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On Neutrosophic Soft Metric Space

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ABSTRACT. In this paper, the notion of neutrosophic soft metric space (NSMS) is introduced in terms of neutrosophic soft points and several related properties, structural characteristics have been investigated. Then the convergence of sequence in neutrosophic soft metric space is defined and illustrated by examples. Further, the concept of Cauchy sequence in NSMS is developed and some related theorems have been established, too.

1 Introduction

Several techniques like probability theory, fuzzy set [1], rough set, intuitionistic fuzzy set [2], interval mathematics have been adopted to handle the various real life problems involving uncertainties in different fields of studies in mathematical modeling, engineering, economics, medical science, social study and many others. But, Molodtsov has shown that each of the above topics suffers from inherent difficulties possibly due to inadequacy of their parametrization tool. In 1999, Molodtsov [3] initiated a novel concept 'soft set theory' for modeling vagueness and uncertainties. It is completely free from the parametrization inadequacy syndrome of different theories dealing with uncertainty. This makes the theory very convenient, efficient and easily applicable in practice. Molodtsov successfully applied several directions for the applications of soft set theory such as smoothness of functions, game theory, operation research, Riemann integration, Perron integration and probability etc. Maji et al. [4-6] defined and studied the several basic operations in soft sets theory over fuzzy sets and intuitionistic fuzzy sets.

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Many authors [7-9] have introduced and studied several notions of fuzzy metric space from different point of view. George and Veeramani [10] have modified the concept of fuzzy metric space given by Kramosil and Michalek [8] and studied some properties [11, 12] upon this concept. Other contributions to the study of fuzzy metric space may be found in [13-17]. Chang [18] has introduced the theory of fuzzy topological spaces, Roy and Samanta [19] have defined open and closed sets on fuzzy topological spaces. Park [20] and Alaca et al. [21] defined the concept of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms as a generalisation of fuzzy metric space, respectively, in 2004 and in 2006. Using the concept of soft sets, Beaula et al. [22, 23] and Yazar et al. [24-27] have proposed the notions on soft metric spaces and soft normed spaces.

The concept of 'Neutrosophic set' (NS) was first introduced by Smarandache [28, 29] which is a generalisation of classical sets, fuzzy set, intuitionistic fuzzy set etc. Later, Maji [30] has combined this notion with soft set theory and introduced a new concept 'Neutrosophic soft set' (NSS). Using this concept, several mathematicians have produced their research works in different mathematical structures, for instance Deli and Broumi [31], Broumi and Smarandache [32]. But, this concept has been modified by Deli and Broumi [33]. Accordingly, Bera and Mahapatra [34-38] studied some algebraic structures upon this modified concept.

This paper presents the notion of NSMS in terms of neutrosophic soft points along with investigation of some related properties and theorems. Section 2 gives some preliminary useful definitions which will be used through out the paper. In Section 3, NSMS is defined and illustrated by examples along with study of some related properties. Section 4 deals with the convergence of sequence and introduction of Cauchy sequence in NSMS. Finally, the conclusion of our work is given in Section 5.

2 Preliminaries

We recall some basic definitions and theorems related to fuzzy set, soft set, neutrosophic soft set for the sake of completeness.

2.1 Definition [37]

1.A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t - norm if $*$ satisfies the following conditions :

(i) $*$ is commutative and associative.

(ii) $*$ is continuous.

(iii) $a * 1 = 1 * a = a,\ \forall a \in [0,1]$.

(iv) $a * b \leq c * d$ if $a \leq c,\ b \leq d$ with $a, b, c, d \in [0,1]$.

A few examples of continuous t-norm are $a * b = ab, a * b = \min\{a, b\}, a * b = \max\{a + b - 1, 0\}$.

2. A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is continuous t - conorm (s - norm) if $\diamond$ satisfies the following conditions:

(i) $\diamond$ is commutative and associative.

(ii) $\diamond$ is continuous.

(iii) $a \diamond 0 = 0 \diamond a = a, \ \forall a \in [0,1]$.

(iv) $a \diamond b \leq c \diamond d$ if $a \leq c, b \leq d$ with $a,b,c,d \in [0,1]$.

A few examples of continuous t-norm are $a \diamond b = a + b - ab, a \diamond b = \max\{a,b\}, a \diamond b = \min\{a+b,1\}$.

2.2 Definition [29]

Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is characterized by a truth-membership function T_A, an indeterminacy-membership function I_A and a falsity-membership function F_A. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]^-0,1^+[$. That is $T_A, I_A, F_A : X \to]^-0,1^+[$. A neutrosophic set (NS) on the universe of discourse X is defined as :

$$A = \{< x, T_A(x), I_A(x), F_A(x) >: x \in X\}$$

There is no restriction on the sum of $T_A(x), I_A(x), F_A(x)$ and so, $^-0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$. Here $1^+ = 1 + \epsilon$, where 1 is it's standard part and ϵ it's non-standard part. Similarly $^-0 = 0 - \epsilon$, where 0 is it's standard part and ϵ it's non-standard part.

From philosophical point of view, the neutrosophic set (NS) takes the value from real standard or nonstandard subsets of $]^-0,1^+[$. But in real life application in scientific and engineering problems, it is difficult to use NS with value from real standard or nonstandard subset of $]^-0,1^+[$. Hence we consider the NS which takes the value from the subset of [0,1].

2.3 Definition [3]

Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denote the power set of U. Then for $A \subseteq E$, a pair (F, A) is called a soft set over U, where $F : A \to P(U)$ is a mapping.

2.4 Definition [30]

Let U be an initial universe set and E be a set of parameters. Let $NS(U)$ denote the set of all NSs of U. Then for $A \subseteq E$, a pair (F, A) is called an NSS over U, where $F : A \to NS(U)$ is a mapping.

This concept has been modified by Deli and Broumi [33] as given below.

2.5 Definition [33]

Let U be an initial universe set and E be a set of parameters. Let $NS(U)$ denote the set of all NSs of U. Then, a neutrosophic soft set N over U is a set defined by a set valued function f_N representing a mapping $f_N : E \to NS(U)$ where f_N is called approximate function of the neutrosophic soft set N. In other words, the neutrosophic soft set is a parameterized family of some elements of the set $NS(U)$ and therefore it can be written as a set of ordered pairs,

$$N = \{(e, \{< x, T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x) >: x \in U\}) : e \in E\}$$

where $T_{f_N(e)}(x), I_{f_N(e)}(x), F_{f_N(e)}(x) \in [0,1]$, respectively called the truth-membership, indeterminacy-membership, falsity-membership function of $f_N(e)$. Since supremum of each T, I, F is 1 so the inequality $0 \leq T_{f_N(e)}(x) + I_{f_N(e)}(x) + F_{f_N(e)}(x) \leq 3$ is obvious.

2.5.1 *Example*

Let $U = \{h_1, h_2, h_3\}$ be a set of houses and $E = \{e_1(\text{beautiful}), e_2(\text{wooden}), e_3(\text{costly})\}$ be a set of parameters with respect to which the nature of houses are described. Let,

$$f_N(e_1) = \{< h_1, (0.5, 0.6, 0.3) >, < h_2, (0.4, 0.7, 0.6) >, < h_3, (0.6, 0.2, 0.3) >\};$$
$$f_N(e_2) = \{< h_1, (0.6, 0.3, 0.5) >, < h_2, (0.7, 0.4, 0.3) >, < h_3, (0.8, 0.6, 0.2) >\};$$
$$f_N(e_3) = \{< h_1, (0.7, 0.4, 0.3) >, < h_2, (0.6, 0.7, 0.2) >, < h_3, (0.7, 0.2, 0.5) >\};$$

Then $N = \{[e_1, f_N(e_1)], [e_2, f_N(e_2)], [e_3, f_N(e_3)]\}$ is an NSS over (U, E). The tabular representation of the NSS N is given in Table 1.

Table 1 : Tabular form of NSS N.

	$f_N(e_1)$	$f_N(e_2)$	$f_N(e_3)$
h_1	(0.5,0.6,0.3)	(0.6,0.3,0.5)	(0.7,0.4,0.3)
h_2	(0.4,0.7,0.6)	(0.7,0.4,0.3)	(0.6,0.7,0.2)
h_3	(0.6,0.2,0.3)	(0.8,0.6,0.2)	(0.7,0.2,0.5)

2.5.2 Definition [33]

The complement of a neutrosophic soft set N is denoted by N^c and is defined as :

$$N^c = \{(e, \{< x, F_{f_N(e)}(x), 1 - I_{f_N(e)}(x), T_{f_N(e)}(x) >: x \in U\}) : e \in E\}$$

2.5.3 Definition [33]

Let N_1 and N_2 be two NSSs over the common universe (U, E). Then N_1 is said to be the neutrosophic soft subset of N_2 if $\forall e \in E$ and $x \in U$,

$$T_{f_{N_1}(e)}(x) \leq T_{f_{N_2}(e)}(x), I_{f_{N_1}(e)}(x) \geq I_{f_{N_2}(e)}(x), F_{f_{N_1}(e)}(x) \geq F_{f_{N_2}(e)}(x).$$

We write $N_1 \subseteq N_2$ and then N_2 is the neutrosophic soft superset of N_1.

2.5.4 Definition [33]

Let N_1 and N_2 be two NSSs over the common universe (U, E). Then their union is denoted by $N_1 \cup N_2 = N_3$ and is defined as :

$$N_3 = \{(e, \{< x, T_{f_{N_3}(e)}(x), I_{f_{N_3}(e)}(x), F_{f_{N_3}(e)}(x) >: x \in U\}) : e \in E\}$$

where $T_{f_{N_3}(e)}(x) = T_{f_{N_1}(e)}(x) \diamond T_{f_{N_2}(e)}(x)$, $I_{f_{N_3}(e)}(x) = I_{f_{N_1}(e)}(x) * I_{f_{N_2}(e)}(x)$,
$F_{f_{N_3}(e)}(x) = F_{f_{N_1}(e)}(x) * F_{f_{N_2}(e)}(x)$;

Their intersection is denoted by $N_1 \cap N_2 = N_4$ and is defined as :

$$N_4 = \{(e, \{< x, T_{f_{N_4}(e)}(x), I_{f_{N_4}(e)}(x), F_{f_{N_4}(e)}(x) >: x \in U\}) : e \in E\}$$

where $T_{f_{N_4}(e)}(x) = T_{f_{N_1}(e)}(x) * T_{f_{N_2}(e)}(x)$, $I_{f_{N_4}(e)}(x) = I_{f_{N_1}(e)}(x) \diamond I_{f_{N_2}(e)}(x)$,
$F_{f_{N_4}(e)}(x) = F_{f_{N_1}(e)}(x) \diamond F_{f_{N_2}(e)}(x)$;

2.6 Definition [38]

1. A neutrosophic soft set N over (U, E) is said to be null neutrosophic soft set if $T_{f_N(e)}(x) = 0, I_{f_N(e)}(x) = 1, F_{f_N(e)}(x) = 1; \forall e \in E, \forall x \in U$. It is denoted by ϕ_u.

2. A neutrosophic soft set N over (U, E) is said to be absolute neutrosophic soft set if $T_{f_N(e)}(x) = 1, I_{f_N(e)}(x) = 0, F_{f_N(e)}(x) = 0; \forall e \in E, \forall x \in U$. It is denoted by 1_u.

Clearly, $\phi_u^c = 1_u$ and $1_u^c = \phi_u$.

2.7 Definition [38]

1. A neutrosophic soft point in an NSS N is defined as an element $(e, f_N(e))$ of N, for $e \in E$ and is denoted by e_N, if $f_N(e) \notin \phi_u$ and $f_N(e') \in \phi_u, \forall e' \in E - \{e\}$.

2. The complement of a neutrosophic soft point e_N is another neutrosophic soft point e_N^c such that $f_N^c(e) = (f_N(e))^c$.

3. A neutrosophic soft point $e_N \in M$, M being an NSS if for $e \in E$, $f_N(e) \leq f_M(e)$ i.e., $T_{f_N(e)}(x) \leq T_{f_M(e)}(x)$, $I_{f_N(e)}(x) \geq I_{f_M(e)}(x)$, $F_{f_N(e)}(x) \geq F_{f_M(e)}(x)$, $\forall x \in U$.

2.7.1 *Example*

Let $U = \{x_1, x_2, x_3\}$ and $E = \{e_1, e_2\}$. Then,
$e_{1N} = \{< x_1, (0.6, 0.4, 0.8) >, < x_2, (0.8, 0.3, 0.5) >, < x_3, (0.3, 0.7, 0.6) >\}$
is a neutrosophic soft point whose complement is :
$e_{1N}^c = \{< x_1, (0.8, 0.6, 0.6) >, < x_2, (0.5, 0.7, 0.8) >, < x_3, (0.6, 0.3, 0.3) >\}$.
For another NSS M defined on same (U, E), let
$f_M(e_1) = \{< x_1, (0.7, 0.4, 0.7) >, < x_2, (0.8, 0.2, 0.4) >, < x_3, (0.5, 0.6, 0.5) >\}$.

Then $f_N(e_1) \leq f_M(e_1)$ i.e., $e_{1N} \in M$.

3 Neutrosophic Soft Metric

Unless otherwise stated, E is treated as the parametric set through out this paper and $e \in E$, an arbitrary parameter.

3.1 Definition

Let $NS(U_E)$ be the collection of all neutrosophic soft points over (U, E). Then the neutrosophic soft metric interm of neutrosophic soft points is defined by a mapping $d : NS(U_E) \times NS(U_E) \to [0, 3]$ satisfying the following conditions :

$NSM1 : d(e_M, e_N) \geq 0, \ \forall e_M, e_N \in NS(U_E)$.

$NSM2 : d(e_M, e_N) = 0 \Leftrightarrow e_M = e_N$.

$NSM3 : d(e_M, e_N) = d(e_N, e_M)$.

$NSM4 : d(e_M, e_N) \leq d(e_M, e_P) + d(e_P, e_N), \ \forall e_M, e_P, e_N \in NS(U_E)$.

Then $NS(U_E)$ is said to form an NSMS with respect to the neutrosophic soft metric 'd ' over (U, E) and is denoted by $(NS(U_E), d)$. Here $e_M = e_N$ in the sense that $T_{e_M}(x_i) = T_{e_N}(x_i)$, $I_{e_M}(x_i) = I_{e_N}(x_i)$, $F_{e_M}(x_i) = F_{e_N}(x_i)$, $\forall x_i \in U$.

3.1.1 *Example*

1. Define $d(e_M, e_N) = min_{x_i}\{(|T_{e_M}(x_i) - T_{e_N}(x_i)|^k + |I_{e_M}(x_i) - I_{e_N}(x_i)|^k + |F_{e_M}(x_i) - F_{e_N}(x_i)|^k)^{\frac{1}{k}}\} \quad (k \geq 1)$ on $NS(U_E)$.

Evidently, $d(e_M, e_N) \geq 0$ and $d(e_M, e_N) = 0$ iff $e_M = e_N$. Also $d(e_M, e_N) = d(e_N, e_M)$. To verify the final condition, we shall use Minkowski inequality for sum.

$$
\begin{aligned}
& d(e_M, e_N) \\
= \ & \min_{x_i}\{(|T_{e_M}(x_i) - T_{e_N}(x_i)|^k + |I_{e_M}(x_i) - I_{e_N}(x_i)|^k + |F_{e_M}(x_i) - F_{e_N}(x_i)|^k)^{\frac{1}{k}}\} \\
= \ & \min_{x_i}\{(|T_{e_M}(x_i) - T_{e_P}(x_i) + T_{e_P}(x_i) - T_{e_N}(x_i)|^k + |I_{e_M}(x_i) - I_{e_P}(x_i) \\
& + I_{e_P}(x_i) - I_{e_N}(x_i)|^k + |F_{e_M}(x_i) - F_{e_P}(x_i) + F_{e_P}(x_i) - F_{e_N}(x_i)|^k)^{\frac{1}{k}}\} \\
\leq \ & \min_{x_i}\{(|T_{e_M}(x_i) - T_{e_P}(x_i)|^k + |I_{e_M}(x_i) - I_{e_P}(x_i)|^k + |F_{e_M}(x_i) - F_{e_P}(x_i)|^k)^{\frac{1}{k}}\} \\
& + \min_{x_i}\{(|T_{e_P}(x_i) - T_{e_N}(x_i)|^k + |I_{e_P}(x_i) - I_{e_N}(x_i)|^k + |F_{e_P}(x_i) - F_{e_N}(x_i)|^k)^{\frac{1}{k}}\} \\
= \ & d(e_M, e_P) + d(e_P, e_N)
\end{aligned}
$$

Thus 'd ' defined above is called a neutrosophic soft metric over (U, E).

2. Let 'd ' be a neutrosophic soft metric on $NS(U_E)$. Suppose $d_1(e_M, e_N) = \frac{d(e_M, e_N)}{1+d(e_M, e_N)}$; Then '$d_1$' satisfies the first three conditions. It is required to verify the fourth condition for 'd_1'. For $e_M, e_N, e_P \in N$,

$$\begin{aligned} d_1(e_M, e_N) &= \frac{d(e_M, e_N)}{1 + d(e_M, e_N)} \\ &= 1 - \frac{1}{1 + d(e_M, e_N)} \\ &\leq 1 - \frac{1}{1 + d(e_M, e_P) + d(e_P, e_N)} \\ &= \frac{d(e_M, e_P) + d(e_P, e_N)}{1 + d(e_M, e_P) + d(e_P, e_N)} \\ &= \frac{d(e_M, e_P)}{1 + d(e_M, e_P) + d(e_P, e_N)} + \frac{d(e_P, e_N)}{1 + d(e_M, e_P) + d(e_P, e_N)} \\ &\leq \frac{d(e_M, e_P)}{1 + d(e_M, e_P)} + \frac{d(e_P, e_N)}{1 + d(e_P, e_N)} \\ &= d_1(e_M, e_P) + d_1(e_P, e_N) \end{aligned}$$

So, $(NS(U_E), d_1)$ is an NSMS with respect to the neutrosophic soft metric d_1.

3.2 Definition

1. Let $(NS(U_E), d)$ be a neutrosophic soft metric space and $t \in (0, 3]$. Then the neutrosophic soft open ball and the neutrosophic soft closed ball having center at $e_N \in NS(U_E)$ and radius 't' are defined by following sets, respectively.

$B(e_N, t) = \{e_{iN} \in NS(U_E) : d(e_N, e_{iN}) < t\}$,

$B[e_N, t] = \{e_{iN} \in NS(U_E) : d(e_N, e_{iN}) \leq t\}$.

2. A neighbourhood of $e_N \in NS(U_E)$ is defined by an open ball $B(e_N, t)$ with center at e_N and radius $t \in (0, 3]$.

3.3 Definition

1. In an NSMS $(NS(U_E), d)$ over (U, E), a neutrosophic soft point e_N is called an interior point of $NS(U_E)$ if there exist an open ball $B(e_N, t)$ such that $B(e_N, t) \subset NS(U_E)$.

2. For an NSMS $(NS(U_E), d)$ over (U, E), an NSS M is called open if each of it's points is an interior point.

3.3.1 *Example*

1. Consider an NSMS $(NS(U_E), d)$ with respect to the distance function 'd' defined in (1) of 3.1.1 for $k = 1$ where $NS(U_E) = \{e_M, e_N, e_P\}$ is given as following :

$$\begin{aligned} e_M &= \{< x, (0.5, 0.6, 0.3) >, < y, (0.4, 0.7, 0.6) >, < z, (0.6, 0.2, 0.3) >\}; \\ e_N &= \{< x, (0.6, 0.3, 0.5) >, < y, (0.7, 0.4, 0.3) >, < z, (0.8, 0.6, 0.2) >\}; \\ e_P &= \{< x, (0.7, 0.4, 0.3) >, < y, (0.6, 0.7, 0.2) >, < z, (0.7, 0.2, 0.5) >\}; \end{aligned}$$

Let us define an arbitrary neutrosophic soft point $e_{1Q} \notin NS(U_E)$ [by sense of 2.7] as following :

$$e_{1Q} = \{< x, (0.5, 0.7, 0.6) >, < y, (0.3, 0.6, 0.7) >, < z, (0.2, 0.4, 0.8) >\};$$

Then for $t = 0.4$, we have $e_{1Q} \in B(e_M, 0.4)$ as $d(e_M, e_{1Q}) = 0.3 < 0.4$ and thus $B(e_M, 0.4) \not\subset NS(U_E)$.

Next, let us verify for the radius $t = 0.3$; Consider a neutrosophic soft point e_{2S} defined as following :

$$e_{2S} = \{< x, (0.7, 0.4, 0.4) >, < y, (0.5, 0.7, 0.5) >, < z, (0.9, 0.1, 0.3) >\}$$

Then $e_{2S} \notin NS(U_E)$ [by sense of 2.7] but $e_{2S} \in B(e_M, 0.3)$ as $d(e_M, e_{2S}) = 0.2 < 0.3$; Hence, $B(e_M, 0.3) \not\subset NS(U_E)$ also. Similar conclusion can be drawn in taking different radii t. Hence, e_M is not a neutrosophic soft interior point of $NS(U_E)$ i.e., it is not open.

2. Let $E = \mathbf{N}$ (the set of natural numbers) be the parametric set and $U = \mathbf{Z}$ (the set of all integers) be the universal set. Define a mapping $f_M : \mathbf{N} \to NS(\mathbf{Z})$ by :

$T_{f_M(n)}(x) = \frac{1}{n},\ I_{f_M(n)}(x) = \frac{1}{n+1},\ F_{f_M(n)}(x) = \frac{1}{n+2};\ \ \forall x \in \mathbf{Z}, n \in \mathbf{N}$

where $\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}$ are respectively the n-th, (n+1)-th, (n+2)-th rational numbers in $Q^I \subset (0, 1)$ [Q^I being a set of rational numbers] and $T_{f_M(n)}(x), I_{f_M(n)}(x), F_{f_M(n)}(x) \in Q^I$, $\forall x \in \mathbf{Z}$, $\forall n \in \mathbf{N}$. Then all the neutrosophic soft points of NSS M are interior points and consequently, M is open over $(\mathbf{Z}, \mathbf{N})$.

3. Every absolute NSS and null NSS are open.

3.4 Definition

1. A neutrosophic soft point e_N in an NSMS $(NS(U_E), d)$ is called a limit point/ accumulation point of an NSS $M \subset NS(U_E)$ if for every $t \in (0, 3]$, $B(e_N, t)$ contains at least one neutrosophic soft point of M distinct from e_N.

2. Collection of all limit points of M is called derived NSS of M and is denoted by $D(M)$. An NSS $M \subset NS(U_E)$ in an NSMS $(NS(U_E), d)$ over (U, E) is closed NSS if $D(M) \subset M$ or M has no limit point.

3.4.1 *Example*

1. Let $U = \{h_1, h_2, h_3\}$ and $E = \{e_1, e_2, e_3\}$; Now consider the Table 1 and the NSS M over (U, E) given in Table 2.

Table 2 : Tabular form of NSS M.

	$f_M(e_1)$	$f_M(e_2)$	$f_M(e_3)$
h_1	(0.4,0.7,0.4)	(0.6,0.4,0.6)	(0.7,0.6,0.4)
h_2	(0.2,0.8,0.8)	(0.4,0.6,0.7)	(0.5,0.8,0.5)
h_3	(0.5,0.5,0.6)	(0.3,0.8,0.2)	(0.3,0.4,0.7)

By the distance function 'd' as defined in (1) of 3.1.1 for $k = 1$,

$$d(e_{1N}, e_{1M}) = 0.3,\ d(e_{1N}, e_{2M}) = 0.2,\ d(e_{1N}, e_{3M}) = 0.3$$
$$d(e_{2N}, e_{1M}) = 0.7,\ d(e_{2N}, e_{2M}) = 0.2,\ d(e_{2N}, e_{3M}) = 0.5$$
$$d(e_{3N}, e_{1M}) = 0.6,\ d(e_{3N}, e_{2M}) = 0.4,\ d(e_{3N}, e_{3M}) = 0.3$$

If $t = 0.1$, then each of $B(e_{1N}, t), B(e_{2N}, t), B(e_{3N}, t)$ contains no point of M. Thus any of e_{1N}, e_{2N}, e_{3N} is not a limit point of M. Similarly, either of e_{1M}, e_{2M}, e_{3M} is not also a limit point of M. Thus M has no limit point i.e.,

$D(M) = \phi \subset M$. Hence, M is a closed NSS.

2. Let $E = \mathbf{N}$ (the set of natural numbers) be the parametric set and $U = \mathbf{Z}$ (the set of all integers) be the universal set. Define a mapping $f_M : \mathbf{N} \to NS(\mathbf{Z})$ where, for any $n \in \mathbf{N}$ and $x \in \mathbf{Z}$,

$$T_{f_M(n)}(x) = \begin{cases} \frac{1}{n^2} & \text{if } x \text{ is odd} \\ 0 & \text{if } x \text{ is even.} \end{cases}$$

$$I_{f_M(n)}(x) = \begin{cases} 1 - \frac{1}{n} & \text{if } x \text{ is odd} \\ 1 & \text{if } x \text{ is even.} \end{cases}$$

$$F_{f_M(n)}(x) = \begin{cases} \frac{1}{1+n} & \text{if } x \text{ is odd} \\ 0 & \text{if } x \text{ is even.} \end{cases}$$

The limit point of NSS M over $(\mathbf{Z}, \mathbf{N})$ is $(0, 1, 0) \in M$ and so M is closed.

3. For the above NSS M, define truth-membership (T), indeterminacy-membership (I) and falsity-membership (F) functions as following :

$$T_{f_M(n)}(x) = \frac{1}{n},\ I_{f_M(n)}(x) = \frac{1}{2n},\ F_{f_M(n)}(x) = 1 - \frac{1}{n}\ \forall x \in \mathbf{Z}.$$

It's limit point $(0, 0, 1) \notin M$. It is neither closed nor open NSS.

3.5 Theorem

In an NSMS $(NS(U_E), d)$, every neutrosophic soft open ball $B(e_N, t)$ is open and every neutrosophic soft closed ball $B[e_N, t]$ is closed.

Proof. Let $e_P \in B(e_N, t)$. Then $d(e_N, e_P) < t$. Let $r = t - d(e_N, e_P)$ and choose another open ball $B(e_P, r)$. It is necessary to show $B(e_P, r) \subset B(e_N, t)$ i.e., e_P is an interior point of $B(e_N, t)$.
Let $e_M \in B(e_P, r)$. Then $d(e_P, e_M) < r$. Now

$$\begin{aligned} & d(e_N, e_M) \leq d(e_N, e_P) + d(e_P, e_M) \\ \Rightarrow\ & d(e_N, e_M) < d(e_N, e_P) + r \\ \Rightarrow\ & d(e_N, e_M) < t \\ \Rightarrow\ & e_M \in B(e_N, t) \end{aligned}$$

Hence $B(e_P, r) \subset B(e_N, t)$.

Next, let $e_P \in NS(U_E) - B[e_N, t]$. Then $e_P \notin B[e_N, t]$ i.e., $d(e_N, e_P) > t$. Let $r = d(e_N, e_P) - t$. Then $r > 0$. Choose an open neutrosophic soft ball $B(e_P, r)$. It is required to show that $B(e_P, r) \cap B[e_N, t] = \phi$.

If possible $e_M \in B(e_P, r) \cap B[e_N, t]$. Then $d(e_P, e_M) < r$, $d(e_N, e_M) \leq t$. Now

$$\begin{aligned} & d(e_N, e_P) \leq d(e_N, e_M) + d(e_M, e_P) \\ \Rightarrow\quad & d(e_N, e_M) \geq d(e_N, e_P) - d(e_M, e_P) \\ \Rightarrow\quad & d(e_N, e_M) > d(e_N, e_P) - r \\ \Rightarrow\quad & d(e_N, e_M) > t \\ \Rightarrow\quad & e_M \notin B[e_N, t] \end{aligned}$$

It is a contradiction to the fact that $e_M \in B[e_N, t]$. Hence $B(e_P, r) \cap B[e_N, t] = \phi$.

3.6 Definition

Let M be an NSS over (U, E) and e_N be an arbitrary neutrosophic soft point. Then,

1. $e_N \in M$ strictly, if for $e \in E,\ e_N = e_M$ holds i.e.,
$T_{f_N(e)}(x) = T_{f_M(e)}(x)$, $I_{f_N(e)}(x) = I_{f_M(e)}(x)$, $F_{f_N(e)}(x) = F_{f_M(e)}(x)$, $\forall x \in U$.

2. $e_N \in M$ pseudonymously, if for $e \in E,\ e_N \subset e_M$ holds i.e.,
$T_{f_N(e)}(x) < T_{f_M(e)}(x)$, $I_{f_N(e)}(x) > I_{f_M(e)}(x)$, $F_{f_N(e)}(x) > F_{f_M(e)}(x)$, $\forall x \in U$.

3.6.1 *Example*

Let $e_M = \{< h_1, (0.6, 0.3, 0.5) >, < h_2, (0.7, 0.4, 0.3) >, < h_3, (0.8, 0.6, 0.2) >\}$ and $e_P = \{< h_1, (0.6, 0.5, 0.4) >, < h_2, (0.5, 0.8, 0.3) >, < h_3, (0.3, 0.3, 0.6) >\}$ be two neutrosophic soft points in $NS(U_E)$. Consider the NSS $N \subset NS(U_E)$ defined in Table 1. Clearly $e_M = e_{2N}$ and $e_P \subset e_{3N}$. Thus $e_M \in N$ strictly but $e_P \in N$ pseudonymously.

3.7 Proposition

Let $(NS(U_E), d)$ be an NSMS and $N_1, N_2 \subset NS(U_E)$. Then by sense of 2.7,

1. $e_N \in N_1$ or N_2 or both $\Rightarrow$ $e_N \in N_1 \cup N_2$.

2. $e_N \in N_1 \cup N_2$ $\Rightarrow$ $e_N \notin N_1$ or N_2 or both necessarily.

For the strict belongingness of e_N, '$\Leftrightarrow$' occurs always.

3. $e_N \in N_1 \cap N_2$ $\Leftrightarrow$ $e_N \in N_1, N_2$ both.

The above results can be easily verified by taking two arbitrary NSSs.

These are also true for arbitrary number of NSSs in an NSMS.

3.8 Theorem

Let $(NS(U_E), d)$ be an NSMS over (U, E). Then,

1. the intersection of finite number of open NSSs in $(NS(U_E), d)$ is open.

2. the intersection of any family of closed NSSs in $(NS(U_E), d)$ is closed.

Proof. **1.** Let $\{M_i : 1 \le i \le k\} \subset NS(U_E)$ and they are open. Suppose $e_M \in \cap_{i=1}^{k} M_i$. Then $e_M \in M_i$, $\forall i$ by sense of 2.7. Since each M_i is open, then $B(e_M, t_i) \subset M_i$ for $t_i \in \mathbf{R}^+, 1 \le i \le k$. Let $t = \min\{t_1, t_2, \cdots, t_k\}$. Then $B(e_M, t) \subset B(e_M, t_i) \subset M_i$, $\forall i$ i.e., $B(e_M, t) \subset \cap_{i=1}^{k} M_i$. Thus e_M is an interior neutrosophic soft point of $\cap_{i=1}^{k} M_i$. Since e_M is arbitrary, so $\cap_{i=1}^{k} M_i$ is open.

2. Let $\{Q_i | i \in \Delta\} \subset NS(U_E)$ and they are closed. Suppose e_Q be an arbitrary limit point of $(\cap Q_i)$. Then there exists an open neutrosophic soft ball $B(e_Q, r)$ such that $e_{Q_1} \in B(e_Q, r) \cap (\cap Q_i)$, say. This implies $e_{Q_1} \in B(e_Q, r) \cap Q_i$ for each i. Thus e_Q is a limit point for each Q_i. Now since each Q_i is closed, so $e_Q \in Q_i$ for each i and hence $e_Q \in \cap Q_i$.

3.9 Theorem

Let $(NS(U_E), d)$ be an NSMS. Then $M \subset NS(U_E)$ is an open NSS iff it can be expressed as an intersection of a finite number of neutrosophic soft open balls.

Proof. The first part is obvious. we shall prove only the reverse part.

Since each open ball in an NSMS is open and the intersection of a finite number of open NSSs in $(NS(U_E), d)$ is open, so the proof is completed.

3.10 Theorem

Let $(NS(U_E), d)$ be an NSMS. Then $Q \subset NS(U_E)$ is a closed NSS iff it can be expressed as an intersection of a family of neutrosophic soft closed balls.

Proof. Straight forward.

3.11 Theorem

1. Let $\{M_i : i \in \Delta\}$ be a family of open NSSs in an NSMS $(NS(U_E), d)$. Then $\cup M_i$ is open if $e_M \in \cup M_i \Rightarrow e_M$ strictly belongs to at least one M_i, holds.

2. Let $\{Q_i : i \in \Delta\}$ be a family of closed NSSs in an NSMS $(NS(U_E), d)$. Then $\cup Q_i$ is closed if $e_q \in \cup Q_i \Rightarrow e_q$ strictly belongs to at least one Q_i, holds.

Proof. **1.** Let an arbitrary neutrosophic soft point $e_M \in \cup M_i$. Then $e_M \in M_k$ strictly for some $k \in \Delta$. Since M_k is open NSS, so e_M is an interior neutrosophic soft point of M_k i.e., $B(e_M, t) \subset M_k \subset \cup M_i$. Hence e_M is an interior neutrosophic soft point of $\cup M_i$. Since e_M is arbitrary, so $\cup M_i$ is open NSS.

2. Let an arbitrary neutrosophic soft point e_Q be a limit point of $\cup Q_i$. Then $B(e_Q, r) \cap (\cup Q_i) \neq \phi$ for every r. Suppose $e_q \in B(e_Q, r) \cap (\cup Q_i)$. Then $e_q \in B(e_Q, r)$ and $e_q \in \cup Q_i$. This implies $e_q \in Q_k$ strictly for some $k \in \Delta$ i.e., $e_q \in B(e_Q, r) \cap Q_k$. This shows e_Q is a limit point of Q_k and since Q_k is closed, so $e_Q \in Q_k$. Hence $e_Q \in \cup Q_i$ and so $\cup Q_i$ is closed.

3.12 Theorem

Any two distinct neutrosophic soft points in an NSMS $(NS(U_E), d)$ have disjoint neighbourhoods.

Proof. Let us consider two distinct neutrosophic soft points e_{N_1}, e_{N_2} in $NS(U_E)$. Then $d(e_{N_1}, e_{N_2}) > 0$. Suppose $r = \frac{1}{2}d(e_{N_1}, e_{N_2})$. Now consider two neutrosophic soft open balls $B(e_{N_1}, r)$ and $B(e_{N_2}, r)$ such that $e_M \in B(e_{N_1}, r) \cap B(e_{N_2}, r)$.

Then $e_M \in B(e_{N_1}, r)$, $e_M \in B(e_{N_2}, r)$ and so $d(e_M, e_{N_1}) < r$, $d(e_M, e_{N_2}) < r$. Now by $NSM4$, $d(e_{N_1}, e_{N_2}) \leq d(e_{N_1}, e_M) + d(e_M, e_{N_2}) < r + r = 2r \Rightarrow d(e_{N_1}, e_{N_2}) < 2r$. It is a contradiction to the fact that $d(e_{N_1}, e_{N_2}) = 2r$. So, $B(e_{N_1}, r) \cap B(e_{N_2}, r) = \phi$.

3.13 Theorem

Every finite neutrosophic soft subset of an NSMS is closed.

Proof. Let $(NS(U_E), d)$ be an NSMS and $M \subset NS(U_E)$. Then following cases arise.

(i) Let $M = \{e_M\}$ i.e., M is singleton and $e_N \in M^c$. Then $e_N \neq e_M$ and so $d(e_N, e_M) > 0$. Suppose $0 < r < d(e_N, e_M)$. Then there exists an open ball $B(e_N, r)$ which does not contain e_M i.e., $B(e_N, r) \cap M = \phi$. Hence $e_N \in M^c$ is not a limit point of M. Since e_N is arbitrary, so $D(M) = \phi \subset M$ i.e., M is closed.

(ii) If $M = \{e_{1M}, e_{2M}, \cdots, e_{nM}\}$ then $M = \{e_{1M}\} \cup \{e_{2M}\} \cup \cdots \cup \{e_{nM}\}$. Since each $\{e_{iM}\}, 1 \leq i \leq n$ is closed and arbitrary union of neutrosophic soft closed sets is closed with respect to the strict belongingness of neutrosophic soft point, thus M is closed.

This ends the theorem.

3.14 Theorem

A neutrosophic soft point e_N in an NSMS $(NS(U_E), d)$ is a limit point of an NSS $M \subset NS(U_E)$ iff every neighbourhood of e_N contains infinitely many neutrosophic soft points of M, provided E being an infinite parametric set.

Proof. First suppose that every neighbourhood of e_N contains infinitely many points of M. Then obviously every neighbourhood of e_N contains at least one point of M distinct from e_N. So e_N is a limit point of M. Next, let e_N be a limit point of M. Then for $r \in (0, 3]$ there is an open ball $B(e_N, r)$ such that $e_{1M} \in B(e_N, r) \cap M$ with $e_N \neq e_{1M}$. Let $r_1 = d(e_N, e_{1M})$. For that there exists another open ball $B(e_N, r_1)$ such that $e_{2M} \in B(e_N, r_1) \cap M$ with $e_N \neq e_{1M} \neq e_{2M}$. Proceeding in the manner, we have successively $r_k = d(e_N, e_{kM})$ with $e_{(k+1)M} \in B(e_N, r_k) \cap M$ with $e_N \neq e_{1M} \neq \cdots \neq e_{(k+1)M}$. Extending this process infinitely, there is infinite number of distinct neutrosophic soft points in M which are contained in the neighbourhood of e_N.

3.15 Definition

Let $(NS(U_E), d)$ be an NSMS and $M \subset NS(U_E)$. Then the distance between a neutrosophic soft point $e_N \in NS(U_E) - M$ and M is defined by :

$d(e_N, M) = \inf\{d(e_N, e_M) : e_M \in M\}$.

3.16 Definition

Let $(NS(U_E), d)$ be an NSMS. Then the diameter of $NS(U_E)$ is defined as :

$$\delta(NS(U_E)) = \sup\{d(e_{1N}, e_{2N}) : e_{1N}, e_{2N} \in NS(U_E)\}.$$

An NSS $M \subset NS(U_E)$ is bounded if it has a finite diameter i.e., if $d(e_{1M}, e_{2M}) \leq r$, for $r \in (0, 3]$ and $\forall e_{1M}, e_{2M} \in M$.

3.17 Theorem

Let $(NS(U_E), d)$ be an NSMS and $M \subset NS(U_E)$ is bounded. Then for each $e_N \in NS(U_E)$, there is a $r > 0$ such that $M \subset B(e_N, r)$. If M, P are bounded subsets of $NS(U_E)$, then $M \cup P$ is also bounded with respect to strict belongingness of a neutrosophic soft point.

Proof. For $e_M, e_{1M} \in M$,

$$d(e_N, e_M) \leq d(e_N, e_{1M}) + d(e_{1M}, e_M) \Rightarrow d(e_N, e_M) < d(e_N, e_{1M}) + \delta(M);$$

Since $\delta(M)$ is finite and fixed, let $d(e_N, e_{1M}) + \delta(M) = r$. Hence $d(e_N, e_M) < r$, $\forall e_M \in M \Rightarrow e_M \in B(e_N, r)$. Thus $M \subset B(e_N, r)$.

Next let $r_1, r_2 > 0$ such that $M \subset B(e_N, r_1)$ and $P \subset B(e_N, r_2)$. Suppose $r = \max\{r_1, r_2\}$ and $e_Q \in M \cup P$. If $e_Q \in M$ strictly, then $e_Q \in B(e_N, r_1) \Rightarrow d(e_N, e_Q) < r_1 \leq r$. If $e_Q \in P$ strictly, then $d(e_N, e_Q) < r_2 \leq r$. Thus $e_Q \in M \cup P \Rightarrow d(e_N, e_Q) \leq r \Rightarrow e_Q \in B[e_N, r]$. Hence $M \cup P \subset B[e_N, r]$ and so is bounded NSS in $(NS(U_E), d)$.

3.18 Definition

Let $(NS(U_E), d)$ be an NSMS having at least two neutrosophic soft points e_{1N}, e_{2N} such that $d(e_{1N}, e_{2N}) > 0$. Then $(NS(U_E), d)$ is called Hausdorff space if there exists two neutrosophic soft open balls $B(e_{1N}, t)$ and $B(e_{2N}, t)$ with center at e_{1N}, e_{2N} respectively and radius $t \in (0, 3]$ such that $B(e_{1N}, t) \cap B(e_{2N}, t) = \phi$.

3.19 Theorem

Every NSMS is Hausdorff.

Proof. Let $(NS(U_E), d)$ be an NSMS having two distinct neutrosophic soft points e_{1N}, e_{2N} such that $d(e_{1N}, e_{2N}) > 0$. Choose $t \in (0, 3]$ such that $0 < t < \frac{1}{2}d(e_{1N}, e_{2N})$. We consider two neutrosophic soft open balls $B(e_{1N}, t) = \{e'_N : d(e_{1N}, e'_N) < t\}$ and $B(e_{2N}, t) = \{e''_N : d(e_{2N}, e''_N) < t\}$. If possible $B(e_{1N}, t) \cap B(e_{2N}, t) \neq \phi$. Let $e_P \in B(e_{1N}, t) \cap B(e_{2N}, t)$. Then $e_P \in B(e_{1N}, t)$ and $e_P \in B(e_{2N}, t)$ i.e., $d(e_{1N}, e_P) < t$ and $d(e_{2N}, e_P) < t$. Then by

$NSM4$, $d(e_{1N}, e_{2N}) \leq d(e_{1N}, e_P) + d(e_P, e_{2N}) < t + t = 2t \Rightarrow t > \frac{1}{2}d(e_{1N}, e_{2N})$. This contradicts our assumption. Hence $B(e_{1N}, t) \cap B(e_{2N}, t) = \phi$ and so $(NS(U_E), d)$ is Hausdorff.

3.20 Definition

Let $(NS(U_E), d)$ and $(NS(V_E), d)$ be two NSMSs. Suppose $N_1 \subset NS(U_E)$ and $N_2 \subset NS(V_E)$ be two NSSs. Then their cartesian product is $N_1 \times N_2 = N_3$ where
$f_{N_3}(a, b) = f_{N_1}(a) \times f_{N_2}(b)$ for $(a, b) \in E \times E$. Analytically,

$f_{N_3}(a, b) = \{< (x, y), T_{f_{N_3}(a,b)}(x, y), I_{f_{N_3}(a,b)}(x, y), F_{f_{N_3}(a,b)}(x, y) >: (x, y) \in U \times V\}$ with

$$\begin{cases} T_{f_{N_3}(a,b)}(x, y) = T_{f_{N_1}(a)}(x) * T_{f_{N_2}(b)}(y) \\ I_{f_{N_3}(a,b)}(x, y) = I_{f_{N_1}(a)}(x) \diamond I_{f_{N_2}(b)}(y) \\ F_{f_{N_3}(a,b)}(x, y) = F_{f_{N_1}(a)}(x) \diamond F_{f_{N_2}(b)}(y). \end{cases}$$

This definition can be extended for more than two NSSs.

3.21 Theorem

Cartesian product of two neutrosophic soft Hausdorff metric spaces is Hausdorff.

Proof. For two Hausdorff NSMs $((NS(U_E), d)$ and $((NS(V_E), d)$, let (e_{1M}, e_{1N}) and (e_{2M}, e_{2N}) be two points in $NS(U_E) \times NS(V_E)$ such that $d((e_{1M}, e_{1N}), (e_{2M}, e_{1N})) > 0$. Then at least one of $e_{1M} \neq e_{2M}, \ e_{1N} \neq e_{2N}$ occurs.

Suppose $e_{1M} \neq e_{2M}$ holds. Since $((NS(U_E), d)$ is a neutrosophic soft Hausdorff metric space, so there exists two neutrosophic soft open balls $B(e_{1M}, t_1)$ and $B(e_{2M}, t_2)$ where $t_1, t_2 \in (0, 3]$ such that $0 < t_1, t_2 < \frac{1}{2}d(e_{1M}, e_{2M})$ and $B(e_{1M}, t_1) \cap B(e_{2M}, t_2) = \phi$. Since every metric space is metrizable, each $NS(U_E)$ and $NS(V_E)$ are open. So $B(e_{1M}, t_1) \times NS(V_E)$ and $B(e_{2M}, t_2) \times NS(V_E)$ are the neutrosophic soft open sets on $NS(U_E) \times NS(V_E)$. Hence, $(B(e_{1M}, t_1) \times (NS(V_E)) \cap (B(e_{2M}, t_2) \times (NS(V_E)) = \phi$ and this ends the theorem.

4 Sequence in Neutrosophic soft metric space

4.1 Definition

A sequence of neutrosophic soft points $\{e_{nN}\}$ in an NSMS $(NS(U_E), d)$ is said to converge in $(NS(U_E), d)$ if there exists a neutrosophic soft point $e_N \in NS(U_E)$ such that $d(e_{nN}, e_N) \to 0$ as $n \to \infty$ or $e_{nN} \to e_N$ as $n \to \infty$. Analytically, for every $\epsilon > 0$ there exists a natural number n_0 such that $d(e_{nN}, e_N) < \epsilon, \ \forall n \geq n_0$.

4.1.1 *Example*

Let $E = \mathbf{N}$ (the set of natural number) be the parametric set and $U = \mathbf{Z}$ (the set of all integers) be the universal set. Define a mapping $f_M : \mathbf{N} \to NS(\mathbf{Z})$ where, for any $n \in \mathbf{N}$ and $x \in \mathbf{Z}$,

$$T_{f_M(n)}(x) = \begin{cases} 0 & \text{if } x \text{ is odd} \\ \frac{1}{n} & \text{if } x \text{ is even.} \end{cases}$$

$$I_{f_M(n)}(x) = \begin{cases} \frac{1}{2n} & \text{if } x \text{ is odd} \\ 0 & \text{if } x \text{ is even.} \end{cases}$$

$$F_{f_M(n)}(x) = \begin{cases} 1 - \frac{1}{n} & \text{if } x \text{ is odd} \\ 0 & \text{if } x \text{ is even.} \end{cases}$$

The tabular representation of the above sequence is given in Table 3.

Table 3 : Tabular form of sequence $\{e_{nM}\}$.

	e_{1M}	e_{2M}	e_{3M}	...
odd integers	$(0, \frac{1}{2}, 0)$	$(0, \frac{1}{4}, \frac{1}{2})$	$(0, \frac{1}{6}, \frac{2}{3})$	...
even integers	(1,0,0)	$(\frac{1}{2}, 0, 0)$	$(\frac{1}{3}, 0, 0)$	...

Clearly, $\{e_{nM}\} \to (0, 0, 1)$ for odd integers and $\{e_{nM}\} \to (0, 0, 0)$ for even integers. Hence, $\{e_{nM}\}$ is divergent neutrosophic soft sequence over $(\mathbf{Z}, \mathbf{N})$.

Now, if we construct the falsity membership function of the above sequence in the following manners :

$$F^1_{f_M(n)}(x) = \begin{cases} \frac{1}{1+n} & \text{if } x \text{ is odd} \\ 0 & \text{if } x \text{ is even.} \end{cases}$$

$$F^2_{f_M(n)}(x) = \begin{cases} 1 - \frac{1}{n} & \text{if } x \text{ is odd} \\ \frac{n}{1+n} & \text{if } x \text{ is even.} \end{cases}$$

then in 1st case $\{e_{nM}\} \to (0, 0, 0)$ and in 2nd case $\{e_{nM}\} \to (0, 0, 1)$ for all integers. So in any case, $\{e_{nM}\}$ is a convergent neutrosophic soft sequence over $(\mathbf{Z}, \mathbf{N})$.

4.2 Theorem

The limit of a sequence of points in an NSMS is unique.

Proof. Let $\{e_{nN}\}$ be a sequence of points in an NSMS $(NS(U_E), d)$ such that $e_{nN} \to e_N$ and $e_{nN} \to e'_N$ as $n \to \infty$. Then for $\epsilon > 0$ (so small chosen) there exists natural numbers n_0, n'_0 such that

$$d(e_{nN}, e_N) < \tfrac{\epsilon}{2},\ \forall n \geq n_0 \quad \text{and} \quad d(e_{nN}, e'_N) < \tfrac{\epsilon}{2},\ \forall n \geq n'_0.$$

Let $N_0 = \max\{n_0, n'_0\}$. Then $d(e_N, e'_N) \leq d(e_N, e_{nN}) + d(e_{nN}, e'_N) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$,
$\forall n \geq N_0$. This shows that $e_N = e'_N$.

4.3 Definition

A sequence $\{e_{nN}\}$ of neutrosophic soft point in an NSMS $(NS(U_E), d)$ is said to be a Cauchy sequence if to every $\epsilon > 0$ there exists an $n_0 \in \mathbf{N}$ (set of natural numbers) such that $d(e_{mN}, e_{nN}) < \epsilon,\ \forall m, n \geq n_0$ i.e., $d(e_{mN}, e_{nN}) \to 0$ as $m, n \to \infty$.

4.3.1 *Example*

Consider the Example (3) of 3.4.1 and the distance function defined in (1) of 3.1.1 for $k = 1$; Then,

$$
\begin{aligned}
& d(e_{mN}, e_{nN}) \\
= \;& \min_{x_i}\{|T_{e_{mN}}(x_i) - T_{e_{nN}}(x_i)| + |I_{e_{mN}}(x_i) - I_{e_{nN}}(x_i)| + |F_{e_{mN}}(x_i) - F_{e_{nN}}(x_i)|\} \\
= \;& |T_{e_{mN}}(x) - T_{e_{nN}}(x)| + |I_{e_{mN}}(x) - I_{e_{nN}}(x)| + |F_{e_{mN}}(x) - F_{e_{nN}}(x)| \\
= \;& |\frac{1}{m} - \frac{1}{n}| + |\frac{1}{2m} - \frac{1}{2n}| + |(1 - \frac{1}{m}) - (1 - \frac{1}{n})| \\
= \;& |\frac{1}{m} - \frac{1}{n}| + |\frac{1}{2m} - \frac{1}{2n}| + |\frac{1}{m} - \frac{1}{n}| \\
\to \;& 0 + 0 + 0 = 0; \quad \text{as } m, n \to \infty
\end{aligned}
$$

Hence, $\{e_{nN}\}$ defined in (3) of 3.4.1 is a Cauchy sequence.

4.4 Theorem

Every neutrosophic soft convergent sequence in an NSMS is a Cauchy sequence.

Proof. Let $\{e_{nN}\}$ be a neutrosophic soft convergent sequence in an NSMS $(NS(U_E), d)$ and converges to a neutrosophic soft point e_N. Then for $\epsilon > 0$ there exists $n_0 \in \mathbf{N}$ (set of natural numbers) such that $d(e_{nN}, e_N) < \frac{\epsilon}{2},\ \forall n \geq n_0$.

Now, $d(e_{mN}, e_{nN}) \leq d(e_{mN}, e_N) + d(e_N, e_{nN}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\ \forall m, n \geq n_0$.

Hence, $\{e_{nN}\}$ is a Cauchy sequence.

4.4.1 *Note*

Converse of the above theorem may not be true.

Take the Example 4.3.1; Let $e_{kN} = (\frac{1}{k}, \frac{1}{2k}, 1 - \frac{1}{k})\ \forall x \in \mathbf{Z}$. Now,

$$\begin{aligned}
& d(e_{nN}, e_{kN}) \\
= \;& |T_{e_{nN}}(x) - T_{e_{kN}}(x)| + |I_{e_{nN}}(x) - I_{e_{kN}}(x)| + |F_{e_{nN}}(x) - F_{e_{kN}}(x)| \\
= \;& |\frac{1}{n} - \frac{1}{k}| + |\frac{1}{2n} - \frac{1}{2k}| + |(1 - \frac{1}{n}) - (1 - \frac{1}{k})| \\
= \;& |\frac{1}{n} - \frac{1}{k}| + |\frac{1}{2n} - \frac{1}{2k}| + |\frac{1}{n} - \frac{1}{k}| \\
= \;& |\frac{1}{n} - \frac{1}{k}| + \frac{1}{2}|\frac{1}{n} - \frac{1}{k}| + |\frac{1}{n} - \frac{1}{k}| \\
= \;& \frac{5}{2}|\frac{1}{n} - \frac{1}{k}| \\
\to \;& \frac{5}{2k} \neq 0; \quad \text{as } n \to \infty
\end{aligned}$$

Thus, the Cauchy sequence $\{e_{nN}\}$ is not convergent.

4.5 Definition

An NSMS $(NS(U_E), d)$ is said to be complete if every Cauchy sequence in $(NS(U_E), d)$ converges to a neutrosophic soft point of $NS(U_E)$.

4.5.1 Example

Let $U = \{x_1, x_2, x_3, \cdots \infty\} \subset \mathbf{R}$ and $E = \mathbf{N}$. Then, an $NS(U_E)$ having the soft points in a sequence is given by the Table 4.

Table 4 : Tabular form of NSS M.

	e_{1M}	e_{2M}	e_{3M}	$\cdots$
x_1	$(T_1(x_1), I_1(x_1), F_1(x_1))$	$(T_2(x_1), I_2(x_1), F_2(x_1))$	$(T_3(x_1), I_3(x_1), F_3(x_1))$	$\cdots$
x_2	$(T_1(x_2), I_1(x_2), F_1(x_2))$	$(T_2(x_2), I_2(x_2), F_2(x_2))$	$(T_3(x_2), I_3(x_2), F_3(x_2))$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Consider a Cauchy sequence $\{e_{nM}\}$ of neutrosophic soft points in the NSMS $(NS(U_E), d)$ with respect to 'd ' as defined in (1) of 3.1.1 for $k = 2$;

Then for arbitrary $\epsilon > 0$, there exists a natural number n_0 such that

$$
\begin{aligned}
& d(e_{mM}, e_{nM}) < \frac{\epsilon}{3} \text{ if } m, n \geq n_0 \\
\Rightarrow \quad & \min_{x_i}\{\sqrt{(|T_{e_{mM}}(x_i) - T_{e_{nM}}(x_i)|^2 + |I_{e_{mM}}(x_i) - I_{e_{nM}}(x_i)|^2 + |F_{e_{mM}}(x_i) - F_{e_{nM}}(x_i)|^2)}\} < \frac{\epsilon}{3} \\
\Rightarrow \quad & \sqrt{(|T_{e_{mM}}(x_k) - T_{e_{nM}}(x_k)|^2 + |I_{e_{mM}}(x_k) - I_{e_{nM}}(x_k)|^2 + |F_{e_{mM}}(x_k) - F_{e_{nM}}(x_k)|^2)} < \frac{\epsilon}{3}, \\
& (\text{ for } i = k, \text{ say }) \\
\Rightarrow \quad & |T_{e_{mM}}(x_k) - T_{e_{nM}}(x_k)|^2 + |I_{e_{mM}}(x_k) - I_{e_{nM}}(x_k)|^2 + |F_{e_{mM}}(x_k) - F_{e_{nM}}(x_k)|^2 < \frac{\epsilon^2}{9} \\
& \text{Since each term in L.H.S are positive, so} \\
\Rightarrow \quad & |T_{e_{mM}}(x_k) - T_{e_{nM}}(x_k)|^2 < \frac{\epsilon^2}{9}, |I_{e_{mM}}(x_k) - I_{e_{nM}}(x_k)|^2 < \frac{\epsilon^2}{9}, |F_{e_{mM}}(x_k) - F_{e_{nM}}(x_k)|^2 < \frac{\epsilon^2}{9} \\
\Rightarrow \quad & |T_{e_{mM}}(x_k) - T_{e_{nM}}(x_k)| < \frac{\epsilon}{3}, |I_{e_{mM}}(x_k) - I_{e_{nM}}(x_k)| < \frac{\epsilon}{3}, |F_{e_{mM}}(x_k) - F_{e_{nM}}(x_k)| < \frac{\epsilon}{3}
\end{aligned}
$$

This shows that for each fixed x_i $(i = 1, 2, 3, \cdots)$, each of the sequences $\{T_{e_{nM}}(x_i)\}$, $\{I_{e_{nM}}(x_i)\}$ and $\{F_{e_{nM}}(x_i)\}$ satisfies Cauchy's criterion for real number sequence. Hence, each sequence is convergent and converges to $T_{e_M}(x_i), I_{e_M}(x_i), F_{e_M}(x_i)$, (say) respectively. Clearly, $e_M = \{< x_i, (T_{e_M}(x_i), I_{e_M}(x_i), F_{e_M}(x_i)) >: x_i \in \mathbf{R}\} \in NS(U_E)$. Now,

$$
\begin{aligned}
& d(e_{nM}, e_M) \\
= \quad & \min_{x_i}\{\sqrt{(|T_{e_{nM}}(x_i) - T_{e_M}(x_i)|^2 + |I_{e_{nM}}(x_i) - I_{e_M}(x_i)|^2 + |F_{e_{nM}}(x_i) - F_{e_M}(x_i)|^2)}\} \\
\leq \quad & \min_{x_i}\{\sqrt{|T_{e_{nM}}(x_i) - T_{e_M}(x_i)|^2} + \sqrt{|I_{e_{nM}}(x_i) - I_{e_M}(x_i)|^2} + \sqrt{|F_{e_{nM}}(x_i) - F_{e_M}(x_i)|^2}\} \\
& (\text{ by Minkowski inequality for sum }) \\
= \quad & \min_{x_i}\{|T_{e_{nM}}(x_i) - T_{e_M}(x_i)| + |I_{e_{nM}}(x_i) - I_{e_M}(x_i)| + |F_{e_{nM}}(x_i) - F_{e_M}(x_i)|\} \\
= \quad & |T_{e_{nM}}(x_k) - T_{e_M}(x_k)| + |I_{e_{nM}}(x_k) - I_{e_M}(x_k)| + |F_{e_{nM}}(x_k) - F_{e_M}(x_k)| \\
& (\text{ for } i = k, \text{ say }) \\
< \quad & \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \text{ if } n \geq n_0
\end{aligned}
$$

Thus $\{e_{nM}\} \to e_M \in NS(U_E)$ as $n \to \infty$ and so $(NS(U_E), d)$ is a complete NSMS.

4.6 Theorem

An NSMS $(NS(U_E), d)$ is complete if every Cauchy sequence in $(NS(U_E), d)$ has a convergent subsequence.

Proof. Let $\{e_{n_kN}\}$ be a subsequence of a Cauchy sequence $\{e_{nN}\}$ in $(NS(U_E), d)$. It is necessary to show that if $\{e_{n_kN}\}$ converges to a neutrosophic soft point e_N then $\{e_{nN}\}$ itself converges to e_N.

Since $\{e_{nN}\}$ is Cauchy so for $\epsilon > 0$ there exists $n_0 \in \mathbf{N}$ (set of natural numbers) such that $d(e_{mN}, e_{nN}) < \frac{\epsilon}{2}$, $\forall m, n \geq n_0$. Then $d(e_{n_kN}, e_N) < \frac{\epsilon}{2}$, $\forall n_k \geq n_0$.

Now $d(e_{nN}, e_N) \leq d(e_{nN}, e_{n_kN}) + d(e_{n_kN}, e_N) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, $\forall n \geq n_0$ and this completes the theorem.

4.7 Theorem

Every closed subset of a complete NSS in an NSMS is complete.

Proof. Let M be a complete NSS in an NSMS $(NS(U_E), d)$ and P be a closed subset of M. Suppose $\{e_{nP}\}$ be a Cauchy sequence in P. Since $P \subset M$ and $\{e_{nP}\} \in P$, so $\{e_{nP}\} \in M$. But as M is complete, so $\{e_{nP}\} \to e_M \in M$, say. Now since P is closed and limit of a sequence of point in $(NS(U_E), d)$ is unique, then $e_M \in P$, too. Hence P is complete in $(NS(U_E), d)$.

4.8 Theorem

Let $(NS(U_E), d)$ be an NSMS and τ_u denote the set of all neutrosophic soft open sets in $NS(U_E)$. Then τ_u has the following properties.
(i) $\phi_u,\ 1_u \in \tau_u$.
(ii) $N_1, N_2 \in \tau_u \Rightarrow N_1 \cap N_2 \in \tau_u$.
(iii) $\{N_i : i \in \Gamma\} \in \tau_u \Rightarrow \cup_{i\in\Gamma} N_i \in \tau_u$.

This τ_u is called the neutrosophic soft topology determined by the neutrosophic soft metric d.

Proof. (i) By the definition of absolute neutrosophic soft set (1_u), null neutrosophic soft set (ϕ_u) in 2.6 and by the definition of neutrosophic soft open ball $B(e_N, t),\ t \in (0, 3]$ for $e_N \in 1_u$, the first property is obvious.

The other two properties follow from Theorems 3.8 and 3.11;

5 Conclusion

The theoretical point of view of neutrosophic soft metric space in terms of neutrosophic soft points has been discussed and illustrated with suitable examples in the present paper. The notion of convergence of a neutrosophic soft sequence and the complete NSMS have been proposed here. Some related theorems have been developed also.

References

[1] L. A. Zadeh, Fuzzy sets, information and control, 8, 338-353, (1965).

[2] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy sets and systems, 20(1), 87-96, (1986).

[3] D. Molodtsov, Soft set theory- First results, Computer and Mathematics with Applications, 37(4-5), 19-31, (1999).

[4] P. K. Maji, R. Biswas and A. R. Roy, Fuzzy soft sets, The journal of fuzzy mathematics, 9(3), 589-602, (2001).

[5] P. K. Maji, R. Biswas and A. R. Roy, Intuitionistic fuzzy soft sets, The journal of fuzzy mathematics, 9(3), 677-692, (2001).

[6] P. K. Maji, R. Biswas and A. R. Roy, On intuitionistic fuzzy soft sets, The journal of fuzzy mathematics, 12(3), 669-683, (2004).

[7] O. Kalseva and S. Seikkala, On fuzzzy metric spaces, Fuzzy sets and systems, 12, 215-229, (1984).

[8] I. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetika, 11, 326-334, (1975).

[9] M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy sets and systems, 27, 385-389, (1989).

[10] A. George and P. Veeramani, On some results in fuzzy metric spaces, Fuzzy sets and systems, 64, 395-399, (1994).

[11] A. George and P. Veeramani, Some theorems in fuzzy metric spaces, J. Fuzzy Math., 3, 933-940, (1995).

[12] A. George and P. Veeramani, On some results of analysis for fuzzy metric spaces, Fuzzy sets and systems, 90, 365-368, (1997).

[13] K. Chakrabarty, R. Biswas and S. Nanda, On fuzzy metric spaces, Fuzzy sets and systems, 99, 111-114, (1998).

[14] V. Gregori and S. Romaguera, Some properties of fuzzy metric spaces, Fuzzy sets and systems, 115, 485-489, (2000).

[15] V. Gregori and S. Romaguera, On completion of fuzzy metric spaces, Fuzzy sets and systems, 130, 399-404, (2002).

[16] J. Rodrigues-Lopez, S. Ramaguera, The Hausdorff fuzzy metric on compact sets, Fuzzy sets and systems, 147, 273-283, (2004).

[17] G. A. Afrouzi, S. Shakeri and S. H. Rasouli, On the fuzzy metric spaces, J. Math. and Comput. Sc., 2(3), 475-482, (2011).

[18] C. Chang, Fuzzy topological spaces, J. Math. Anal. Appl., 24, 182-190, (1968).

[19] S. Roy and T. K. Samanta, A note on fuzzy soft topological spaces, Ann. Fuzzy Math. Inform., 3(2), 305-311, (2012).

[20] J. H. Park, Intuitionistic fuzzy metric spaces, Chaos, Solitons and Fractals, 22(5), 1039-1046, (2004).

[21] C. Alaca, D. Turkoglu and C. Yildiz, Fixed points in intuitionistic fuzzy metric spaces, Chaos, Solitons and Fractals, 29(5), 1073-1078, (2006).

[22] T. Beaula and C. Gunaseeli, On fuzzy soft metric spaces, Malaya J. Mat., 2(3), 197-202, (2014).

[23] T. Beaula and R. Raja, Completeness in fuzzy soft metric spaces, Malaya J. Mat., S(2), 438-442, (2015).

[24] M. I. Yazar, C. Gunduz (Aras) and S. Bayramov, Fixed point theorems of soft contractive mappings, Filomat 30 (2), 269-279, (2016).

[25] C. Gunduz (Aras), M. I. Yazar and S. Bayramov, Some notes on compact sets in soft metric spaces, AFMI, accepted (March, 2017).

[26] M. I. Yazar, T. Bilgin, S. Bayramov and C. Gunduz (Aras), A new view on soft normed spaces, Computer Networks, International Mathematical Forum 9 (24), 1149-1159, (2014).

[27] M. I. Yazar, On soft functionals and soft duality, AIP Conference Proceedings, Vol (1726), issue 1, (April, 2016), id. 020036, http://doi.org/10.1063/1.4945862.

[28] F. Smarandache, Neutrosophic set, a generalisation of the intuitionistic fuzzy sets, Inter. J. Pure Appl. Math., 24, 287-297, (2005).

[29] F. Smarandache, Neutrosophy, Neutrosophic Probability, Set and Logic, Amer. Res. Press, Rehoboth, USA., (1998), p. 105, http://fs.gallup.unm.edu/eBook-neutrosophics4.pdf (fourth version).

[30] P. K. Maji, Neutrosophic soft set, Annals of Fuzzy Mathematics and Informatics, 5(1), 157-168, (2013).

[31] I. Deli and S. Broumi, Neutrosophic soft relations and some properties, Annals of Fuzzy Mathematics and Informatics, 9(1), 169-182, (2015).

[32] S. Broumi and F. Smarandache, Intuitionistic neutrosophic soft set, Journal of Information and Computing Science, 8(2), 130-140, (2013).

[33] I. Deli and S. Broumi, Neutrosophic Soft Matrices and NSM-decision Making, Journal of Intelligent and Fuzzy Systems, 28(5), 2233-2241, (2015).

[34] T. Bera and N. K. Mahapatra, Introduction to neutrosophic soft groups, Neutrosophic Sets and Systems, 13, (2016), 118-127, doi.org/10.5281/zenodo.570845.

[35] T. Bera and N. K. Mahapatra, On neutrosophic normal soft groups, Int. J. Appl. Comput. Math., 2(4), (2016), DOI 10.1007/s40819-016-0284-2.

[36] T. Bera and N. K. Mahapatra, On neutrosophic soft rings, OPSEARCH, 1-25, (2016), DOI 10.1007/s12597-016-0273-6 .

[37] T. Bera and N. K. Mahapatra, (α, β, γ)-cut of neutrosophic soft set and it's application to neutrosophic soft groups, Asian Journal of Math. and Compt. Research, 12(3), 160-178, (2016).

[38] T. Bera and N. K. Mahapatra, Introduction to neutrosophic soft topological space, OPSEARCH, (March, 2017), DOI 10.1007/s12597-017-0308-7.