n + y^n = z^n .$$

Theorem 1.8. There are no positive integer solutions for equation (1-1) when $n \to \infty$ and x, y, z in equation (1-1) meet

$$z < \sqrt[n]{2}x, x > 2, y > 1, z > 3.$$

Proof: Since $x^n + y^n = z^n$, let $x > y$, we get

$$\left(\frac{z}{x}\right)^n - \left(\frac{y}{x}\right)^n = 1,$$

since

$$z > x > y,$$

so we have

$$z < \sqrt[n]{2}x,$$

and

$$\lim_{n\to\infty}\left(\frac{z}{x}\right)^n-\left(\frac{y}{x}\right)^n=\infty>1$$

which means there are no positive integer solutions for equation (1-1) when $n\to\infty$.

According to **Theorem 1.6, 1.7** we have $x>2, y>1, z>3$.

Theorem 1.9. There are no positive integer solutions for equation (1-1) when $x,y,z\le 10^4$.

Proof: Using the method of which we prove **Theorem 1.6, 1.7** we can prove when $x,y\le 10^4$, there are no positive integer solutions for equation (1-1), since means $z>x>y$ so when $z\le 10^4$ there are no positive integer solutions.

Theorem 1.10. In the equation of (1-1), x,y,z meet

$$x^{n-i}+y^{n-i}>z^{n-i},$$

$$x^{n+i}+y^{n+i}<z^{n+i},$$

where

$$n>i\ge 1.$$

This theorem holds true when x,y,z are positive real numbers but n must be a positive integer.

Proof: From equation (1-1), since

$$x^n+y^n=z^n,$$

from **Theorem 1.1**, since $z>x>y$, we have

$$x^{n-i}+y^{n-i}>\left[\left(\frac{x}{z}\right)^i x^{n-i}+\left(\frac{y}{z}\right)^i y^{n-i}=z^{n-i}\right],$$

$$x^{n+i}+y^{n+i}<\left(z^i x^{n-i}+z^i y^{n-i}=z^{n+i}\right),$$

so we have

$$x^{n-i}+y^{n-i}>z^{n-i}.$$

$$x^{n+i}+y^{n+i}<z^{n+i}.$$

This theorem means given x,y,z if equation (1-1) has one positive integer solution then this solution is the only one.

Theorem 1.11. There are no positive integer solutions for equation (1-1) when

$$\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}}\le 1.$$

And in order to have positive integer solutions for equation (1-1),

$$\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}}>4000$$

must be satisfied.

Proof: In equation (1-1), let

$$\begin{cases} a=x^{n-2} \\ b=y^{n-2} \\ c=z^{n-2} \end{cases},$$

we have

$$\begin{cases} ax^2+by^2=cz^2 \\ a^{\frac{n-1}{n-2}}x+b^{\frac{n-1}{n-2}}y=c^{\frac{n-1}{n-2}}z \end{cases}.$$

Since we reduce the order of equation so the method is called "Order reducing method for equations". Let $x>y$ and

$$\begin{cases} y=x-f \\ z=x+e \end{cases},$$

we have

$$\begin{cases} ax^2+b(x-f)^2=c(x+e)^2 \\ a^{\frac{n-1}{n-2}}x+b^{\frac{n-1}{n-2}}(x-f)=c^{\frac{n-1}{n-2}}(x+e) \end{cases}$$

and

$$\begin{cases} (a+b-c)x^2-2(bf+ce)x+(bf^2-ce^2)=0 \\ a^{\frac{n-1}{n-2}}x+b^{\frac{n-1}{n-2}}(x-f)-c^{\frac{n-1}{n-2}}(x+e)=0 \end{cases},$$

the roots are

$$x=\frac{(bf+ce)\pm\sqrt{(bf+ce)^2-(a+b-c)(bf^2-ce^2)}}{x^{n-2}+y^{n-2}-z^{n-2}}, \qquad (1\text{-}2)$$

and

$$x=\frac{c^{\frac{n-1}{n-2}}e+b^{\frac{n-1}{n-2}}f}{a^{\frac{n-1}{n-2}}+b^{\frac{n-1}{n-2}}-c^{\frac{n-1}{n-2}}}=\frac{bfy+cez}{x^{n-1}+y^{n-1}-z^{n-1}}. \qquad (1\text{-}3)$$

There are two cases for bf^2, ce^2 when $bf^2\geq ce^2$ and $bf^2<ce^2$.

Case A: If $bf^2\geq ce^2$, from (1-2) when

$$x=\frac{(bf+ce)+\sqrt{(bf+ce)^2-(a+b-c)(bf^2-ce^2)}}{x^{n-2}+y^{n-2}-z^{n-2}},$$

from **Theorem 1.10** we know $a+b-c=x^{n-2}+y^{n-2}-z^{n-2}>0$, so we have

$$x\le\frac{2(bf+ce)}{x^{n-2}+y^{n-2}-z^{n-2}},$$

also from **Theorem 1.10** we have $x^{n-1}+y^{n-1}-z^{n-1}>0$, compare to (1-3) we get

$$\frac{bfy+cez}{x^{n-1}+y^{n-1}-z^{n-1}}\le\frac{2(bf+ce)}{x^{n-2}+y^{n-2}-z^{n-2}}.$$

When $\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}}\le 1$, we have

$$bfy+cez\le 2(bf+ce)$$

that is impossible since from **Theorem 1.8** we know $y\ge 2$ and $z>3$.

When

$$x=\frac{(bf+ce)-\sqrt{(bf+ce)^2-(a+b-c)(bf^2-ce^2)}}{x^{n-2}+y^{n-2}-z^{n-2}}.$$

we have

$$x\le\frac{bf+ce}{x^{n-2}+y^{n-2}-z^{n-2}},$$

compare to (1-3) we get

$$\frac{bfy+cez}{x^{n-1}+y^{n-1}-z^{n-1}}\le\frac{bf+ce}{x^{n-2}+y^{n-2}-z^{n-2}}.$$

When $\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}}\le 1$, we have

$$bfy+cez\le bf+ce$$

that is impossible since from **Theorem 1.8** we have already known $y\ge 2$ and $z>3$.

Case B: If $bf^2<ce^2$, from (1-2) when

$$x=\frac{(bf+ce)+\sqrt{(bf+ce)^2+(a+b-c)(ce^2-bf^2)}}{x^{n-2}+y^{n-2}-z^{n-2}},$$

we can prove $(bf+ce)^2>(a+b-c)(ce^2-bf^2)$ since if not we have

$$(bf+ce)^2\le(a+b-c)(ce^2-bf^2)$$

and

$$[(2b+a)-c]bf^2+2bfce+[2c-(a+b)]ce^2\le 0$$

that is impossible since $a+b-c>0$ and $c>a, c>b, 2c-(a+b)>0.$ So we have

$$x<\frac{(bf+ce)(1+\sqrt{2})}{x^{n-2}+y^{n-2}-z^{n-2}}$$

compare to (2-4) we get

$$\frac{bfy+cez}{x^{n-1}+y^{n-1}-z^{n-1}}<\frac{(bf+ce)(1+\sqrt{2})}{x^{n-2}+y^{n-2}-z^{n-2}}.$$

When $\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}}\le 1$, we have

$$bfy+cez<(bf+ce)(1+\sqrt{2})<2.5(bf+ce)$$

and

$$bf(x-f)+ce(x+e)<2.5(bf+ce)$$

that leads to

$$x<\left[\frac{2.5(bf+ce)+bf^2-ce^2}{bf+ce}=2.5-\frac{ce^2-bf^2}{bf+ce}\right]<2.5$$

where possible values for x are 1, 2 but according to **Theorem 1.6**, **1.7** we know there are no positive integer solutions.

When

$$x=\frac{(bf+ce)-\sqrt{(bf+ce)^2+(a+b-c)(ce^2-bf^2)}}{x^{n-2}+y^{n-2}-z^{n-2}}$$

is not possible since $x\le 0$.

So we have the conclusion of there are no positive integer solutions for equation (1-1) when $\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}}\le 1.$

Obviously we have

$$bfy + cez < 2.5\left(\frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}}\right)(bf + ce),$$

from **Theorem 1.9** we know $x, y, z \le 10^4$ there are no positive integer solutions for equation (1-1), so we have

$$\frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}} > 4000,$$

which must be satisfied to have positive integer solutions for equation (1-1).

2. Proving Method

From **Theorem 1.11** we know in order to have positive integer solutions for this equation, $\frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}} > 1$ must be satisfied. We give the graph of this equation as showed in

Figure 2-1 when $\frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}} > 1$, where $AB // CD'$.

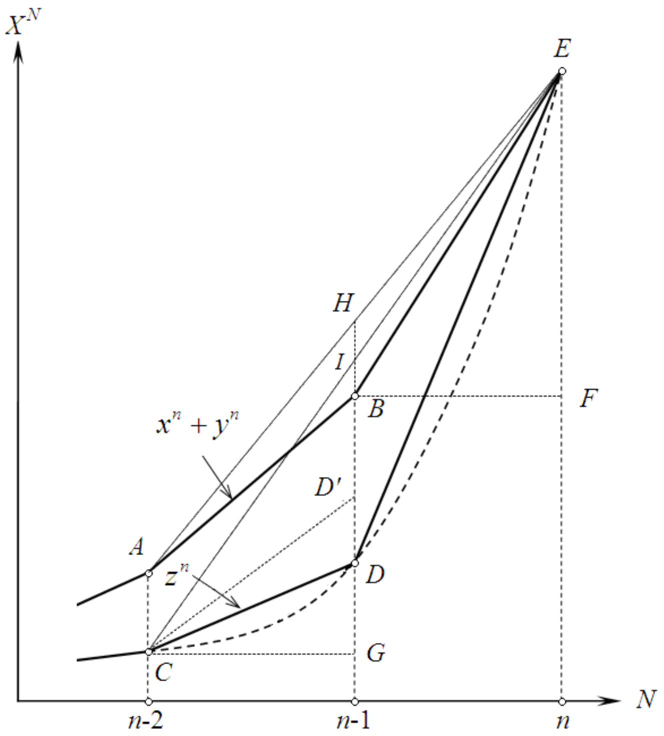

Figure 2-1 *Graph of* $x^n + y^n = z^n$ *when* $\frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}} > 1$

2.1. In **Figure 2-1** we have

$$\angle CDE = 360^0 - \arctan\left(\frac{z^n - z^{n-1}}{1}\right) - \arctan\left(\frac{1}{z^{n-1} - z^{n-2}}\right) - 90^0,$$

and

$$BD = x^{n-1} + y^{n-1} - z^{n-1},$$
$$AC = x^{n-2} + y^{n-2} - z^{n-2}.$$

When $\frac{BD}{AC} > 1$ we have

$$\angle ABE - \angle CDE = \angle D'CD + \angle BED > 0,$$

which means

$$\angle ABE > \angle CDE.$$

It is also very clear that if $\angle ABE \le \angle CDE$ then $\frac{BD}{AC} < 1$.

From **Theorem 1.9** we know if $z \le 10^4$ then there are no positive integer solutions for equation (1-1), when $n = 3$ (*which is the worst case*) we have

$$\angle CDE = 270^0 - \arctan\left(\frac{z^n - z^{n-1}}{1}\right) - \arctan\left(\frac{1}{z^{n-1} - z^{n-2}}\right)$$
$$= 270^0 - \arctan\left(10000^3 - 10000^2\right) - \arctan\left(\frac{1}{10000^2 - 10000}\right) > 179.999999^0,$$

and

$$\angle ABE > \angle CDE > 179.999999^0,$$

which means $\angle ABE, \angle CDE \to 180^0$, so ABE, CDE are almost lines with $z > 10^4, n \ge 3$, that leads to $\frac{BD}{AC} \to \frac{1}{2} < 1$, which contradicts against $BD > AC$. So when z^n is large enough then $\frac{BD}{AC} = \frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}} < 1$, from **Theorem 1.11** we know there are no positive integer solutions for equation (1-1).

2.2. For function

$$f(z) = \angle CDE = 270^0 - \arctan\left(\frac{z^n - z^{n-1}}{1}\right) - \arctan\left(\frac{1}{z^{n-1} - z^{n-2}}\right)$$
$$= \frac{3}{2}\pi - \arctan\left(\frac{z^n - z^{n-1}}{1}\right) - \arctan\left(\frac{1}{z^{n-1} - z^{n-2}}\right),$$

we give the function plot for it in **Figure 2-2**.

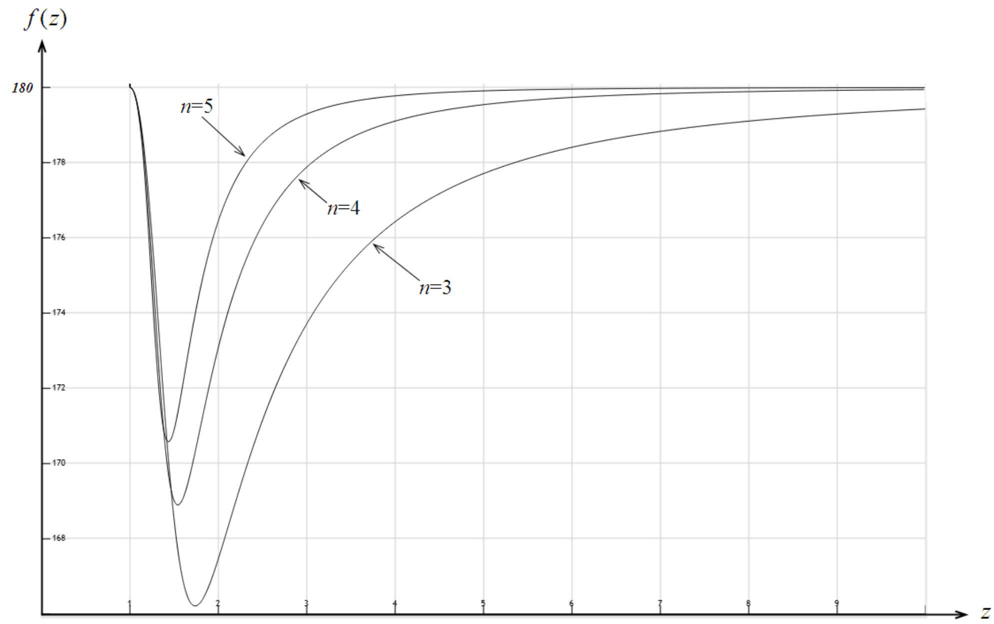

Figure 2-2 *Graph of* $f(z) = \angle CDE = 270^0 - \arctan\left(\frac{z^n - z^{n-1}}{1}\right) - \arctan\left(\frac{1}{z^{n-1} - z^{n-2}}\right)$

where we take $\pi = 3.1415926535897932$

Obviously $f(z) = \angle CDE$ is a "Monotonically increasing function" when $z \geq 3$, and with the increasing of z the value of $f(z) = \angle CDE$ is close to 180^0. It is very clear that $\angle ABE - \angle CDE$ is decreasing with the increasing of z, since

$$\left(\angle ABE - \angle CDE = \angle D'CD + \angle BED\right) < 180^0 - \angle CDE,$$

where $\angle CDE$ is increasing. When $n = 3$ since $\angle CDE > 179.999999^0$, so we have

$$\left(\angle D'CD + \angle BED\right) < 180^0 - \angle CDE < 180^0 - 179.999999^0 < 0.000001^0,$$

which means

$$\angle BED, \angle D'CD < 0.000001^0,$$

and when z^n is large enough, we have

$$\angle ABE - \angle CDE = \left(\angle BED + \angle D'CD\right) \to 0,$$

which means $BD < AC$ that contradicts against $BD > AC$. So when z^n is large enough then $\frac{BD}{AC} = \frac{x^{n-1} + y^{n-1} - z^{n-1}}{x^{n-2} + y^{n-2} - z^{n-2}} < 1$, from **Theorem 1.11** we know there are no positive integer solutions for equation (1-1).

2.3. In **Figure 2-1** we have

$$\angle ABE = \arccos\frac{AB^2 + BE^2 - AE^2}{2\times AB\times BE}$$

$$= \arccos\frac{\begin{bmatrix}\left(x^{n-1}+y^{n-1}-x^{n-2}-y^{n-2}\right)^2+1+\left(x^n+y^n-x^{n-1}-y^{n-1}\right)^2+1\\ -\left(x^n+y^n-x^{n-2}-y^{n-2}\right)^2-4\end{bmatrix}}{2\times\sqrt{\left(x^{n-1}+y^{n-1}-x^{n-2}-y^{n-2}\right)^2+1}\times\sqrt{\left(x^n+y^n-x^{n-1}-y^{n-1}\right)^2+1}},$$

$$= \arccos\frac{\begin{bmatrix}\left(x^{n-1}+y^{n-1}-x^{n-2}-y^{n-2}\right)^2+1+\left(z^n-x^{n-1}-y^{n-1}\right)^2+1\\ -\left(z^n-x^{n-2}-y^{n-2}\right)^2-4\end{bmatrix}}{2\times\sqrt{\left(x^{n-1}+y^{n-1}-x^{n-2}-y^{n-2}\right)^2+1}\times\sqrt{\left(z^n-x^{n-1}-y^{n-1}\right)^2+1}}$$

and

$$\angle CDE = \arccos\frac{CD^2 + DE^2 - CE^2}{2\times CD\times DE},$$

$$= \arccos\frac{\left(z^{n-1}-z^{n-2}\right)^2+1+\left(z^n-z^{n-1}\right)^2+1-\left(z^n-z^{n-2}\right)^2-4}{2\times\sqrt{\left(z^{n-1}-z^{n-2}\right)^2+1}\times\sqrt{\left(z^n-z^{n-1}\right)^2+1}}$$

from (1-1) we have

$$y = \left(z^n - x^n\right)^{\frac{1}{n}},$$

we give the plot of $f(z,x) = \angle ABE - \angle CDE$ using Excel VBA program that is showed below:

```
Dim x As Long
Dim y As Double
Dim z As Long
Dim i As Long
Dim j As Long
Dim AB As Double
Dim BE As Double
Dim CD As Double
Dim DE As Double
Dim AE As Double
Dim CE As Double
Dim AB2 As Double
Dim BE2 As Double
Dim CD2 As Double
```

```
Dim DE2 As Double
Dim AE2 As Double
Dim CE2 As Double
Dim A_CDE As Double
Dim A_ABE As Double
Dim R As Double

n = 3
j = 1
For z = 3 To 10 ^ 7 Step 1
    For x = z / (2 ^ (1 / n)) To z - 1
        y = (z ^ n - x ^ n) ^ (1 / n)
        AB2 = (x ^ (n - 1) + y ^ (n - 1) - x ^ (n - 2) - y ^ (n - 2)) ^ 2 + 1
        AB = Sqr(AB2)
        BE2 = (z ^ n - x ^ (n - 1) - y ^ (n - 1)) ^ 2 + 1
        BE = Sqr(BE2)
        AE2 = (z ^ n - x ^ (n - 2) - y ^ (n - 2)) ^ 2 + 4
        AE = Sqr(AE2)
        CD2 = (z ^ (n - 1) - z ^ (n - 2)) ^ 2 + 1
        CD = Sqr(CD2)
        DE2 = (z ^ n - z ^ (n - 1)) ^ 2 + 1
        DE = Sqr(DE2)
        CE2 = (z ^ n - z ^ (n - 2)) ^ 2 + 4
        CE = Sqr(CE2)
        A_ABE = Application.Acos((AB2 + BE2 - AE2) / (2 * AB * BE))
        A_CDE = Application.Acos((CD2 + DE2 - CE2) / (2 * CD * DE))
        R = A_ABE - A_CDE
        Cells(i, j) = R
        i = i + 1
        If i = 65535 Then j = j + 1: i = 0
        If j = 99 Then End
    Next x
Next z
```

Figure 2-3 shows $f(z,x) = \angle ABE - \angle CDE, n = 3$ is decreasing when z is small, when z is large enough then $f(z,x)$ is between positive and negative at very small amplitude, which means $f(z,x)$ is close to 0.

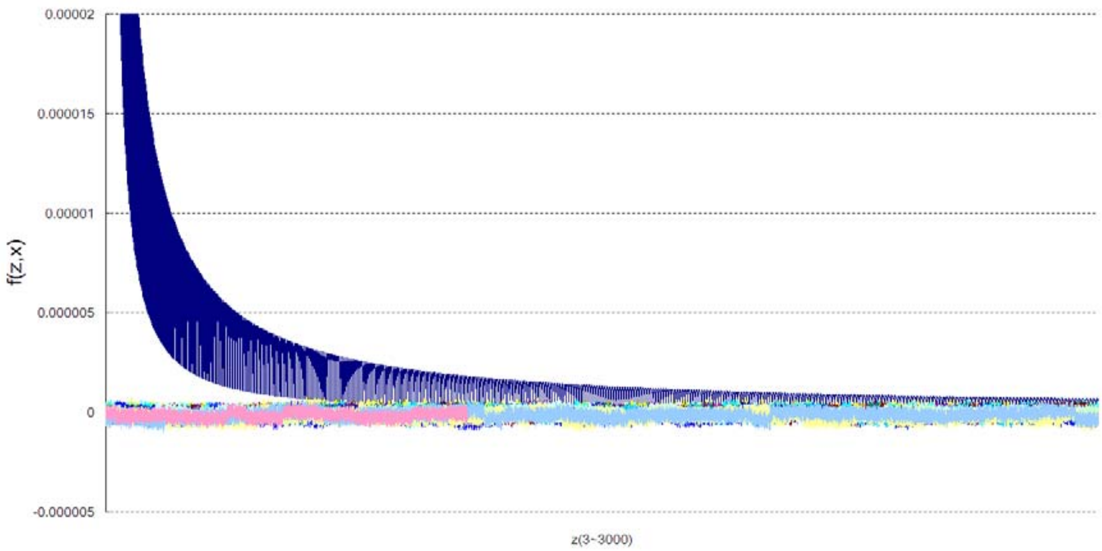

Figure 2-3 *Graph of* $f(z,x)=\angle ABE-\angle CDE, n=3$

2.4. In **Figure 2-1** we have

$$BD^2=BE^2+DE^2-2BE\times DE\times\cos(\angle BED)$$
$$=\left[\begin{array}{l}\left(z^n-z^{n-1}\right)^2+1+\\ \left(x^n+y^n-x^{n-1}-y^{n-1}\right)^2+1\\ -2\sqrt{\left(z^n-z^{n-1}\right)^2+1}\times\sqrt{\left(x^n+y^n-x^{n-1}-y^{n-1}\right)^2+1}\times\\ \cos\left(\arctan\left(\dfrac{1}{x^n+y^n-x^{n-1}-y^{n-1}}\right)-\arctan\left(\dfrac{1}{z^n-z^{n-1}}\right)\right)\end{array}\right],$$

and

$$AC^2=AE^2+CE^2-2AE\times CE\times\cos(\angle AEC)$$
$$=\left[\begin{array}{l}\left(z^n-z^{n-2}\right)^2+4+\\ \left(x^n+y^n-x^{n-2}-y^{n-2}\right)^2+4\\ -2\sqrt{\left(z^n-z^{n-2}\right)^2+4}\times\sqrt{\left(x^n+y^n-x^{n-2}-y^{n-2}\right)^2+4}\times\\ \cos\left(\arctan\left(\dfrac{2}{x^n+y^n-x^{n-2}-y^{n-2}}\right)-\arctan\left(\dfrac{2}{z^n-z^{n-2}}\right)\right)\end{array}\right],$$

from (1-1) we have

$$y=\left(z^n-x^n\right)^{\frac{1}{n}}.$$

We give the plot of $f(z,x)=\dfrac{BD}{AC}$ using Excel VBA program that is showed below:

```
Dim x As Long
Dim y As Double
```

```
Dim z As Long
Dim k As Long
Dim t1 As Double
Dim t2 As Double
Dim t3 As Double
Dim t4 As Double
Dim BD As Double
Dim AC As Double
Dim R As Double
Dim j As Long

n = 3
k = 1
For z = 10 ^ 1 To 10 ^ 9 Step 1
   For x = z / (2 ^ (1 / n)) To z - 1 Step 1
      y = (z ^ n - x ^ n) ^ (1 / n)
      t1 = z ^ n - z ^ (n - 1)
      t2 = x ^ n + y ^ n - x ^ (n - 1) - y ^ (n - 1)
      t3 = z ^ n - z ^ (n - 2)
      t4 = x ^ n + y ^ n - x ^ (n - 2) - y ^ (n - 2)
      BD = (t1 ^ 2 + t2 ^ 2 + 2 - 2 * Sqr((t1 ^ 2 + 1) * (t2 ^ 2 + 1)) *
           Cos(Application.Atan2(t2, 1) - Application.Atan2(t1, 1)))
      AC = (t3 ^ 2 + t4 ^ 2 + 8 - 2 * Sqr((t3 ^ 2 + 4) * (t4 ^ 2 + 4)) *
           Cos(Application.Atan2(t4, 2) - Application.Atan2(t3, 2)))
      R = (BD / AC) ^ 0.5
      Cells(j, k) = R
      j = j + 1
      If j = 65535 Then j = 0: k = k + 1
      If k = 100 then End
   Next x
Next z
```

We give the plot of $f(z,x)=\dfrac{BD}{AC}, n=3$ when $z=3\sim 9999, x=\dfrac{z}{\sqrt[n]{2}}\sim z$, it is showed in **Figure 2-4**. Obviously the maximum value of $f(z,x)=\dfrac{BD}{AC}$ is about 4000 at which $z\approx 9000$. If z increases, $f(z,x)=\dfrac{BD}{AC}$ will be smaller until $f(z,x)=\dfrac{BD}{AC}<1$ if $z>3\times 10^7$, which can be showed in **Figure 2-5** to **Figure 2-11**, from **Theorem 1.9** we know there are no positive integer solutions for equation (1-1) when $n=3$. So we have the conclusion of when $z^n>\left(3\times 10^7\right)^3$ then there are no positive integer solutions for equation (1-1), if $n>3$

then when $\frac{BD}{AC} < 1$, z is of a value less than 3×10^7.

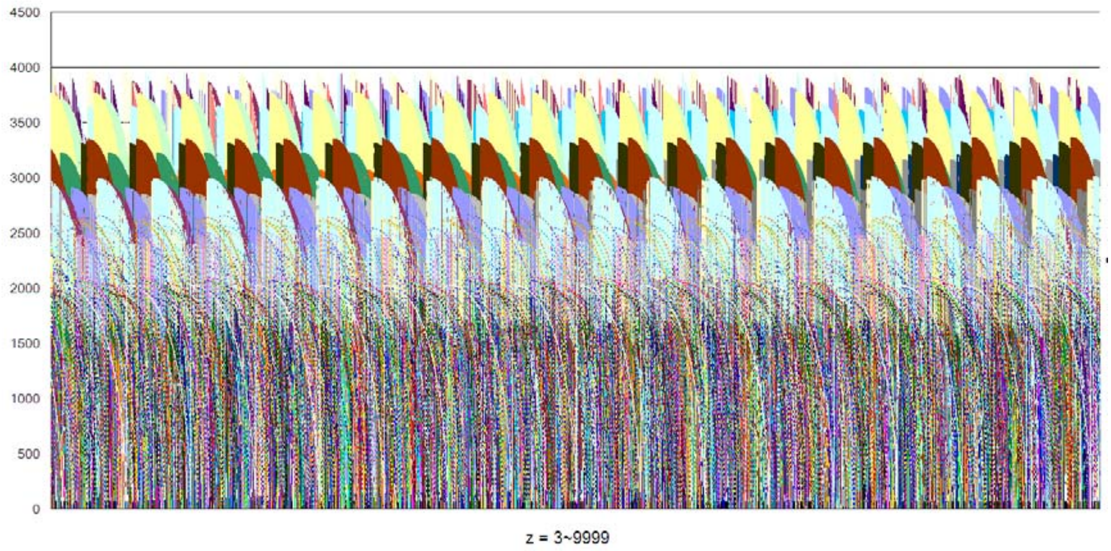

Figure 2-4 *Graph of* $f(z,x) = \frac{BD}{AC}, n = 3, z = 3 \sim 9999$

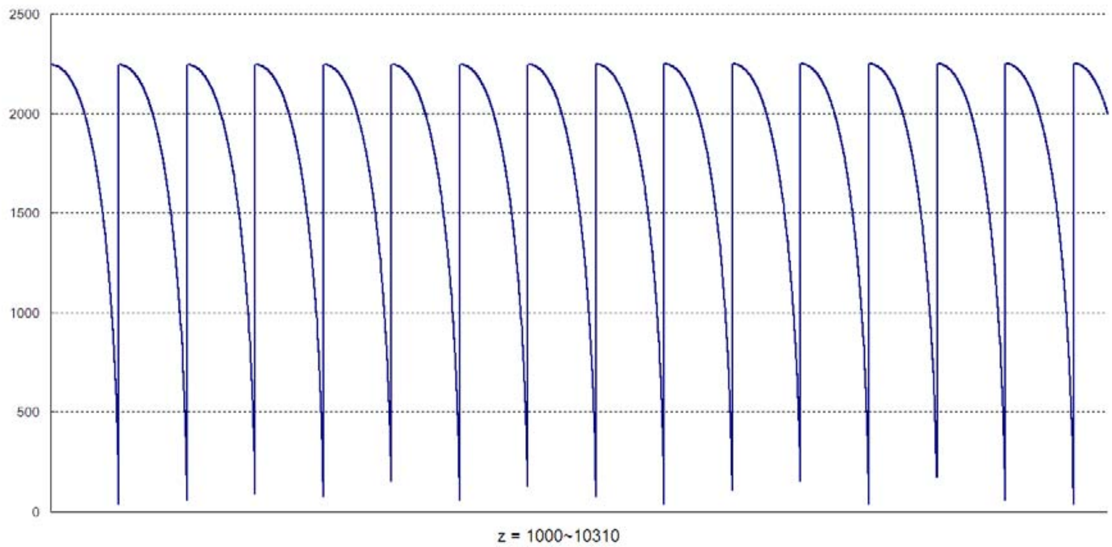

Figure 2-5 *Graph of* $f(z,x) = \frac{BD}{AC}, n = 3, z = 10000 \sim 10310$

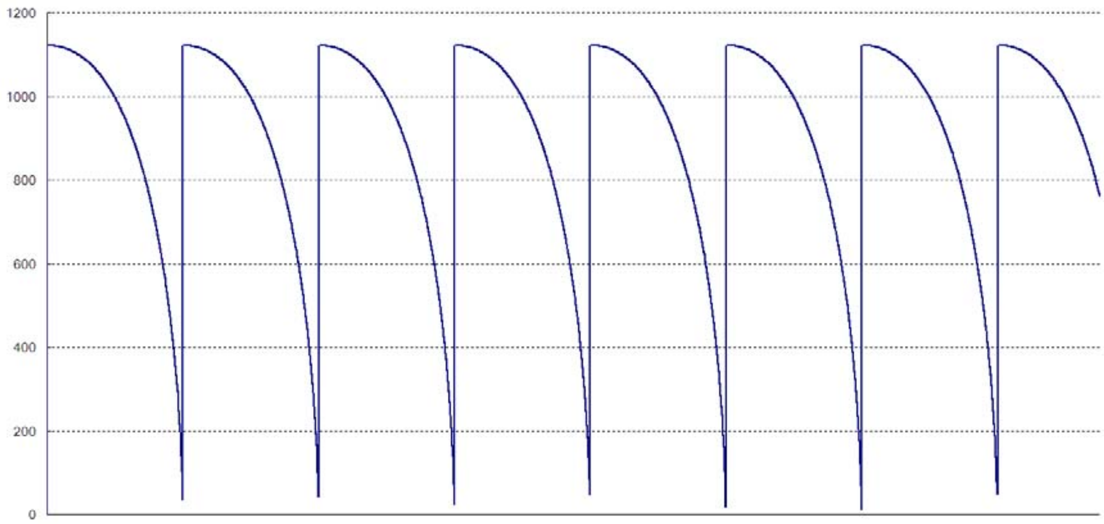

Figure 2-6 *Graph of* $f(z,x)=\dfrac{BD}{AC}, n=3, z=20000$

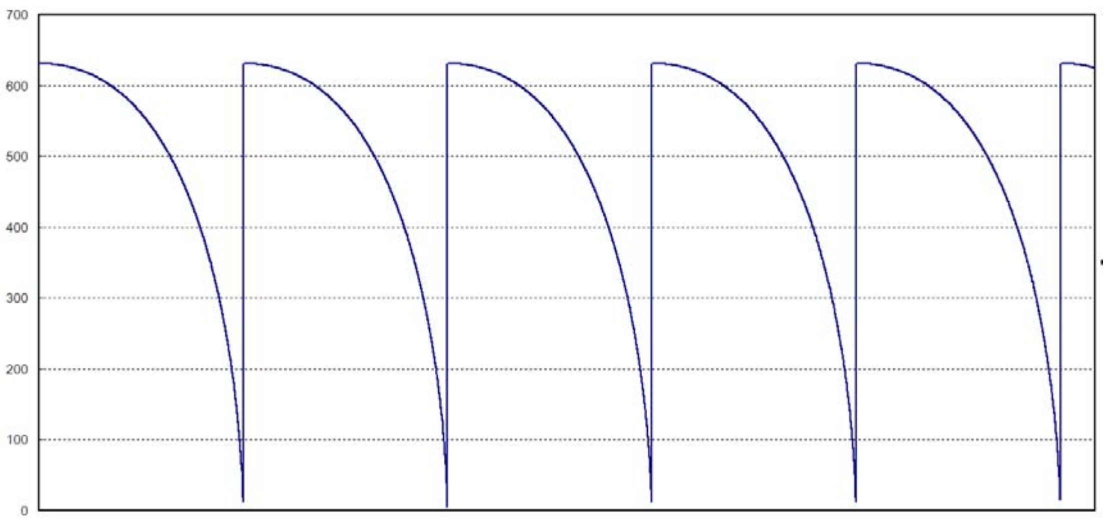

Figure 2-7 *Graph of* $f(z,x)=\dfrac{BD}{AC}, n=3, z=30000$

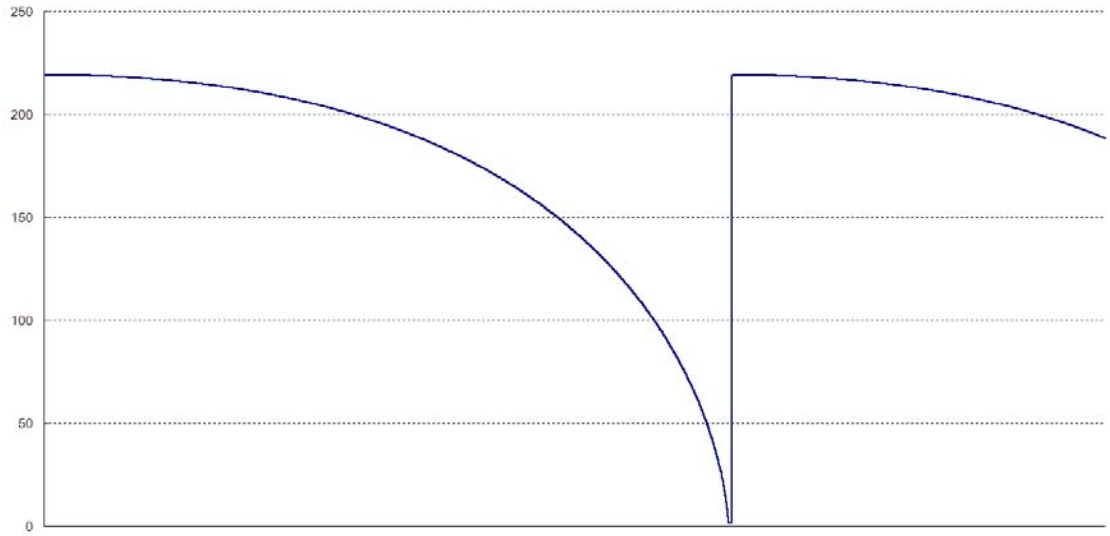

Figure 2-8 *Graph of* $f(z,x)=\dfrac{BD}{AC}, n=3, z=10^5$

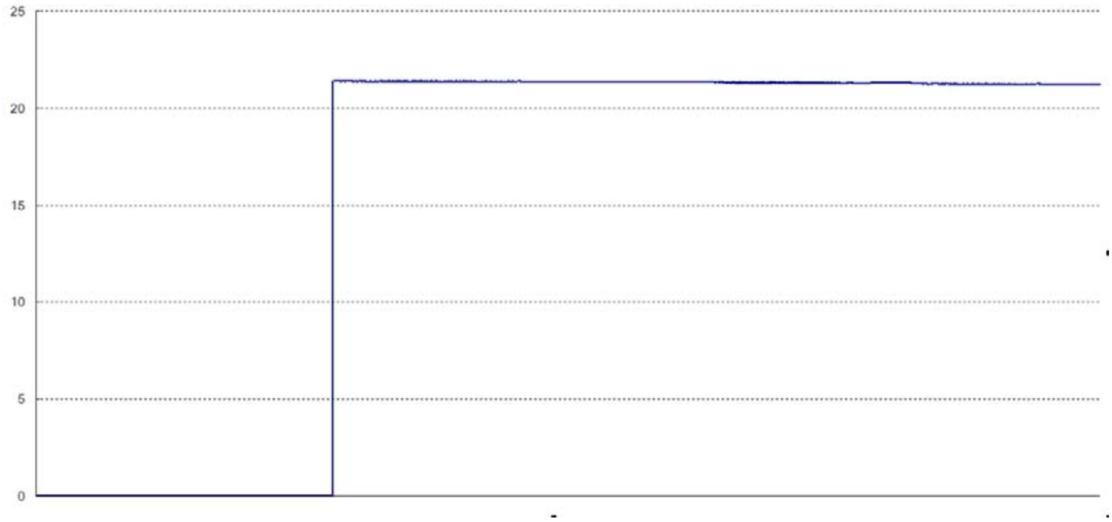

Figure 2-9 *Graph of* $f(z,x)=\dfrac{BD}{AC}, n=3, z=10^6$

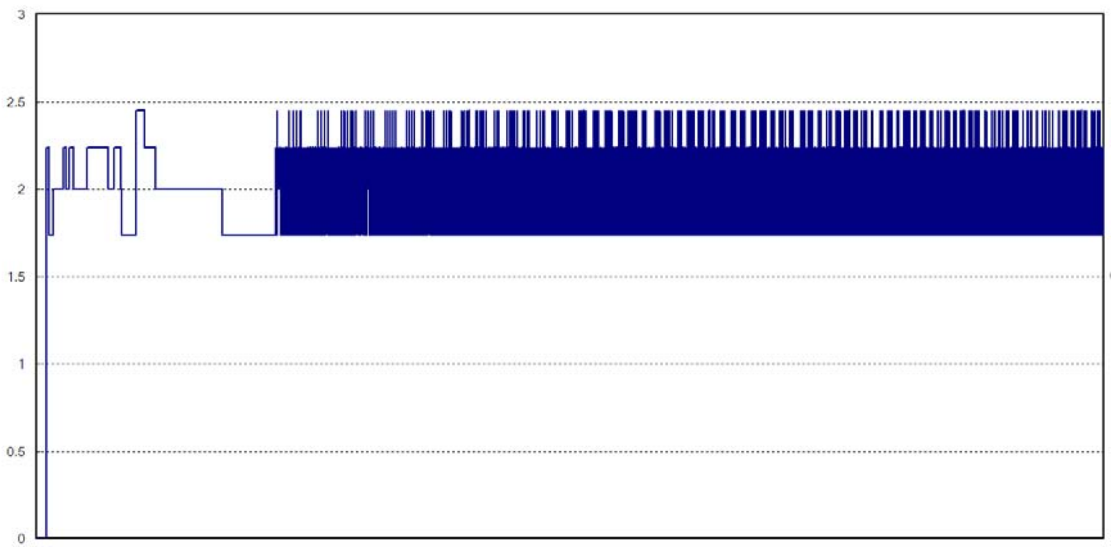

Figure 2-10 *Graph of* $f(z,x)=\dfrac{BD}{AC}, n=3, z=10^7$

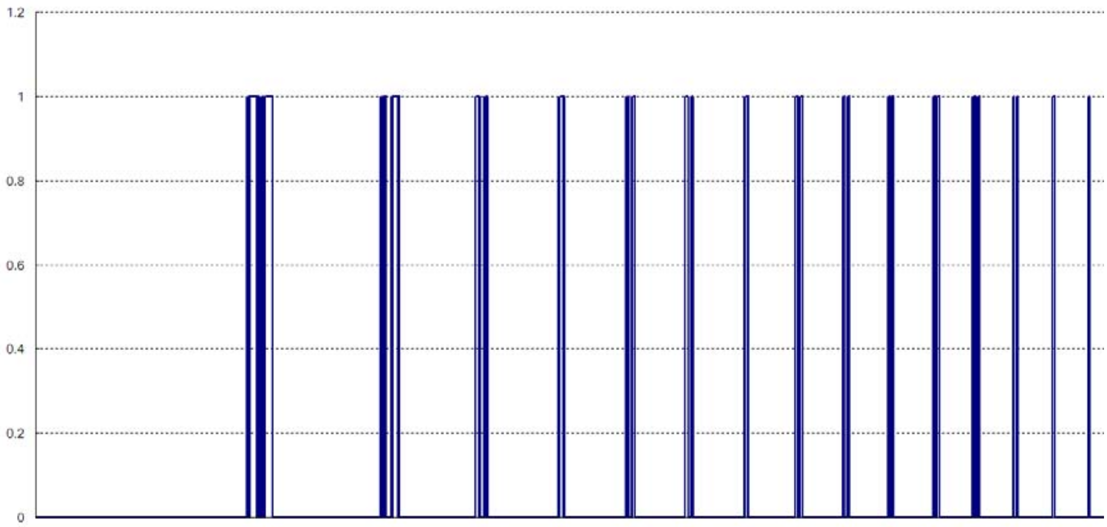

Figure 2-11 *Graph of* $f(z,x)=\dfrac{BD}{AC}, n=3, z=3\times 10^7$

If $n=4,5,7,11$ then from the results of this program we find the maximum values of $f(z,x)=\dfrac{BD}{AC}$ is less than 4000 which means there are no positive integer solutions for equation (1-1).

3. Conclusion

In this paper we first prove there are no positive integer solutions for equation (1-1) when $\dfrac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}}\le 1$, and then prove with the increasing of *x* the conclusion still holds when

$$\frac{x^{n-1}+y^{n-1}-z^{n-1}}{x^{n-2}+y^{n-2}-z^{n-2}}>1.$$