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On single valued neutrosophic refined rough set model and its application

Yan-Ling Bao and Hai-Long Yang*
College of Mathematics and Information Science, Shaanxi Normal University, Xi'an, P.R. China

Abstract. Neutrosophic set (NS) theory was originally established by Smarandache for handling indeterminate and inconsistent information. In this paper, we introduce single valued neutrosophic refined rough sets by combining single valued neutrosophic refined sets with rough sets and further study the hybrid model from two perspectives—constructive viewpoint and axiomatic viewpoint. We also give single valued neutrosophic refined rough sets on two universes and an available algorithm for handling multi-attribute decision making problem based on single valued neutrosophic refined rough sets on two universes. In addition, we illustrate the validity of the single valued neutrosophic refined rough set model by an example.

Keywords: Neutrosophic sets, single valued neutrosophic refined sets, rough sets, single valued neutrosophic refined rough sets, multi-attribute decision making

1. Introduction

To resolve indeterminate and inconsistent information, Smarandache [1, 2] initiated neutrosophic sets (NSs) by combining non-standard analysis and tri-component sets. A neutrosophic set consists of three membership functions (truth-membership function T, indeterminacy-membership function I and falsity-membership function F) whose range is the nonstandard unit interval $]0^-, 1^+[$. In a neutrosophic set, indeterminacy is expressed explicitly, and the three membership functions are independent of each other.

Since the neutrosophic set theory established, many scholars have flung themselves into its development [3–10]. Rivieccio [11] proposed neutrosophic logics by introducing neutrosophic idea to logic. Neutrosophic vague soft expert sets as well as their basic operations were defined by Al-Quran and Hassan [12]. Deli and Broumi [13] presented neutrosophic soft matrix and its operators in a novel neutrosophic soft set theory. In order to conveniently employ neutrosophic sets in real problems, Wang et al. [14] put forward interval neutrosophic sets (INSs) by simplying neutrosophic sets. Zhang et al. [15] studied properties of INSs and their application in multicriteria decision making problems. Ye [16] proposed correlation coefficient of INSs and further applied it to interval neutrosophic decision-making problems. Subsequently, Wang et al. [17] raised single valued neutrosophic sets (SVNSs). Yang et al. [18] discussed single valued neutrosophic relations (SVNRs) and explored their properties in detail. In order to describe more accurate information, Ye [19] introduced single valued neutrosophic refined sets in which the three neutrosophic components T, I, F are refined (divided) into $T_1, T_2 \cdots, T_p, I_1, I_2 \cdots, I_p$ and $F_1, F_2 \cdots, F_p$, respectively. Later on, Ye et al. [20] presented distance and similarity measure of single valued neutrosophic refined sets and applied the measure to medical diagnosis problems. Until now, the research about single valued neutrosophic refined sets is still insufficient.

To deal with imprecise information, Pawlak [21, 22] initiated rough set theory which has been successfully applied to many fields. Since established,

*Corresponding author. Hai-Long Yang, College of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710119, P.R. China. Tel.: +86 13572106227; Fax: +86 2985310277; E-mail: yanghailong@snnu.edu.cn.

the theory has attracted the attention of many researchers [23–27]. Yao [28] proposed arbitrary binary relation-based rough sets by extending equivalence relations to arbitrary binary relations. Zakowski [29] put forward concept covering-based rough sets. Later on, Dubois and Prade [30] combined rough sets with fuzzy sets and further proposed fuzzy rough sets and rough fuzzy sets. Cornelis et al. [31] studied intuitionistic fuzzy rough sets. Yao [32] systematically investigated axiomatic characterizations of crisp rough sets. The axiomatic characterizations of fuzzy rough sets were studied by Mi and Zhang [33]. Wu et al. [34] explored axiomatic characterizations of (S, T)-fuzzy rough sets based on a triangular norm T and a conorm S. Zhou and Hu [35] studied axiomatic characterizations of rough approximation operators on complete completely distributive lattices.

Both neutrosophic sets and rough sets play important role in handling imprecise information. In the past few years, many researchers have focused their attention on combining neutrosophic sets with rough sets. Salama and Broumi [36] investigated the roughness of neutrosophic sets. Broumi and Smarandache put forward rough neutrosophic sets [37] as well as interval neutrosophic rough sets [38]. Yang et al. [39] proposed single valued neutrosophic rough sets which is a hybrid model of single valued neutrosophic sets and rough sets. So far, the study on single valued neutrosophic refined rough sets is still vacant. In this paper, we will introduce single valued neutrosophic refined rough sets and explore the model from both constructive and axiomatic approaches. Furthermore, We will apply this novel model to multi-attribute decision making problems.

The paper proceeds as follows. In Section 2, we briefly recall some basic definitions and operations related to single valued neutrosophic refined sets. In Section 3, we propose single valued neutrosophic refined rough sets and study its properties in detail. Moreover, we investigate connections between special single valued neutrosophic refined relations and single valued neutrosophic refined lower (upper) approximation operators. In Section 4, the axiomatic characterizations of the proposed single valued neutrosophic refined approximation operators are systematically explored. In Section 5, we introduce single valued neutrosophic refined rough sets on two universes as well as an algorithm for handling multi-attribute decision making problem. Furthermore, we demonstrate the feasibility of the single valued neutrosophic refined rough set model with a medical diagnosis example. The last section draws the conclusion of the paper.

2. Preliminaries

In this section, we briefly retrospect some basic definitions which will be used in the paper.

2.1. SVNSs and SVNRSs

Definition 2.1. [17] Let U be a space of points (objects), with a generic element in U denoted by x. A SVNS A in U is characterized by a truth-membership function T_A, an indeterminacy-membership function I_A, and a falsity-membership function F_A, where $\forall x \in U$, $T_A(x), I_A(x), F_A(x) \in [0, 1]$ and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$. A SVNS A can be expressed as $A = \{\langle x, T_A(x), I_A(x), F_A(x) | x \in U\}$ or $A = (T_A, I_A, F_A)$. $\forall x \in U, A(x) = (T_A(x), I_A(x), F_A(x))$.

Definition 2.2. [19] Let U be a space of points (objects), with a generic element in U denoted by x. A single valued neutrosophic refined set (SVNRS) A in U is characterized by three membership functions: a truth-membership function T_A, an indeterminacy-membership function I_A, and a falsity-membership function F_A as follows:

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in U\},$$

where $T_A(x) = \{T_{1A}(x), T_{2A}(x), \cdots, T_{pA}(x)\}$, $I_A(x) = \{I_{1A}(x), I_{2A}(x), \cdots, I_{pA}(x)\}$, $F_A(x) = \{F_{1A}(x), F_{2A}(x), \cdots, F_{pA}(x)\}$, p is a positive integer, $T_{iA}(x), I_{iA}(x), F_{iA}(x) \in [0, 1]$ and $0 \leq T_{iA}(x) + I_{iA}(x) + F_{iA}(x) \leq 3$ for $i = 1, 2, \cdots, p$. Also, p is referred to as the dimension of A and $\langle T_A(x), I_A(x), F_A(x)\rangle$ is termed as a single valued neutrosophic refined element of A.

Remark 2.1. (1) In [19], Ye called the above neutrosophic set as a single valued neutrosophic multiset rather than the single valued neutrosophic refined set. In accordance with [40], we call it the single valued neutrosophic refined set in the present paper.

(2) A SVNRS is a generalization of set single valued neutrosophic set. In fact, when $p = 1$ in a SVNRS, then the SVNRS will degenerate into a SVNS.

Let U be a space of points (objects), then the family of all single valued neutrosophic refined sets in U is denoted by SVNRS(U). For convenience, we take SVNRS_p to represent a p-dimension single valued neutrosophic refined set and $\text{SVNRS}_p(U)$ to represent the family of all SVNRS_p in U. Moreover, $\forall A \in \text{SVNRS}_p(U)$,

• A is referred to as an empty single valued neutrosophic refined set if and only if $T_{iA}(x) = 0,\ I_{iA}(x) = F_{iA}(x) = 1\ \ (i = 1, 2, \cdots, p)$ for all $x \in U$, the p-dimension empty single valued neutrosophic refined set is denoted by $\emptyset_p$.

• A is referred to as a full single valued neutrosophic refined set if and only if $T_{iA}(x) = 1,\ I_{iA}(x) = F_{iA}(x) = 0\ \ (i = 1, 2, \cdots, p)$ for all $x \in U$, the p-dimension full single valued neutrosophic refined set is denoted by U_p.

• A is referred to as a p-dimension constant single valued neutrosophic refined set if $T_{iA}(x) = a_i, I_{iA}(x) = b_i, F_{iA}(x) = c_i\ \ (i = 1, 2, \cdots, p)$ for all $x \in U$. Let $\alpha = \{a_1, a_2, \cdots, a_p\}, \beta = \{b_1, b_2, \cdots, b_p\}, \gamma = \{c_1, c_2, \cdots, c_p\}$, then the constant single valued neutrosophic refined set is denoted by $\widehat{\alpha, \beta, \gamma}$. Obviously, both $\emptyset_p$ and U_p are special p-dimension constant single valued neutrosophic refined sets.

Definition 2.3. [19] Let U be a space of points (objects). $\forall A, B \in \text{SVNRS}_p(U)$, then

(1) The complement of A is denoted by A^c and defined as:

$$A^c = \{\langle x, T_{A^c}(x), I_{A^c}(x), F_{A^c}(x)\rangle | x \in U\},$$

where

$$\begin{aligned} T_{A^c}(x) &= F_A(x) = \{F_{1A}(x), F_{2A}(x), \cdots, F_{pA}(x)\}, \\ I_{A^c}(x) &= \sim I_A(x) = \{1 - I_{1A}(x), 1 - I_{2A}(x), \\ &\quad \cdots, 1 - I_{pA}(x)\}, \\ F_{A^c}(x) &= T_A(x) = \{T_{1A}(x), T_{2A}(x), \cdots, T_{pA}(x)\}, \end{aligned}$$

(2) The intersection of A and B is denoted by $A \sqcap B$ and defined as:

$$A \sqcap B = \{\langle x, T_{A\sqcap B}(x), I_{A\sqcap B}(x), F_{A\sqcap B}(x)\rangle | x \in U\},$$

where

$$\begin{aligned} T_{A\sqcap B}(x) &= T_A(x)\widetilde{\wedge}T_B(x) = \{T_{1A}(x) \wedge T_{1B}(x), \\ &\quad T_{2A}(x) \wedge T_{2B}(x), \cdots, T_{pA}(x) \wedge T_{pB}(x)\}, \\ I_{A\sqcap B}(x) &= I_A(x)\widetilde{\vee}I_B(x) = \{I_{1A}(x) \vee I_{1B}(x), \\ &\quad I_{2A}(x) \vee I_{2B}(x), \cdots, I_{pA}(x) \vee I_{pB}(x)\}, \\ F_{A\sqcap B}(x) &= F_A(x)\widetilde{\vee}F_B(x) = \{F_{1A}(x) \vee F_{1B}(x), \\ &\quad F_{2A}(x) \vee F_{2B}(x), \cdots, F_{pA}(x) \vee F_{pB}(x)\}, \end{aligned}$$

(3) The union of A and B is denoted by $A \sqcup B$ and defined as:

$$A \sqcup B = \{\langle x, T_{A\sqcup B}(x), I_{A\sqcup B}(x), F_{A\sqcup B}(x)\rangle | x \in U\},$$

where

$$\begin{aligned} T_{A\sqcup B}(x) &= T_A(x)\widetilde{\vee}T_B(x), \\ I_{A\sqcup B}(x) &= I_A(x)\widetilde{\wedge}I_B(x), \\ F_{A\sqcup B}(x) &= F_A(x)\widetilde{\wedge}F_B(x). \end{aligned}$$

For any $y \in U$, a SVNRS_p 1_y and its complement $1_{U-\{y\}}$ are given as follows: $\forall x \in U$,

$$T_{1_y}(x) = \begin{cases} \{1, 1, \cdots, 1\}, & x = y \\ \{0, 0, \cdots, 0\}, & x \neq y \end{cases},$$

$$I_{1_y}(x) = F_{1_y}(x) = \begin{cases} \{0, 0, \cdots, 0\}, & x = y \\ \{1, 1, \cdots, 1\}, & x \neq y \end{cases};$$

$$T_{1_{U-\{y\}}}(x) = \begin{cases} \{0, 0, \cdots, 0\}, & x = y \\ \{1, 1, \cdots, 1\}, & x \neq y \end{cases},$$

$$I_{1_{U-\{y\}}}(x) = F_{1_{U-\{y\}}}(x) = \begin{cases} \{1, 1, \cdots, 1\}, & x = y \\ \{0, 0, \cdots, 0\}, & x \neq y \end{cases}.$$

Definition 2.4. [19] Let A, B be two p-dimension SVNRSs in U. If for any $x \in U$, $T_A(x) \prec T_B(x), I_B(x) \prec I_A(x), F_B(x) \prec F_A(x)$, i.e. $T_{iA}(x) \leq T_{iB}(x), I_{iB} \leq I_{iA}(x), F_{iB}(x) \leq F_{iA}(x)$ for all $i = 1, 2, \cdots, p$, then we say A is contained in B, denoted by $A \sqsubset B$.

Proposition 2.1. *Let A and B be two p-dimension SVNRSs in U, the following properties can be obtained:*

(1) Idempotency: $A \sqcap A = A,\ A \sqcup A = A$;
(2) Commutativity: $A \sqcap B = B \sqcap A,\ \ A \sqcup B = B \sqcup A$;
(3) Associativity: $A \sqcap (B \sqcap C) = (A \sqcap B) \sqcap C$, $A \sqcup (B \sqcup C) = (A \sqcup B) \sqcup C$;
(4) Distributivity: $A \sqcap (B \sqcup C) = (A \sqcap B) \sqcup (A \sqcap C),\ A \sqcup (B \sqcap C) = (A \sqcup B) \sqcap (A \sqcup C)$;
(5) De Morgan's laws: $(A \sqcap B)^c = A^c \sqcup B^c$, $(A \sqcup B)^c = A^c \sqcap B^c$;
(6) Double negation law: $(A^c)^c = A$.

Proof. The results are straightforward from Definition 2.3. □

2.2. Pawlak rough sets and single valued neutrosophic rough sets

Definition 2.5. [21, 22] Let $\mathcal{R}$ be an equivalence relation on a non-empty finite universe U. Then the pair $(U, \mathcal{R})$ is referred to as a Pawlak approximation space. $\forall X \subseteq U$, the lower and upper approximations of X with respect to $(U, \mathcal{R})$ are defined as follows:

$$\underline{\mathcal{R}}(X) = \{x \in U \mid [x]_{\mathcal{R}} \subseteq X\},$$

$$\overline{\mathcal{R}}(X) = \{x \in U \mid [x]_{\mathcal{R}} \cap X \neq \emptyset\},$$

where $[x]_{\mathcal{R}} = \{y \in U \mid (x, y) \in \mathcal{R}\}$. The pair $(\underline{\mathcal{R}}(X), \overline{\mathcal{R}}(X))$ is called a Pawlak rough set. $\underline{\mathcal{R}}$ and $\overline{\mathcal{R}}$ are called lower and upper approximation operators, respectively.

A SVNS $\mathcal{R}$ in $U \times U$ is referred to as a single valued neutrosophic relation (SVNR) in U, denoted by $\mathcal{R} = \{\langle (x, y), T_{\mathcal{R}}(x, y), I_{\mathcal{R}}(x, y), F_{\mathcal{R}}(x, y)\rangle \mid (x, y) \in U \times U\}$, where $T_{\mathcal{R}} : U \times U \longrightarrow [0, 1]$, $I_{\mathcal{R}} : U \times U \longrightarrow [0, 1]$ and $F_{\mathcal{R}} : U \times U \longrightarrow [0, 1]$ represent the truth-membership function, indeterminacy-membership function and falsity-membership function of $\mathcal{R}$, respectively.

Based on a SVNR, Yang et al. [39] gave the notion of single valued neutrosophic rough set as follows.

Definition 2.6. [39] Let $\mathcal{R}$ be a SVNR in U, the tuple $(U, \mathcal{R})$ is called a single valued neutrosophic approximation space. $\forall A \in \mathrm{SVNS}(U)$, the lower and upper approximations of A with respect to $(U, \mathcal{R})$, denoted by $\underline{\mathcal{R}}(A)$ and $\overline{\mathcal{R}}(A)$, are two SVNSs whose membership functions are defined as: $\forall x \in U$,

$$T_{\underline{\mathcal{R}}(A)}(x) = \bigwedge_{y \in U} (F_{\mathcal{R}}(x, y) \vee T_A(y)),$$

$$I_{\underline{\mathcal{R}}(A)}(x) = \bigvee_{y \in U} ((1 - I_{\mathcal{R}}(x, y)) \wedge I_A(y)),$$

$$F_{\underline{\mathcal{R}}(A)}(x) = \bigvee_{y \in U} (T_{\mathcal{R}}(x, y) \wedge F_A(y));$$

$$T_{\overline{\mathcal{R}}(A)}(x) = \bigvee_{y \in U} (T_{\mathcal{R}}(x, y) \wedge T_A(y)),$$

$$I_{\overline{\mathcal{R}}(A)}(x) = \bigwedge_{y \in U} (I_{\mathcal{R}}(x, y) \vee I_A(y)),$$

$$F_{\overline{\mathcal{R}}(A)}(x) = \bigwedge_{y \in U} (F_{\mathcal{R}}(x, y) \vee F_A(y)).$$

The pair $(\underline{\mathcal{R}}(A), \overline{\mathcal{R}}(A))$ is called a single valued neutrosophic rough set of A with respect to $(U, \mathcal{R})$. $\underline{\mathcal{R}}$ and $\overline{\mathcal{R}}$ are referred to as single valued neutrosophic lower and upper approximation operators, respectively.

3. The constructive approach of single valued neutrosophic refined rough sets

3.1. The notion of single valued neutrosophic refined rough sets

Ye [19] presented single valued neutrosophic refined sets as a generalization of single valued neutrosophic sets. In this subsection, we will introduce single valued neutrosophic refined relations and single valued neutrosophic refined rough sets to extend the notions and results in [18, 39].

Definition 3.1. A SVNRS$_p$ $\mathcal{R}$ in $U \times U$ is termed as a p-dimension single valued neutrosophic refined relation (SVNRR$_p$) in U, which is characterized by three membership functions: a truth-membership function $T_{\mathcal{R}}$, an indeterminacy-membership function $I_{\mathcal{R}}$, and a falsity-membership function $F_{\mathcal{R}}$ as follows:

$$\mathcal{R} = \{\langle (x, y), T_{\mathcal{R}}(x, y), I_{\mathcal{R}}(x, y), F_{\mathcal{R}}(x, y)\rangle \mid (x, y) \in U \times U\},$$

where

$$T_{\mathcal{R}}(x, y) = \{T_{1\mathcal{R}}(x, y), T_{2\mathcal{R}}(x, y), \cdots, T_{p\mathcal{R}}(x, y)\},$$

$$I_{\mathcal{R}}(x, y) = \{I_{1\mathcal{R}}(x, y), I_{2\mathcal{R}}(x, y), \cdots, I_{p\mathcal{R}}(x, y)\},$$

$$F_{\mathcal{R}}(x, y) = \{F_{1\mathcal{R}}(x, y), F_{2\mathcal{R}}(x, y), \cdots, F_{p\mathcal{R}}(x, y)\},$$

p is a positive integer, $T_{i\mathcal{R}}(x, y), I_{i\mathcal{R}}(x, y), F_{i\mathcal{R}}(x, y) \in [0, 1]$ for $i = 1, 2, \cdots, p$.

Let $\mathcal{R}$ be a SVNRR$_p$ in U. If $T_{i\mathcal{R}}(x, x) = 1$ and $I_{i\mathcal{R}}(x, x) = F_{i\mathcal{R}}(x, x) = 0$ $(i = 1, 2, \cdots, p)$ for all $x \in U$, then we say $\mathcal{R}$ is reflexive. If $T_{i\mathcal{R}}(x, y) = T_{i\mathcal{R}}(y, x)$, $I_{i\mathcal{R}}(x, y) = I_{i\mathcal{R}}(y, x)$ and $F_{i\mathcal{R}}(x, y) = F_{i\mathcal{R}}(y, x)$ $(i = 1, 2, \cdots, p)$ for all $x, y \in U$, then we say $\mathcal{R}$ is symmetric. If $\widetilde{\bigvee}_{y \in U} T_{\mathcal{R}}(x, y) = \{1, 1, \cdots, 1\}$ and $\widetilde{\bigwedge}_{y \in U} I_{\mathcal{R}}(x, y) = \widetilde{\bigwedge}_{y \in U} F_{\mathcal{R}}(x, y) = \{0, 0, \cdots, 0\}$ for all $x \in U$, then we say $\mathcal{R}$ is serial. If $\widetilde{\bigvee}_{y \in U}(T_{\mathcal{R}}(x, y) \widetilde{\wedge} T_{\mathcal{R}}(y, z)) \prec T_{\mathcal{R}}(x, z)$, $I_{\mathcal{R}}(x, z) \prec \widetilde{\bigwedge}_{y \in U}(I_{\mathcal{R}}(x, y) \widetilde{\vee} T_{\mathcal{R}}(y, z))$ and $F_{\mathcal{R}}(x, z) \prec \widetilde{\bigwedge}_{y \in U} (F_{\mathcal{R}}(x, y) \widetilde{\vee} F_{\mathcal{R}}(y, z))$ for all $x, y, z \in U$, then we say $\mathcal{R}$ is transitive.

Definition 3.2. Let $\mathcal{R}$ be a SVNRR$_p$ in U, the tuple $(U, \mathcal{R})$ is termed as a p-dimension single valued neutrosophic refined approximation space. $\forall A \in$ SVNRS$_p(U)$, the lower and upper approximations

Table 1
A 2-dimension single valued neutrosophic refined relation $\mathcal{R}$

$\mathcal{R}$	x_1	x_2
x_1	$\langle\{0.2, 0.5\}, \{0.3, 0.2\}, \{0.8, 0.9\}\rangle$	$\langle\{0.2, 0.7\}, \{0.2, 0.1\}, \{0.9, 0.8\}\rangle$
x_2	$\langle\{0.4, 0.6\}, \{0.3, 0.4\}, \{0.2, 0.4\}\rangle$	$\langle\{0.8, 0.9\}, \{0.2, 0.3\}, \{0, 0.1\}\rangle$
x_3	$\langle\{0.8, 0.9\}, \{0.1, 0.2\}, \{0.1, 0.3\}\rangle$	$\langle\{0.7, 0.9\}, \{0.3, 0.5\}, \{0.2, 0.2\}\rangle$
x_4	$\langle\{0.5, 0.8\}, \{0.3, 0.4\}, \{0.2, 0.3\}\rangle$	$\langle\{0, 0.1\}, \{0, 0.2\}, \{0.8, 1\}\rangle$
$\mathcal{R}$	x_3	x_4
x_1	$\langle\{0.8, 1\}, \{0.2, 0.4\}, \{0, 0.1\}\rangle$	$\langle\{0.1, 0.3\}, \{0.3, 0.4\}, \{0.8, 1\}\rangle$
x_2	$\langle\{0.9, 1\}, \{0.5, 0.8\}, \{0, 0.1\}\rangle$	$\langle\{0, 0.1\}, \{0.2, 0.3\}, \{0.9, 1\}\rangle$
x_3	$\langle\{0.5, 0.8\}, \{0.8, 0.5\}, \{0.3, 0.5\}\rangle$	$\langle\{0.9, 1\}, \{0.4, 0.6\}, \{0.1, 0.3\}\rangle$
x_4	$\langle\{0, 0.1\}, \{0.3, 0.4\}, \{0.8, 0.9\}\rangle$	$\langle\{0, 0.1\}, \{0.1, 0.2\}, \{0.7, 0.9\}\rangle$

of A with respect to $(U, \mathcal{R})$ are two p-dimension SVNRSs, denoted by $\underline{\mathcal{R}}(A)$ and $\overline{\mathcal{R}}(A)$, whose membership functions are defined as follows: $\forall x \in U$,

$$T_{\underline{\mathcal{R}}(A)}(x) = \widetilde{\bigwedge_{y\in U}} (F_{\mathcal{R}}(x, y)\widetilde{\vee}T_A(y)),$$

$$I_{\underline{\mathcal{R}}(A)}(x) = \widetilde{\bigvee_{y\in U}} ((\sim I_{\mathcal{R}}(x, y))\widetilde{\wedge}I_A(y)),$$

$$F_{\underline{\mathcal{R}}(A)}(x) = \widetilde{\bigvee_{y\in U}} (T_{\mathcal{R}}(x, y)\widetilde{\wedge}F_A(y));$$

$$T_{\overline{\mathcal{R}}(A)}(x) = \widetilde{\bigvee_{y\in U}} (T_{\mathcal{R}}(x, y)\widetilde{\wedge}T_A(y)),$$

$$I_{\overline{\mathcal{R}}(A)}(x) = \widetilde{\bigwedge_{y\in U}} (I_{\mathcal{R}}(x, y)\widetilde{\vee}I_A(y)),$$

$$F_{\overline{\mathcal{R}}(A)}(x) = \widetilde{\bigwedge_{y\in U}} (F_{\mathcal{R}}(x, y)\widetilde{\vee}F_A(y)).$$

The pair $(\underline{\mathcal{R}}(A), \overline{\mathcal{R}}(A))$ is termed as the single valued neutrosophic refined rough set of A with respect to $(U, \mathcal{R})$. $\underline{\mathcal{R}}$ and $\overline{\mathcal{R}}$ are termed as single valued neutrosophic refined lower and upper approximation operators, respectively.

Example 3.1. Let $U = \{x_1, x_2, x_3, x_4\}$. $\mathcal{R} \in$ SVNRS$_2(U \times U)$ is a SVNRR$_2$ given in Table 1. Assume $A \in$ SVNRS$_2(U)$ is given as follows:

$$\begin{aligned} A = \{&\langle x_1, \{0.5, 0.8\}, \{0.2, 0.4\}, \{0.1, 0.3\}\rangle,\\ &\langle x_2, \{0.7, 0.9\}, \{0.2, 0.4\}, \{0.5, 0.6\}\rangle,\\ &\langle x_3, \{0.2, 0.4\}, \{0.6, 0.3\}, \{0.7, 0.5\}\rangle,\\ &\langle x_4, \{0.2, 0.6\}, \{0.3, 0.5\}, \{0.1, 0.4\}\rangle\}.\end{aligned}$$

By Definition 3.2, we can obtain the lower and upper approximations of A with respect to $(U, \mathcal{R})$ as follows:

$$\underline{\mathcal{R}}(A)(x_1) = \langle\{0.2, 0.4\}, \{0.6, 0.5\}, \{0.7, 0.6\}\rangle,$$
$$\overline{\mathcal{R}}(A)(x_1) = \langle\{0.2, 0.7\}, \{0.2, 0.4\}, \{0.7, 0.5\}\rangle,$$
$$\underline{\mathcal{R}}(A)(x_2) = \langle\{0.2, 0.4\}, \{0.5, 0.5\}, \{0.7, 0.6\}\rangle,$$
$$\overline{\mathcal{R}}(A)(x_2) = \langle\{0.7, 0.9\}, \{0.2, 0.4\}, \{0.2, 0.4\}\rangle,$$
$$\underline{\mathcal{R}}(A)(x_3) = \langle\{0.2, 0.5\}, \{0.3, 0.4\}, \{0.5, 0.6\}\rangle,$$
$$\overline{\mathcal{R}}(A)(x_3) = \langle\{0.7, 0.9\}, \{0.2, 0.4\}, \{0.1, 0.3\}\rangle,$$
$$\underline{\mathcal{R}}(A)(x_4) = \langle\{0.5, 0.8\}, \{0.6, 0.5\}, \{0.1, 0.3\}\rangle,$$
$$\overline{\mathcal{R}}(A)(x_4) = \langle\{0.5, 0.8\}, \{0.2, 0.4\}, \{0.2, 0.3\}\rangle.$$

Remark 3.1. If $\mathcal{R}$ in Definition 3.2 is a single valued neutrosophic relation and A is a single valued neutrosophic set, then Definition 3.2 is consistent to the notion of single valued neutrosophic rough sets proposed in [39], which means that single valued neutrosophic rough sets proposed in [39] is a special case of single valued neutrosophic refined rough sets.

3.2. The properties of single valued neutrosophic refined approximation operators

This subsection is devoted to the properties of single valued neutrosophic refined lower and upper approximation operators.

Theorem 3.1. *Let $(U, \mathcal{R})$ be a p-dimension single valued neutrosophic refined approximation space. The single valued neutrosophic refined lower and upper approximation operators defined in Definition 3.2 have the following properties: $\forall A, B, \widehat{\alpha, \beta, \gamma} \in$ SVNRS$_p(U)$,*

(1) $\underline{\mathcal{R}}(U) = U,\ \overline{\mathcal{R}}(\emptyset) = \emptyset$;
(2) If $A \sqsubset B$, *then* $\underline{\mathcal{R}}(A) \sqsubset \underline{\mathcal{R}}(B)$ *and* $\overline{\mathcal{R}}(A) \sqsubset \overline{\mathcal{R}}(B)$;
(3) $\underline{\mathcal{R}}(A \sqcap B) = \underline{\mathcal{R}}(A) \sqcap \underline{\mathcal{R}}(B)$, $\overline{\mathcal{R}}(A \sqcup B) = \overline{\mathcal{R}}(A) \sqcup \overline{\mathcal{R}}(B)$;

(4) $\underline{\mathcal{R}}(A) \sqcup \underline{\mathcal{R}}(B) \sqsubset \underline{\mathcal{R}}(A \sqcup B)$, $\overline{\mathcal{R}}(A \sqcap B) \sqsubset \overline{\mathcal{R}}(A) \sqcap \overline{\mathcal{R}}(B)$;

(5) $\underline{\mathcal{R}}(A^c) = (\overline{\mathcal{R}}(A))^c$, $\overline{\mathcal{R}}(A^c) = (\underline{\mathcal{R}}(A))^c$;

(6) $\underline{\mathcal{R}}(A \sqcup \widehat{\alpha, \beta, \gamma}) = \underline{\mathcal{R}}(A) \sqcup \widehat{\alpha, \beta, \gamma}$, $\overline{\mathcal{R}}(A \sqcap \widehat{\alpha, \beta, \gamma}) = \overline{\mathcal{R}}(A) \sqcap \widehat{\alpha, \beta, \gamma}$;

(7) $\underline{\mathcal{R}}(\emptyset_p) = \emptyset_p \Longleftrightarrow \underline{\mathcal{R}}(\widehat{\alpha, \beta, \gamma}) = \widehat{\alpha, \beta, \gamma}$, $\overline{\mathcal{R}}(U_p) = U_p \Longleftrightarrow \overline{\mathcal{R}}(\widehat{\alpha, \beta, \gamma}) = \widehat{\alpha, \beta, \gamma}$.

Proof. All claims are immediate results of the corresponding definitions. □

Theorem 3.2. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two p-dimension SVNRRs in U. $\forall A \in SVNRS_p(U)$, we have*

(1) $\underline{\mathcal{R}_1 \sqcup \mathcal{R}_2}(A) = \underline{\mathcal{R}_1}(A) \sqcap \underline{\mathcal{R}_2}(A)$;

(2) $\overline{\mathcal{R}_1 \sqcup \mathcal{R}_2}(A) = \overline{\mathcal{R}_1}(A) \sqcup \overline{\mathcal{R}_2}(A)$.

Proof. (1) According to Definitions 2.3 and 3.2, the result can be easily proved.

(2) According to Proposition 2.1 (5) and Theorem 3.1 (5), the result can be directly obtained. □

Theorem 3.3. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two p-dimension SVNRRs in U. $\forall A \in SVNRS_p(U)$, we have*

(1) $\underline{\mathcal{R}_1}(A) \sqcap \underline{\mathcal{R}_2}(A) \sqsubset \underline{\mathcal{R}_1}(A) \sqcup \underline{\mathcal{R}_2}(A) \sqsubset \underline{\mathcal{R}_1 \sqcap \mathcal{R}_2}(A)$;

(2) $\overline{\mathcal{R}_1 \sqcap \mathcal{R}_2}(A) \sqsubset \overline{\mathcal{R}_1}(A) \sqcap \overline{\mathcal{R}_2}(A) \sqsubset \overline{\mathcal{R}_1}(A) \sqcup \overline{\mathcal{R}_2}(A)$.

Proof. (1) According to Definition 3.2, $\forall x \in U$,

$$\begin{aligned}
T_{\underline{\mathcal{R}_1 \sqcap \mathcal{R}_2}(A)}(x) &= \widetilde{\bigwedge_{y \in U}} (F_{\mathcal{R}_1 \sqcap \mathcal{R}_2}(x, y) \widetilde{\vee} T_A(y)) \\
&= \widetilde{\bigwedge_{y \in U}} ((F_{\mathcal{R}_1}(x, y) \widetilde{\vee} T_A(y)) \widetilde{\vee} \\
&\quad (F_{\mathcal{R}_2}(x, y) \widetilde{\vee} T_A(y))) \\
&\geq \left(\widetilde{\bigwedge_{y \in U}} (F_{\mathcal{R}_1}(x, y) \widetilde{\vee} T_A(y)) \right) \\
&\quad \widetilde{\vee} \left(\widetilde{\bigwedge_{y \in U}} (F_{\mathcal{R}_2}(x, y) \widetilde{\vee} T_A(y)) \right) \\
&= T_{\underline{\mathcal{R}_1}(A)}(x) \widetilde{\vee} T_{\underline{\mathcal{R}_2}(A)}(x) \\
&= T_{\underline{\mathcal{R}_1}(A) \sqcup \underline{\mathcal{R}_2}(A)}(x),
\end{aligned}$$

$$\begin{aligned}
I_{\underline{\mathcal{R}_1 \sqcap \mathcal{R}_2}(A)}(x) &= \widetilde{\bigvee_{y \in U}} ((\sim I_{\mathcal{R}_1 \sqcap \mathcal{R}_2}(x, y)) \widetilde{\wedge} I_A(y)) \\
&= \widetilde{\bigvee_{y \in U}} ((\sim I_{\mathcal{R}_1}(x, y)) \widetilde{\wedge} (\sim I_{\mathcal{R}_2}(x, y)) \\
&\quad \widetilde{\wedge} I_A(y)) \\
&= \widetilde{\bigvee_{y \in U}} (((\sim I_{\mathcal{R}_1}(x, y)) \widetilde{\wedge} I_A(y)) \\
&\quad \widetilde{\wedge} ((\sim I_{\mathcal{R}_2}(x, y)) \widetilde{\wedge} I_A(y))) \\
&\leq \left(\widetilde{\bigvee_{y \in U}} ((\sim I_{\mathcal{R}_1}(x, y)) \widetilde{\wedge} I_A(y)) \right) \\
&\quad \widetilde{\wedge} \left(\widetilde{\bigvee_{y \in U}} ((\sim I_{\mathcal{R}_2}(x, y)) \widetilde{\wedge} I_A(y)) \right) \\
&= I_{\underline{\mathcal{R}_1}(A)}(x) \widetilde{\wedge} I_{\underline{\mathcal{R}_2}(A)}(x) \\
&= I_{\underline{\mathcal{R}_1}(A) \sqcup \underline{\mathcal{R}_2}(A)}(x),
\end{aligned}$$

Similarly, we can show

$$F_{\underline{\mathcal{R}_1 \sqcap \mathcal{R}_2}(A)}(x) \leq F_{\underline{\mathcal{R}_1}(A) \sqcup \underline{\mathcal{R}_2}(A)}(x).$$

It is obvious $\underline{\mathcal{R}_1}(A) \sqcap \underline{\mathcal{R}_2}(A) \sqsubset \underline{\mathcal{R}_1}(A) \sqcup \underline{\mathcal{R}_2}(A)$. Hence, we get that $\underline{\mathcal{R}_1}(A) \sqcap \underline{\mathcal{R}_2}(A) \sqsubset \underline{\mathcal{R}_1}(A) \sqcup \underline{\mathcal{R}_2}(A) \sqsubset \underline{\mathcal{R}_1 \sqcap \mathcal{R}_2}(A)$.

(2) According to (1) and Theorem 3.1 (5), we have

$$\begin{aligned}
\overline{\mathcal{R}_1 \sqcap \mathcal{R}_2}(A) &= (\underline{\mathcal{R}_1 \sqcap \mathcal{R}_2}(A^c))^c \\
&\sqsubset (\underline{\mathcal{R}_1}(A^c) \sqcup \underline{\mathcal{R}_2}(A^c))^c \\
&= (\underline{\mathcal{R}_1}(A^c))^c \sqcap (\underline{\mathcal{R}_2}(A^c))^c \\
&= \overline{\mathcal{R}_1}(A) \sqcap \overline{\mathcal{R}_2}(A).
\end{aligned}$$

Consequently, $\overline{\mathcal{R}_1 \sqcap \mathcal{R}_2}(A) \sqsubset \overline{\mathcal{R}_1}(A) \sqcap \overline{\mathcal{R}_2}(A) \sqsubset \overline{\mathcal{R}_1}(A) \sqcup \overline{\mathcal{R}_2}(A)$. □

Remark 3.2. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two p-dimension SVNRRs in U. $\forall A \in \mathrm{SVNRS}_p(U)$. If $\mathcal{R}_1 \sqsubset \mathcal{R}_2$, then $\underline{\mathcal{R}_2}(A) \sqsubset \underline{\mathcal{R}_1}(A)$ and $\overline{\mathcal{R}_1}(A) \sqsubset \overline{\mathcal{R}_2}(A)$.

Proof. According to Theorem 3.3, the result is obvious. □

Next, we study the connections between special SVNRRs and single valued neutrosophic refined approximation operators.

Theorem 3.4. *Let $(U, \mathcal{R})$ be a p-dimension single valued neutrosophic refined approximation space. $\underline{\mathcal{R}}$ and $\overline{\mathcal{R}}$ are the lower and upper approximation operators defined in Definition 3.2, then we have the following results:*

$$
\begin{aligned}
(1)\quad \mathcal{R} \text{ is serial} &\iff \underline{\mathcal{R}}(\widehat{\alpha,\beta,\gamma}) = \widehat{\alpha,\beta,\gamma},\\
&\quad \forall\, \widehat{\alpha,\beta,\gamma} \in \text{SVNRS}_p(U),\\
&\iff \underline{\mathcal{R}}(\emptyset_p) = \emptyset_p,\\
&\iff \overline{\mathcal{R}}(\widehat{\alpha,\beta,\gamma}) = \widehat{\alpha,\beta,\gamma},\\
&\quad \forall\, \widehat{\alpha,\beta,\gamma} \in \text{SVNRS}_p(U),\\
&\iff \overline{\mathcal{R}}(U_p) = U_p;\\
(2)\quad \mathcal{R} \text{ is reflexive} &\iff \underline{\mathcal{R}}(A) \sqsubset A,\\
&\quad \forall A \in \text{SVNRS}_p(U),\\
&\iff A \sqsubset \overline{\mathcal{R}}(A),\\
&\quad \forall A \in \text{SVNRS}_p(U);\\
(3)\ \mathcal{R} \text{ is symmetric} &\iff \underline{\mathcal{R}}(1_{U-\{x\}})(y)\\
&= \underline{\mathcal{R}}(1_{U-\{y\}})(x),\ \forall x, y \in U,\\
&\iff \overline{\mathcal{R}}(1_x)(y)\\
&= \overline{\mathcal{R}}(1_y)(x),\ \forall x, y \in U;\\
(4)\quad \mathcal{R} \text{ is transitive} &\iff \underline{\mathcal{R}}(A) \sqsubset \underline{\mathcal{R}}(\underline{\mathcal{R}}(A)),\\
&\quad \forall A \in \text{SVNRS}_p(U),\\
&\iff \overline{\mathcal{R}}(\overline{\mathcal{R}}(A)) \sqsubset \overline{\mathcal{R}}(A),\\
&\quad \forall A \in \text{SVNRS}_p(U).
\end{aligned}
$$

Proof. According to Theorem 3.1 (5), we can know that $\underline{\mathcal{R}}$ and $\overline{\mathcal{R}}$ is a pair of dual operators. Thus, we only need to consider the properties of the lower approximation operator.

(1) By Theorem 3.1 (7), it suffices to verify that $\mathcal{R}$ is serial$\iff \underline{\mathcal{R}}(\widehat{\alpha,\beta,\gamma}) = \widehat{\alpha,\beta,\gamma},\ \forall\, \widehat{\alpha,\beta,\gamma} \in \text{SVNRS}_p(U)$.

"$\Longrightarrow$" If $\mathcal{R}$ is serial, then for any $x \in U$, $\widetilde{\bigvee}_{y\in U} T_{\mathcal{R}}(x,y) = \{1,1,\cdots,1\}$ and $\widetilde{\bigwedge}_{y\in U} I_{\mathcal{R}}(x,y) = \widetilde{\bigwedge}_{y\in U} F_{\mathcal{R}}(x,y) = \{0,0,\cdots,0\}$. $\forall\, \widehat{\alpha,\beta,\gamma} \in \text{SVNRS}_p(U)$, $\forall x \in U$, by Definition 3.2, we can conclude

$$
T_{\underline{\mathcal{R}}(\widehat{\alpha,\beta,\gamma})}(x) = \alpha,\ I_{\underline{\mathcal{R}}(\widehat{\alpha,\beta,\gamma})}(x) = \beta,
$$
$$
F_{\underline{\mathcal{R}}(\widehat{\alpha,\beta,\gamma})}(x) = \gamma.
$$

"$\Longleftarrow$" If $\underline{\mathcal{R}}(\widehat{\alpha,\beta,\gamma}) = \widehat{\alpha,\beta,\gamma}$ for any $\widehat{\alpha,\beta,\gamma} \in \text{SVNRS}_p(U)$. Take $\alpha = \{0,0,\cdots,0\}$, $\beta = \gamma = \{1,1,\cdots,1\}$, then we have

$$
\widetilde{\bigvee_{y\in U}} T_{\mathcal{R}}(x,y) = \{1,1,\cdots,1\},
$$
$$
\sim \widetilde{\bigwedge_{y\in U}} I_{\mathcal{R}}(x,y) = \{1,1,\cdots,1\},
$$

which implies that

$$
\widetilde{\bigwedge_{y\in U}} I_{\mathcal{R}}(x,y) = \{0,0,\cdots,0\},
$$
$$
\widetilde{\bigwedge_{y\in U}} F_{\mathcal{R}}(x,y) = \{0,0,\cdots,0\}.
$$

Thus, R is serial.

(2) "$\Longrightarrow$" If $\mathcal{R}$ is reflexive, then $T_{\mathcal{R}}(x,x) = \{1,1,\cdots,1\}$ and $I_{\mathcal{R}}(x,x) = F_{\mathcal{R}}(x,x) = \{0,0,\cdots,0\}$ for any $x \in U$. By Definition 3.2, $\forall A \in \text{SVNRS}_p(U)$, $\forall x \in U$,

$$
\begin{aligned}
T_{\underline{\mathcal{R}}(A)}(x) &= \widetilde{\bigwedge_{y\in U}} (F_{\mathcal{R}}(x,y) \widetilde{\vee} T_A(y))\\
&\prec F_{\mathcal{R}}(x,x) \widetilde{\vee} T_A(x)\\
&= T_A(x),
\end{aligned}
$$

Similarly, $I_{\underline{\mathcal{R}}(A)}(x) \succ I_A(x)$, $F_{\underline{\mathcal{R}}(A)}(x) \succ F_A(x)$. Therefore,$\underline{\mathcal{R}}(A) \sqsubset A$.

"$\Longleftarrow$" If $\underline{\mathcal{R}}(A) \sqsubset A$ for any $A \in \text{SVNRS}_p(U)$, then $\forall x \in U$, by taking $A = 1_{U-\{x\}}$, we have

$$
\begin{aligned}
T_{\mathcal{R}}(x,x) &= (T_{\mathcal{R}}(x,x) \widetilde{\wedge} \{1,1,\cdots,1\}) \widetilde{\vee} \{0,0,\cdots,0\}\\
&= (T_{\mathcal{R}}(x,x) \widetilde{\wedge} F_{1_{U-\{x\}}}(x)) \widetilde{\vee}\\
&\quad \Big(\widetilde{\bigvee_{y\in U-\{x\}}} (T_{\mathcal{R}}(x,y) \widetilde{\wedge} F_{1_{U-\{x\}}}(y))\Big)\\
&= \widetilde{\bigvee_{y\in U}} (T_{\mathcal{R}}(x,y) \widetilde{\wedge} F_{1_{U-\{x\}}}(y))\\
&= F_{\underline{\mathcal{R}}(1_{U-\{x\}})}(x)\\
&\succ F_{1_{U-\{x\}}}(x)\\
&= \{1,1,\cdots,1\},
\end{aligned}
$$

which means $T_{\mathcal{R}}(x,x) = \{1,1,\cdots,1\}$.

Similarly, we have $\sim I_{\mathcal{R}}(x,x) \succ \{1,1,\cdots,1\}$, i.e. $I_{\mathcal{R}}(x,x) = \{0,0,\cdots,0\}$, and $F_{\mathcal{R}}(x,x) \prec \{0,0,\cdots,0\}$, i.e. $F_{\mathcal{R}}(x,x) = \{0,0,\cdots,0\}$.

Thus, $\mathcal{R}$ is reflexive.

(3) According to Definition 3.2, $\forall x, y \in U$, it follows that

$$
\begin{aligned}
T_{\underline{\mathcal{R}}(1_{U-\{x\}})}(y) &= \widetilde{\bigwedge_{z\in U}} (F_{\mathcal{R}}(y,z) \widetilde{\vee} T_{1_{U-\{x\}}}(z))\\
&= (F_{\mathcal{R}}(y,x) \widetilde{\vee} T_{1_{U-\{x\}}}(x)) \widetilde{\wedge}
\end{aligned}
$$

$$\widetilde{\bigwedge_{z\in U-\{x\}}}(F_{\mathcal{R}}(y,z)\widetilde{\vee}T_{1_{U-\{x\}}}(z))$$
$$= F_{\mathcal{R}}(y,x),$$

Similarly, we can conclude that

$$T_{\underline{\mathcal{R}}(1_{U-\{y\}})}(x) = F_{\mathcal{R}}(x,y),$$
$$I_{\underline{\mathcal{R}}(1_{U-\{x\}})}(y) = \sim I_{\mathcal{R}}(y,x),$$
$$I_{\underline{\mathcal{R}}(1_{U-\{y\}})}(x) = \sim I_{\mathcal{R}}(x,y),$$
$$F_{\underline{\mathcal{R}}(1_{U-\{x\}})}(y) = T_{\mathcal{R}}(y,x),$$
$$F_{\underline{\mathcal{R}}(1_{U-\{y\}})}(x) = T_{\mathcal{R}}(x,y),$$

Therefore, $\mathcal{R}$ is symmetric iff $\forall x, y \in U$, $\underline{\mathcal{R}}(1_{U-\{x\}})(y) = \underline{\mathcal{R}}(1_{U-\{y\}})(x)$.

(4) "$\Longrightarrow$" If $\mathcal{R}$ is transitive, then for all $x, y, z \in U$,

$$\widetilde{\bigvee}_{y\in U}(T_{\mathcal{R}}(x,y)\widetilde{\wedge}T_{\mathcal{R}}(y,z)) \prec T_{\mathcal{R}}(x,z),$$
$$I_{\mathcal{R}}(x,z) \prec \widetilde{\bigwedge}_{y\in U}(I_{\mathcal{R}}(x,y)\widetilde{\vee}I_{\mathcal{R}}(y,z)),$$
$$F_{\mathcal{R}}(x,z) \prec \widetilde{\bigwedge}_{y\in U}(F_{\mathcal{R}}(x,y)\widetilde{\vee}F_{\mathcal{R}}(y,z)).$$

According to Definition 3.2, $\forall x \in U$, we have

$$\begin{aligned}
T_{\underline{\mathcal{R}}(\underline{\mathcal{R}}(A))}(x) &= \widetilde{\bigwedge_{y\in U}}(F_{\mathcal{R}}(x,y)\widetilde{\vee}T_{\underline{\mathcal{R}}(A)}(y)) \\
&= \widetilde{\bigwedge_{z\in U}}\widetilde{\bigwedge_{y\in U}}(F_{\mathcal{R}}(x,y)\widetilde{\vee}F_{\mathcal{R}}(y,z)\widetilde{\vee}T_A(z)) \\
&= \widetilde{\bigwedge_{z\in U}}\left(\widetilde{\bigwedge_{y\in U}}(F_{\mathcal{R}}(x,y)\widetilde{\vee}F_{\mathcal{R}}(y,z) \widetilde{\vee}T_A(z))\right) \\
&\succ \widetilde{\bigwedge_{z\in U}}(F_{\mathcal{R}}(x,z)\widetilde{\vee}T_A(z)) \\
&= T_{\underline{\mathcal{R}}(A)}(x),
\end{aligned}$$

Similarly, we can obtain $I_{\underline{\mathcal{R}}(\underline{\mathcal{R}}(A))}(x) \prec I_{\underline{\mathcal{R}}(A)}(x)$, $F_{\underline{\mathcal{R}}(\underline{\mathcal{R}}(A))}(x) \prec F_{\underline{\mathcal{R}}(A)}(x)$.

Therefore, $\underline{\mathcal{R}}(A) \sqsubset \underline{\mathcal{R}}(\underline{\mathcal{R}}(A))$.

"$\Longleftarrow$" Assume $\underline{\mathcal{R}}(A) \sqsubset \underline{\mathcal{R}}(\underline{\mathcal{R}}(A))$ for all $A \in \text{SVNRS}_p(U)$. $\forall x, y, z \in U$, let $A = 1_{U-\{z\}}$, from the proving process of (3), we have

$$\begin{aligned}
T_{\mathcal{R}}(x,z) &= F_{\underline{\mathcal{R}}(1_{U-\{z\}})}(x) \\
&\succ F_{\underline{\mathcal{R}}(\underline{\mathcal{R}}(1_{U-\{z\}}))}(x) \\
&= \widetilde{\bigvee_{y\in U}}(T_{\mathcal{R}}(x,y)\widetilde{\wedge}T_{\mathcal{R}}(y,z)),
\end{aligned}$$

Similarly, $I_{\mathcal{R}}(x,z) \prec \widetilde{\bigwedge_{y\in U}}(I_{\mathcal{R}}(x,y)\widetilde{\vee}I_{\mathcal{R}}(y,z))$,

$$F_{\mathcal{R}}(x,z) \prec \widetilde{\bigwedge_{y\in U}}(F_{\mathcal{R}}(x,y)\widetilde{\vee}F_{\mathcal{R}}(y,z)).$$

Therefore, $\mathcal{R}$ is transitive. □

4. Axiomatic characterizations of single valued neutrosophic refined approximation operators

In this section, we will study the axiomatic characterizations of single valued neutrosophic refined lower and upper approximation operators by restricting a pair of abstract single valued neutrosophic refined set operators.

Theorem 4.1. *Let $\mathcal{L}$: $SVNRS_p(U) \longrightarrow SVNRS_p(U)$ be a p-dimension single valued neutrosophic refined set operator. Then, there exists a p-dimension SVNRR $\mathcal{R}$ in U such that $\mathcal{L}(A) = \underline{\mathcal{R}}(A)$ for all $A \in SVNRS_p(U)$ iff $\mathcal{L}$ satisfies the following axioms (SVNRSL1) and (SVNRSL2): $\forall A, B, \widehat{\alpha,\beta,\gamma} \in SVNRS\ _p(U)$,*

(SVNRSL1) $\mathcal{L}(A \sqcup \widehat{\alpha,\beta,\gamma}) = \mathcal{L}(A) \sqcup \widehat{\alpha,\beta,\gamma}$;
(SVNRSL2) $\mathcal{L}(A \sqcap B) = \mathcal{L}(A) \sqcap L(B)$.

Proof. "$\Longrightarrow$" It is straightforward from Theorem 3.1.

"$\Longleftarrow$" Suppose $\mathcal{L}$ satisfies axioms (SVNRSL1) and (SVNRSL2). By using $\mathcal{L}$, we define a p-dimension SVNRR $\mathcal{R} = \{\langle (x,y), T_{\mathcal{R}}(x,y), I_{\mathcal{R}}(x,y), F_{\mathcal{R}}(x,y)\rangle\}$ as follows:

$T_{\mathcal{R}}(x,y) = F_{\mathcal{L}(1_{U-\{y\}})}(x)$, $I_{\mathcal{R}}(x,y) = \sim I_{\mathcal{L}(1_{U-\{y\}})}(x)$, $F_{\mathcal{R}}(x,y) = T_{\mathcal{L}(1_{U-\{y\}})}(x)$ for any $x, y \in U$.

Moreover, we can obtain that for all $A \in \text{SVNRS}_p(U)$,

$$A = \underset{y\in U}{\sqcap}(1_{U-\{y\}} \sqcup \widehat{A(y)}),$$

where $A(y) = \langle T_A(y), I_A(y), F_A(y)\rangle$.

In fact, for all $x \in U$, we have

$$\begin{aligned}
&T_{\underset{y\in U}{\sqcap}(1_{U-\{y\}}\sqcup\widehat{A(y)})}(x) \\
&= \widetilde{\bigwedge_{y\in U}}T_{(1_{U-\{y\}}\sqcup\widehat{A(y)})}(x) \\
&= T_{1_{U-\{x\}}}(x)\widetilde{\vee}T_{\widehat{A(x)}}(x)\widetilde{\wedge}
\end{aligned}$$

$$\widetilde{\bigwedge_{y\in U-\{x\}}}(T_{1_{U-\{y\}}}(x)\widetilde{\vee}T_{\widehat{A(y)}}(x))$$
$$= T_A(x),$$

Similarly,

$$I_{\underset{y\in U}{\sqcap}(1_{U-\{y\}}\sqcup\widehat{A(y)})}(x) = I_A(x),$$
$$F_{\underset{y\in U}{\sqcap}(1_{U-\{y\}}\sqcup\widehat{A(y)})}(x) = F_A(x),$$

By Definition 3.2, (SVNRSL1) and (SVNRSL2), we have

$$T_{\underline{\mathcal{R}}(A)}(x) = \widetilde{\bigwedge_{y\in U}}(F_{\mathcal{R}}(x,y)\widetilde{\vee}T_A(y))$$
$$= \widetilde{\bigwedge_{y\in U}}(T_{\mathcal{L}(1_{U-\{y\}}\sqcup\widehat{A(y)})}(x))$$
$$= T_{\underset{y\in U}{\sqcap}(\mathcal{L}(1_{U-\{y\}}\sqcup\widehat{A(y)}))}(x)$$
$$= T_{\mathcal{L}(\underset{y\in U}{\sqcap}(1_{U-\{y\}}\sqcup\widehat{A(y)}))}(x)$$
$$= T_{\mathcal{L}(A)}(x),$$

Similarly, $I_{\underline{\mathcal{R}}(A)}(x) = I_{\mathcal{L}(A)}(x)$, $F_{\underline{\mathcal{R}}(A)}(x) = F_{\mathcal{L}(A)}(x)$.

Thus, $\underline{\mathcal{R}}(A) = \mathcal{L}(A)$. □

Theorem 4.2. *Let $\mathcal{H}$: $SVNRS_p(U) \longrightarrow SVNRS_p(U)$ be a p-dimension single valued neutrosophic refined set operator. Then, there exists a p-dimension SVNRR $\mathcal{R}$ in U such that $\mathcal{H}(A) = \underline{\mathcal{R}}(A)$ for all $A \in SVNRS_p(U)$ iff $\mathcal{H}$ satisfies the following axioms (SVNRSH1) and (SVNRSH2): $\forall A, B, \widehat{\alpha,\beta,\gamma} \in SVNRS_p(U)$,*

(SVNRSH1) $\mathcal{H}(A \sqcap \widehat{\alpha,\beta,\gamma}) = \mathcal{H}(A) \sqcap \widehat{\alpha,\beta,\gamma}$;
(SVNRSH2) $\mathcal{H}(A \sqcup B) = \mathcal{H}(A) \sqcup \mathcal{H}(B)$.

Proof. "$\Longrightarrow$" It is straightforward from Theorem 3.1.

"$\Longleftarrow$" Suppose $\mathcal{H}$ satisfies axioms (SVNRSH1) and (SVNRSH2). By using $\mathcal{H}$, we define a p-dimension SVNRR $\mathcal{R} = \{\langle(x,y), T_{\mathcal{R}}(x,y), I_{\mathcal{R}}(x,y), F_{\mathcal{R}}(x,y)\rangle\}$ as follows:

$$T_{\mathcal{R}}(x,y) = T_{\mathcal{H}(1_y)}(x),\ I_{\mathcal{R}}(x,y) = I_{\mathcal{H}(1_y)}(x),$$
$$F_{\mathcal{R}}(x,y) = F_{\mathcal{H}(1_y)}(x) \text{ for any } x, y \in U$$

Moreover, we can obtain that for all $A \in$ SVNRS$_p(U)$,

$$A = \underset{y\in U}{\sqcup}(1_y \sqcap \widehat{A(y)}).$$

In fact, for all $x \in U$, we have

$$T_{\underset{y\in U}{\sqcup}(1_y\sqcap\widehat{A(y)})}(x) = \widetilde{\bigvee_{y\in U}}T_{1_y\sqcap\widehat{A(y)}}(x)$$
$$= (T_{1_x}(x)\widetilde{\wedge}T_{\widehat{A(x)}}(x))\widetilde{\vee}\left(\widetilde{\bigvee_{y\in U-\{x\}}}(T_{1_y}(x)\widetilde{\wedge}T_{\widehat{A(y)}}(x))\right)$$
$$= T_A(x),$$

Similarly,

$$I_{\underset{y\in U}{\sqcup}(1_y\sqcap\widehat{A(y)})}(x) = I_A(x),$$
$$F_{\underset{y\in U}{\sqcup}(1_y\sqcap\widehat{A(y)})}(x) = F_A(x).$$

By Definition 3.2, (SVNRSH1) and (SVNRSH2), we have

$$T_{\overline{\mathcal{R}}(A)}(x) = \widetilde{\bigvee_{y\in U}}(T_{\mathcal{R}}(x,y)\widetilde{\wedge}T_A(y))$$
$$= \widetilde{\bigvee_{y\in U}}(T_{\mathcal{H}(1_y)}(x)\widetilde{\wedge}T_A(y))$$
$$= \widetilde{\bigvee_{y\in U}}(T_{\mathcal{H}(1_y)\sqcap\widehat{A(y)}}(x))$$
$$= T_{\underset{y\in U}{\sqcup}(\mathcal{H}(1_y\sqcap\widehat{A(y)}))}(x)$$
$$= T_{\mathcal{H}(\underset{y\in U}{\sqcup}(1_y\sqcap\widehat{A(y)}))}(x)$$
$$= T_{\mathcal{H}(A)}(x),$$

Similarly,

$$I_{\overline{\mathcal{R}}(A)}(x) = I_{\mathcal{H}(A)}(x),$$
$$F_{\overline{\mathcal{R}}(A)}(x) = F_{\mathcal{H}(A)}(x).$$

Thus, $\overline{\mathcal{R}}(A) = \mathcal{H}(A)$. □

Remark 4.1. If $\mathcal{L}, H$: SVNRS$_p(U) \longrightarrow$ SVNRS$_p(U)$ satisfy (SVNRSL1), (SVNRSL2) and (SVNRSU1), (SVNRSU2), respectively. Then, $\mathcal{L}(A) = (\mathcal{H}(A^c))^c$ and $\mathcal{H}(A) = (\mathcal{L}(A^c))^c$. In this case, $\mathcal{L}$ and $\mathcal{H}$ are called a pair of dual operators. Furthermore, if $\mathcal{L}$ and $\mathcal{H}$ are dual operators, then (SVNRSL1), (SVNRSL2) are equivalent to (SVNRSU1), (SVNRSU2).

Proof. It follows immediately from Theorem 3.1.

Next, we investigate axiomatic characterizations of other special single valued neutrosophic refined approximation operators. □

Theorem 4.3. *Let $\mathcal{L}, H : SVNRS_p(U) \longrightarrow SVNRS_p(U)$ be a pair of p-dimension single valued neutrosophic refined set operators, then there exists a serial p-dimension SVNRR $\mathcal{R}$ in U such that $\mathcal{L}(A) = \underline{\mathcal{R}}(A)$, $\mathcal{H}(A) = \overline{\mathcal{R}}(A)$ for all $A \in SVNRS_p(U)$ iff $\mathcal{L}$ satisfies axioms (SVNRSL1), (SVNRSL2) and one of the following equivalent axioms about $\mathcal{L}$, or equivalently $\mathcal{H}$ satisfies (SVNRSU1), (SVNRSU2) and one of the following equivalent axioms about $\mathcal{H}$:*

(SVNRSL3) $\mathcal{L}(\emptyset_p) = \emptyset_p$;
(SVNRSU3) $\mathcal{H}(U_p) = U_p$;
(SVNRSL4) $\mathcal{L}(\widehat{\alpha, \beta, \gamma}) = \widehat{\alpha, \beta, \gamma}$ *for all* $\widehat{\alpha, \beta, \gamma} \in SVNRS_p(U)$;
(SVNRSU4) $\mathcal{H}(\widehat{\alpha, \beta, \gamma}) = \widehat{\alpha, \beta, \gamma}$ *for all* $\widehat{\alpha, \beta, \gamma} \in SVNRS_p(U)$.

Proof. It follows immediately from Theorems 3.4 (1), 4.1 and 4.2. □

Theorem 4.4. *Let $\mathcal{L}, H : SVNRS_p(U) \longrightarrow SVNRS_p(U)$ be a pair of p-dimension single valued neutrosophic refined set operators, then there exists a reflexive p-dimension SVNRR $\mathcal{R}$ in U such that $\mathcal{L}(A) = \underline{\mathcal{R}}(A)$, $\mathcal{H}(A) = \overline{\mathcal{R}}(A)$ for all $A \in SVNRS_p(U)$ iff $\mathcal{L}$ satisfies axioms (SVNRSL1), (SVNRSL2) and (SVNRSL5), or equivalently $\mathcal{H}$ satisfies (SVNRSU1), (SVNRSU2) and (SVNRSU5):*

(SVNRSL5) $\mathcal{L}(A) \sqsubset A$;
(SVNRSU5) $A \sqsubset \mathcal{H}(A)$.

Proof. It follows immediately from Theorems 3.4 (2), 4.1 and 4.2. □

Theorem 4.5. *Let $\mathcal{L}, H : SVNRS_p(U) \longrightarrow SVNRS_p(U)$ be a pair of p-dimension single valued neutrosophic refined set operators, then there exists a symmetric p-dimension SVNRR $\mathcal{R}$ in U such that $\mathcal{L}(A) = \underline{\mathcal{R}}(A)$, $\mathcal{H}(A) = \overline{\mathcal{R}}(A)$ for all $A \in SVNRS(U)$ iff $\mathcal{L}$ satisfies axioms (SVNRSL1), (SVNRSL2) and (SVNRSL6), or equivalently $\mathcal{H}$ satisfies (SVNRSU1), (SVNRSU2) and (SVNRSU6):*

(SVNRSL6) $\mathcal{L}(1_{U-\{y\}})(x) = \mathcal{L}(1_{U-\{x\}})(y)$, $\forall x, y \in U$;
(SVNRSU6) $\mathcal{H}(1_y)(x) = \mathcal{H}(1_x)(y)$, $\forall x, y \in U$.

Proof. It follows immediately from Theorems 3.4 (3), 4.1 and 4.2. □

Theorem 4.6. *Let $\mathcal{L}, H : SVNRS_p(U) \longrightarrow SVNRS_p(U)$ be a pair of dual p-dimension single valued neutrosophic refined set operators, then there exists a transitive p-dimension SVNRR $\mathcal{R}$ in U such that $\mathcal{L}(A) = \underline{\mathcal{R}}(A)$, $\mathcal{H}(A) = \overline{\mathcal{R}}(A)$ for all $A \in SVNRS_p(U)$ iff $\mathcal{L}$ satisfies axioms (SVNRSL1), (SVNRSL2) and (SVNRSL7), or equivalently $\mathcal{H}$ satisfies (SVNRSU1), (SVNRSU2) and (SVNRSU7):*

(SVNRSL7) $\mathcal{L}(A) \sqsubset L(\mathcal{L}(A))$, $\forall A \in SVNRS_p(U)$;
(SVNRSU7) $\mathcal{H}(\mathcal{H}(A)) \sqsubset \mathcal{H}(A)$, $\forall A \in SVNRS_p(U)$.

Proof. It follows immediately from Theorems 3.4 (4), 4.1 and 4.2. □

5. An application of single valued neutrosophic refined rough sets in multi-attribute decision making

5.1. An algorithm for medical diagnosis based on single valued neutrosophic refined rough sets

In real life, decision making problems always involve at least two universes of discourse such as symptoms set and diseases set in medical diagnosis. So it is necessary to introduce single valued neutrosophic refined rough sets on two universes of discourse.

Let U, V be two spaces of points (objects). A SVNRS$_p$ $\mathcal{R}$ in $U \times V$ is termed as a p-dimension single valued neutrosophic refined relation (SVNRRp) from U to V, denoted by $\mathcal{R} = \{\langle (x, y), T_{\mathcal{R}}(x, y), I_{\mathcal{R}}(x, y), F_{\mathcal{R}}(x, y) \rangle \mid (x, y) \in U \times V\}$, where

$$T_{\mathcal{R}}(x, y) = \{T_{1\mathcal{R}}(x, y), T_{2\mathcal{R}}(x, y), \cdots, T_{p\mathcal{R}}(x, y)\},$$

$$I_{\mathcal{R}}(x, y) = \{I_{1\mathcal{R}}(x, y), I_{2\mathcal{R}}(x, y), \cdots, I_{p\mathcal{R}}(x, y)\},$$

$$F_{\mathcal{R}}(x, y) = \{F_{1\mathcal{R}}(x, y), F_{2\mathcal{R}}(x, y), \cdots, F_{p\mathcal{R}}(x, y)\}.$$

Definition 5.1. Let $\mathcal{R}$ be a SVNRR$_p$ from U to V, the tuple $(U, V, \mathcal{R})$ is termed as a single valued neutrosophic refined approximation space on two universes. $\forall A \in$ SVNRS$_p(V)$, the lower and upper approximations of A with respect to $(U, V, \mathcal{R})$ are two p-dimension SVNRSs in U, denoted by $\underline{\mathcal{R}}(A)$ and $\overline{\mathcal{R}}(A)$, where $\forall x \in U$:

$$T_{\underline{\mathcal{R}}(A)}(x) = \widetilde{\bigwedge_{y\in V}}(F_{\mathcal{R}}(x, y)\widetilde{\vee}T_A(y)),$$

$$I_{\underline{\mathcal{R}}(A)}(x) = \widetilde{\bigvee_{y\in V}}((\sim I_{\mathcal{R}}(x, y))\widetilde{\wedge}I_A(y)),$$

$$F_{\underline{\mathcal{R}}(A)}(x) = \widetilde{\bigvee_{y\in V}}(T_{\mathcal{R}}(x, y)\widetilde{\wedge}F_A(y));$$

$$T_{\overline{\mathcal{R}}(A)}(x) = \widetilde{\bigvee_{y\in V}}(T_{\mathcal{R}}(x, y)\widetilde{\wedge}T_A(y),$$

$$I_{\overline{\mathcal{R}}(A)}(x) = \widetilde{\bigwedge_{y\in V}}(I_{\mathcal{R}}(x, y)\widetilde{\vee}I_A(y)),$$

$$F_{\overline{\mathcal{R}}(A)}(x) = \widetilde{\bigwedge_{y\in V}}(F_{\mathcal{R}}(x, y)\widetilde{\vee}F_A(y)).$$

The pair $(\underline{\mathcal{R}}(A), \overline{\mathcal{R}}(A))$ is termed as the single valued neutrosophic refined rough set of A with respect to $(U, V, \mathcal{R})$.

Zhang et al. [15] introduced a novel approach to define the operations of interval neutrosophic numbers based on t-norm and t-conorm. Similarly, we introduce the sum of two single valued neutrosophic refined elements by t-norm and t-conorm as follows:

Definition 5.2. Let A and B be two p-dimension single valued neutrosophic refined sets in U. The sum of A and B is a p-dimension single valued neutrosophic refined set, denoted by $A \boxplus B$, defined based on the Archimedean t-norm and t-conorm as follows:

$$A \boxplus B = \{x, A(x) \oplus B(x)|x \in U\},$$

where $A(x) \oplus B(x) = \langle\{l^{-1}(l(T_{1A}(x)) + l(T_{1B}(x))), l^{-1}(l(T_{2A}(x)) + l(T_{2B}(x))), \cdots, l^{-1}(l(T_{pA}(x)) + l(T_{pB}(x)))\}, \{k^{-1}(k(I_{1A}(x)) + k(I_{1B}(x))), k^{-1}(k(I_{2A}(x)) + k(I_{2B}(x))), \cdots, k^{-1}(k(I_{pA}(x))+ k(I_{pB}(x)))\}, \{k^{-1}(k(F_{1A}(x))+k(F_{1B}(x))), k^{-1}(k(F_{2A}(x))+k(F_{2B}(x))), \cdots, k^{-1}(k(F_{pA}(x)) + k(F_{pB}(x)))\}\rangle$.

In [41], Ye introduced the cosine similarity between two single valued neutrosophic numbers for ranking single valued neutrosophic numbers in decision-making procedure. Analogously, we can define the cosine similarity between two single valued neutrosophic refined elements as follows:

Definition 5.3. Let A be a p-dimension single valued neutrosophic refined set in U and $\alpha = \langle T_A(x_i), I_A(x_i), F_A(x_i)\rangle$, $\beta = \langle T_A(x_j), I_A(x_j), F_A(x_j)\rangle$ be its two p-dimension single valued neutrosophic refined elements. The cosine similarity between α and β is defined as follows:

$$S(\alpha, \beta) = \frac{\sum_{k=1}^{p} A_k(x_i) \cdot A_k(x_j)}{\sqrt{\sum_{k=1}^{p} A_k^2(x_i))} \cdot \sqrt{\sum_{k=1}^{p} A_k^2(x_j)}},$$

where $A_k(x_i) \cdot A_k(x_j) = T_{kA}(x_i) \cdot T_{kA}(x_j) + I_{kA}(x_i) \cdot I_{kA}(x_j) + F_{kA}(x_i) \cdot F_{kA}(x_j)$, $A_k^2(x_i) = T_{kA}^2(x_i) + I_{kA}^2(x_i) + F_{kA}^2(x_i)$.

From Definition 5.3, it can be observed that the bigger the similarity measure S, the closer the two single valued neutrosophic refined elements. By comparing the cosine similarity measures between every single valued neutrosophic refined element and an ideal single valued neutrosophic refined element, the rank of all single valued neutrosophic refined elements can be acquired.

In what follows, we will consider medical diagnosis problems based on single valued neutrosophic refined rough sets on two universes. Suppose that the universe $U = \{x_1, x_2, \cdots, x_m\}$ represents a set of diseases, and the universe $V = \{y_1, y_2, \cdots, y_n\}$ represents a set of symptoms. Let $\mathcal{R} \in \mathrm{SVNRR}_p(U \times V)$ be a single valued neutrosophic refined relation from U to V, where $\forall(x, y) \in U \times V$, $\mathcal{R}(x, y)$ represents the degree that the disease x $(x \in U)$ shows the symptom y $(y \in V)$. Given a patient A who has some symptoms in V, and the symptoms of the patient (also denoted by A) are illustrated by a SVNRS A in the universe V. In the following, we propose an algorithm to diagnose which kind of disease the patient A is suffering from.

Algorithm Diagnosing which kind of disease a patient is suffering from

Input: A diseases set U, a symptoms set V and a p-dimension single valued neutrosophic refined relation from U to V, the symptoms of a patient A.

Output:

RES(A)// the disease patient A is suffering from

1: Computing the lower and upper approximation of A, i.e. $\underline{\mathcal{R}}(A)$ and $\overline{\mathcal{R}}(A)$;

2: Computing $\underline{\mathcal{R}}(A) \boxplus \overline{\mathcal{R}}(A)$;

3: Computing $I = \langle \widetilde{\bigvee_{x_i\in U}} T_{\underline{\mathcal{R}}(A)\boxplus\overline{\mathcal{R}}(A)}(x_i), \widetilde{\bigwedge_{x_i\in U}} I_{\underline{\mathcal{R}}(A)\boxplus\overline{\mathcal{R}}(A)}(x_i), \widetilde{\bigwedge_{x_i\in U}} F_{\underline{\mathcal{R}}(A)\boxplus\overline{\mathcal{R}}(A)}(x_i)\rangle$;

4: Computing $S((\underline{\mathcal{R}}(A) \boxplus \overline{\mathcal{R}}(A))(x_i), I)$ for each $x_i \in U$;

5: $\mathrm{RES}(A) = \{x_k|S((\underline{\mathcal{R}}(A) \boxplus \overline{\mathcal{R}}(A))(x_k), I) \geq S((\underline{\mathcal{R}}(A) \boxplus \overline{\mathcal{R}}(A))(x_i), I), x_i \in U\}$;

6: Return RES(A).

Table 2
The 3-dimension single valued neutrosophic refined relation $\mathcal{R}$ from U to V

$\mathcal{R}$	x_1	x_2	x_3	x_4
y_1	$\langle\{0.4, 0.5, 0.2\}, \{0.2, 0.3, 0.8\}, \{0.3, 0.4, 0.2\}\rangle$	$\langle\{0.8, 0.9, 0.9\}, \{0.1, 0.2, 0.2\}, \{0, 0.1, 0.1\}\rangle$	$\langle\{0.8, 1, 0.9\}, \{0.2, 0.4, 0.3\}, \{0, 0.1, 0.1\}\rangle$	$\langle\{0.1, 0.3, 0.2\}, \{0.3, 0.4, 0.1\}, \{0.8, 1, 0.9\}\rangle$
y_2	$\langle\{0.5, 0.6, 0.6\}, \{0.3, 0.4, 0.2\}, \{0.2, 0.3, 0.1\}\rangle$	$\langle\{0.8, 0.9, 0.8\}, \{0.2, 0.3, 0.2\}, \{0, 0.1, 0.1\}\rangle$	$\langle\{0.9, 1, 1\}, \{0.1, 0.3, 0.1\}, \{0, 0.1, 0\}\rangle$	$\langle\{0, 0.1, 0.2\}, \{0.2, 0.3, 0.1\}, \{0.9, 1, 0.8\}\rangle$
y_3	$\langle\{0, 0.1, 0\}, \{0.1, 0.2, 0.2\}, \{0.8, 0.9, 0.8\}\rangle$	$\langle\{0, 0.2, 0.1\}, \{0.1, 0.3, 0.2\}, \{0.7, 0.9, 0.8\}\rangle$	$\langle\{0.7, 0.8, 0.9\}, \{0.4, 0.6, 0.5\}, \{0.2, 0.1, 0.3\}\rangle$	$\langle\{0.9, 0.9, 1\}, \{0.4, 0.5, 0.6\}, \{0.1, 0.3, 0.2\}\rangle$
y_4	$\langle\{0.7, 0.8, 0.8\}, \{0.3, 0.4, 0.3\}, \{0.2, 0.3, 0.1\}\rangle$	$\langle\{0, 0.1, 0.1\}, \{0, 0.2, 0.1\}, \{0.8, 1, 0.9\}\rangle$	$\langle\{0, 0.1, 0.1\}, \{0.3, 0.2, 0.4\}, \{0.8, 0.7, 0.9\}\rangle$	$\langle\{0, 0.1, 0.2\}, \{0.1, 0.2, 0.2\}, \{0.8, 0.7, 0.9\}\rangle$
y_5	$\langle\{0.4, 0.5, 0.4\}, \{0.5, 0.6, 0.6\}, \{0.6, 0.7, 0.8\}\rangle$	$\langle\{0, 0.1, 0.2\}, \{0.1, 0.2, 0.3\}, \{0.9, 1, 0.8\}\rangle$	$\langle\{0, 0.2, 0.1\}, \{0.2, 0.3, 0.4\}, \{0.7, 0.6, 1\}\rangle$	$\langle\{0.1, 0.2, 0.4\}, \{0.2, 0.4, 0.3\}, \{0.7, 0.8, 0.6\}\rangle$

5.2. An illustrative example

In this subsection, an example of medical diagnosis is illustrated to demonstrate the feasibility of the method proposed in Subsection 5.1.

We take into account the medical diagnosis problem partly adopted from [25] and adjust the hesitant fuzzy environment to neutrosophic environment. Let $U = \{x_1, x_2, x_3, x_4\}$ be a set of four diseases, where x_i $(i = 1, 2, 3, 4)$ represents "common cold", "malaria" "typhoid", and "stomach disease" respectively and the universe $V = \{y_1, y_2, y_3, y_4, y_5\}$ be a set of five symptoms, where y_j $(j = 1, 2, 3, 4, 5)$ represents "fever", "headache", "stomachache", "cough", and "chest-pain", respectively. Let $\mathcal{R}$ be a p-dimension SVNRR from U to V which is actually a medical knowledge statistic data of the relationship between the disease x_i $(x_i \in U)$ and the symptom y_j $(y_j \in V)$. The statistic data is provided in Table 2.

The symptoms of a patient A are illustrated by a 3-dimension SVNRS in the universe V which are obtained at different time intervals such as 7:00 am, 12:00 and 6:00 pm as follows:

$$
\begin{aligned}
A = \{&\langle y_1, \{0.8, 0.9, 1\}, \{0.2, 0.3, 0\}, \{0.1, 0.3, 0\}\rangle,\\
&\langle y_2, \{0.7, 0.9, 0.8\}, \{0.1, 0.2, 0.1\}, \{0.1, 0.2, 0.2\}\rangle,\\
&\langle y_3, \{0.7, 0.8, 0.8\}, \{0.2, 0.4, 0.3\}, \{0.1, 0.2, 0.3\}\rangle,\\
&\langle y_4, \{0.1, 0.2, 0.1\}, \{0.3, 0.4, 0.2\}, \{0.8, 0.7, 0.9\}\rangle,\\
&\langle y_5, \{0, 0.1, 0\}, \{0.1, 0.2, 0.3\}, \{0.8, 0.9, 1\}\rangle\}.
\end{aligned}
$$

In what follows, we illustrate the decision-making process by the six steps:

Step 1. According to Definition 5.1, we can obtain that

$$
\begin{aligned}
\underline{\mathcal{R}}(A) = \{&\langle x_1, \{0.2, 0.3, 0.1\}, \{0.3, 0.4, 0.3\}, \{0.7, 0.7, 0.8\}\rangle,\\
&\langle x_2, \{0.7, 0.9, 0.8\}, \{0.3, 0.4, 0.3\}, \{0.1, 0.3, 0.3\}\rangle,\\
&\langle x_3, \{0.7, 0.6, 0.8\}, \{0.3, 0.4, 0.3\}, \{0.1, 0.3, 0.3\}\rangle,\\
&\langle x_4, \{0.7, 0.7, 0.6\}, \{0.3, 0.4, 0.3\}, \{0.1, 0.3, 0.4\}\rangle\},\\
\overline{\mathcal{R}}(A) = \{&\langle x_1, \{0.5, 0.6, 0.6\}, \{0.2, 0.3, 0.2\}, \{0.2, 0.3, 0.2\}\rangle,\\
&\langle x_2, \{0.8, 0.9, 0.9\}, \{0.1, 0.2, 0.2\}, \{0.1, 0.2, 0.1\}\rangle,\\
&\langle x_3, \{0.8, 0.9, 0.9\}, \{0.1, 0.3, 0.1\}, \{0.1, 0.2, 0.1\}\rangle,\\
&\langle x_4, \{0.7, 0.8, 0.8\}, \{0.2, 0.3, 0.1\}, \{0.1, 0.3, 0.3\}\rangle\}.
\end{aligned}
$$

Step 2. Let $k(x) = -\log(x)$, then $k^{-1}(x) = e^{-x}$, $l(x) = -\log(1 - x)$, and $l^{-1}(x) = 1 - e^{-1}(x)$. By Definition 5.2, we have

$$
\begin{aligned}
&\underline{\mathcal{R}}(A) \boxplus \overline{\mathcal{R}}(A)\\
&\quad = \{\langle x_1, \{0.60, 0.72, 0.64\}, \{0.06, 0.12, 0.06\},\\
&\qquad \{0.14, 0.21, 0.16\}\rangle, \langle x_2, \{0.94, 0.99, 0.98\},\\
&\qquad \{0.03, 0.08, 0.06\}, \{0.01, 0.06, 0.03\}\rangle,\\
&\qquad \langle x_3, \{0.94, 0.96, 0.98\}, \{0.03, 0.12, 0.03\},\\
&\qquad \{0.01, 0.06, 0.03\}\rangle, \langle x_4, \{0.91, 0.94, 0.92\},\\
&\qquad \{0.06, 0.12, 0.03\}, \{0.01, 0.09, 0.12\}\rangle\}.
\end{aligned}
$$

Step 3. According to above results, we calculate the ideal single valued neutrosophic refined element

$$
\begin{aligned}
I = \langle&\{0.94, 0.99, 0.98\}, \{0.03, 0.08, 0.03\},\\
&\{0.01, 0.06, 0.03\}\rangle.
\end{aligned}
$$

Step 4. By Definiton 5.3, we can compute that

$$S((\underline{\mathcal{R}}(A) \boxplus \overline{\mathcal{R}}(A))(x_1), I) = 0.9718,$$

$$S((\underline{\mathcal{R}}(A) \boxplus \overline{\mathcal{R}}(A))(x_2), I) = 0.9998,$$

$$S((\underline{\mathcal{R}}(A) \boxplus \overline{\mathcal{R}}(A))(x_3), I) = 0.9996,$$

$$S((\underline{\mathcal{R}}(A) \boxplus \overline{\mathcal{R}}(A))(x_4), I) = 0.9975,$$

It follows that

$$S((\underline{\mathcal{R}}(A) \boxplus \overline{\mathcal{R}}(A))(x_2), I) > S((\underline{\mathcal{R}}(A) \boxplus \overline{\mathcal{R}}(A))(x_3), I) > S((\underline{\mathcal{R}}(A) \boxplus \overline{\mathcal{R}}(A))(x_4), I) > S((\underline{\mathcal{R}}(A) \boxplus \overline{\mathcal{R}}(A))(x_1), I).$$

Step 5. From discussion above, x_2 is the optimal choice.

Step 6. There is only one optimal choice x_2, so the patient A is suffering from x_2–malaria.

Compared with the model and algorithm proposed in [18], the model and algorithm in this paper can deal with information which come from different time intervals or different information providers in the process of decision making. For single valued neutrosophic refined sets is a generalization of single valued neutrosophic sets, the algorithm based on single valued neutrosophic refined rough sets on two universes suits more general decision-making environment.

6. Conclusion

In this paper, we propose the hybrid model of single valued neutrosophic refined sets and rough sets—single valued neutrosophic refined rough sets. Specifically, we investigate the single valued neutrosophic refined rough sets from both constructive and axiomatic approaches. Then, single valued neutrosophic refined rough sets on two universes are introduced for wider application of single valued neutrosophic refined rough sets. In addition, we provide an algorithm to handle decision making problem in medical diagnosis based on single valued neutrosophic refined rough sets on two universes. Finally, a numerical example is employed to demonstrate the validness of the proposed single valued neutrosophic refined rough sets. It should be highly noted that the model and algorithm proposed in this present paper is available not only in medical diagnosis but also in other decision making problems such as investment decision-making, shopping decision-making and so on. For the future prospects, we will devote to explore the application of the proposed model to data mining and attribute reduction.

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