Forum Geometricorum
Volume 18 (2018) 93–97.

FORUM GEOM
ISSN 1534-1178

A Toroidal Approach to the Doubling of the Cube

Gerasimos T. Soldatos

Abstract. A doubling of the cube is attempted as a problem equivalent to the doubling of a horn torus. Both doublings are attained through the circle of Apollonius.

According to Eratosthenes [1], a plague was sent by god Apollo to the Aegean island of Delos around 430 B.C., and when its citizens consulted the oracle of Delphi to learn how to defeat the plague, the answer was to double the altar of Apollo, which was a cube. And, a cube, K, with twice the volume of a given cube of side length 1, has volume equal to 2, which implies that in order to construct K, side length equal to $\sqrt[3]{2}$ has to be constructed first. Several solutions to this problem have been advanced since antiquity, but not within the context of classical geometry, because as Wantzel [2] proved much later, in 1837, the number $\sqrt[3]{2}$ is not constructible with unruled straightedge and compass. This article adds one more solution to the Delian problem by relating it to Euclidean horn torus metrics as follows.

Let the volume of horn torus $\overline{\mathbb{T}}$ be $2\pi^2R^3$, where R is the radius underlying $\overline{\mathbb{T}}$. Let $\mathbb{T}$ be the horn torus whose volume is half the volume of $\overline{\mathbb{T}}$, that is, $2\pi^2x^3 = \pi^2R^3$, where x is the radius defining $\mathbb{T}$. It follows that $2x^3 = R^3$, that is, the volume of the cube $\overline{\mathbb{C}}$ with edge length R is twice the volume of the cube $\mathbb{C}$ with edge x. Consequently, the problem of doubling the cube translates into the problem of doubling the horn torus. The problem of the constructibility of $\sqrt[3]{2}$ remains but it can be circumvented if this toroidal approach to the doubling of the cube is combined with a similar approach to the quadrature by the present author [3] in this *Journal* as follows.

The latter approach enables the construction of the line length z that squares the circle underlying torus $\overline{\mathbb{T}}$. But then

$$z = R\sqrt{\pi} \implies z^3\sqrt{\pi} = \pi^2R^3.$$

Letting

$$z^3\sqrt{\pi} = 2\pi^2x^3 \implies z^3 = 2(x^3\pi\sqrt{\pi}). \tag{1}$$

Publication Date: February 12, 2018. Communicating Editor: Paul Yiu.

If, now, y is the square edge that squares the circle characterizing torus $\mathbb{T}$ so that $y = x\sqrt{\pi} \implies y^3 = x^3\pi\sqrt{\pi}$, inserting this expression in (1) yields

$$z^3 = 2y^3. \tag{2}$$

Hence, in view of (2), to double the cube $\mathbb{C}$ when torus $\mathbb{T}$ with radius x is given, we can start by finding square edge y, and from y proceed to identify square edge z and subsequently, the line segment R as side length of cube $\overline{\mathbb{C}}$ and as radius of the horn torus $\overline{\mathbb{T}}$ whose volume is twice that of $\mathbb{T}$: So,

Problem. Given a horn torus $\mathbb{T}$ with defining circle and tube circle radius x in the 3-dimensional Euclidean space, find horn torus $\overline{\mathbb{T}}$ whose defining and tube circle radius R is such that $R^3 = 2x^3 \implies \pi^2 R^3 = 2\pi^2 x^3$.

Analysis: For educational purposes, consider first this problem from the viewpoint of the square edges y and z, squaring the circles with radiuses x and R, respectively. Suppose that from this viewpoint, this problem has been solved as in Figure 1, with $z = y + \nu$ being subsequently the cube edge doubling the cube with edge equal to y so that $z = y\sqrt[3]{2}$. It follows that

$$z^2 = y^2 \left(\sqrt[3]{2}\right)^2. \tag{3}$$

Since triangle $M\Gamma N$ is a right-angled one with Γ being the vertex of the right angle and with altitude $\Gamma\Xi = y$, we have also that

$$z^2 = y^2 + \beta^2. \tag{4}$$

Equating (3) and (4) yields

$$\beta^2 = y^2 \left[\left(\sqrt[3]{2}\right)^2 - 1\right]. \tag{5}$$

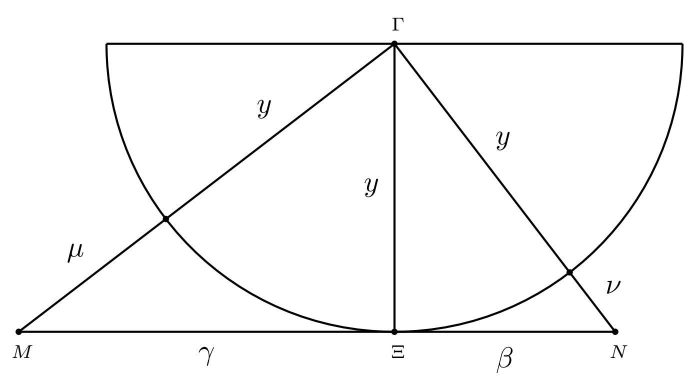

Figure 1.

By the power of point theorem in connection with point N and given that y is also the radius of circle $(\Gamma, \Gamma\Xi = y)$, we have

$$\beta^2 = \nu(\nu + 2y) \tag{6}$$

too, which when equated with (5), gives the quadratic equation

$$\nu^2 + 2y\nu - y^2\left[\left(\sqrt[3]{2}\right)^2 - 1\right] = 0.$$

Solving this equation for $\nu > 0$ gives

$$\nu = y\left(\sqrt[3]{2} - 1\right) \tag{7}$$

and hence

$$z = y + \nu = \sqrt[3]{2}. \tag{8}$$

Note that since ν is as in (7), the sum in (8) obtains regardless the length of y but can be reconciled with the assumption that $z = y\sqrt[3]{2}$ only if $y = 1$. It appears that to double a cube presupposes the normalization of its edge to the value of one. Indeed, since $\beta + \gamma$ is the hypotenuse MN of the right triangle $M\Gamma N$,

$$(\beta + \gamma)^2 = (\mu + y)^2 + z^2. \tag{9}$$

Also, note that from the right triangle $M\Xi\Gamma$,

$$\gamma^2 = (\mu + y)^2 - y^2 \implies (\mu + y)^2 = \gamma^2 + y^2. \tag{10}$$

Inserting (8) and (10) in (9) gives

$$(\beta + \gamma)^2 = \gamma^2 + y^2 + (\sqrt[3]{2})^2$$

which in view of (5) becomes

$$y^2\left[\left(\sqrt[3]{2}\right)^2 - 1\right] + 2\beta\gamma = y^2 + (\sqrt[3]{2})^2. \tag{11}$$

By the power of point theorem, $y^2 = \beta\gamma$, which when inserted in (11), produces the relation

$$y^2\left[\left(\sqrt[3]{2}\right)^2 - 1\right] + 2y^2 = y^2 + \left(\sqrt[3]{2}\right)^2 \implies y^2 = 1 \implies y = 1.$$

That is, if one tried to solve the original problem of the doubling of a cube one should presuppose that its edge is equal to one, indeed.

But, this is not necessary when one more datum to the Analysis is added as for instance is done herein borrowing from torus geometry as follows. Figure 2 presents a version of Apollonius' definition of circle where x, the circle radius of torus $\mathbf{T}$, appears as well. Triangles $\Sigma T\Psi$ and $\Sigma T'\Phi$ are similar isosceles triangles with $TT' = T'T'' = \Phi\Psi = x$ and $\Sigma T' = \Sigma\Phi = y$, satisfying the proportions

$$\frac{\Sigma\Phi}{\Phi\Psi} = \frac{y}{x} = \frac{\Sigma\Omega}{\Psi\Omega} = \frac{\Sigma\Upsilon}{\Psi\Upsilon} = \frac{\Sigma\Upsilon'}{\Psi\Upsilon'}. \tag{12}$$

Points like Υ, Υ' and Ω on the periphery of the circle $(O, O\Upsilon = \varrho)$ satisfy the same ratio of distances in (12) with regard to points Σ and Ψ. Presumably, $\Sigma\Upsilon$ is tangent to this circle at point Υ, $T\Psi \parallel T'\Phi$, $T''\Psi \parallel T'\Omega$, and $\angle\Sigma\Upsilon\Phi = \angle\Phi\Upsilon\Psi$. Methodologically, since $y^2 = \pi x^2 \implies y = x\sqrt{\pi}$ and $z^2 = \pi R^2 \implies z = R\sqrt{\pi}$, it follows that $y/x = z/R$. So, given the endpoints Σ and Ψ of length $y + x = \Sigma\Psi$, with Φ being the point at which $\Sigma\Phi = y$ is extended by $x = \Phi\Psi$, lengths z and R could be searched in terms of the locus of points like Υ and

Ω whose distance from Σ and Ψ satisfies the property $y/x = z/R$. This locus defines Apollonius' circle with candidates for z lengths like $\Sigma\Upsilon$, $\Sigma\Upsilon'$ and $\Sigma\Omega$, and candidates for R lengths like $\Psi\Upsilon$, $\Psi\Upsilon'$ and $\Psi\Omega$, respectively. Actually, it is $\Sigma\Upsilon = z$ and $\Sigma\Upsilon = R$ as follows: From the relation $y/x = \Sigma\Upsilon/\Psi\Upsilon \implies \Sigma\Upsilon = \Psi\Upsilon(y/x) \implies$

$$\Sigma\Upsilon^2 = \Psi\Upsilon^2(y^2/x^2) \tag{13}$$

while power of point theorem in connection with point Σ gives

$$\Sigma\Upsilon^2 = y(y + 2\varrho). \tag{14}$$

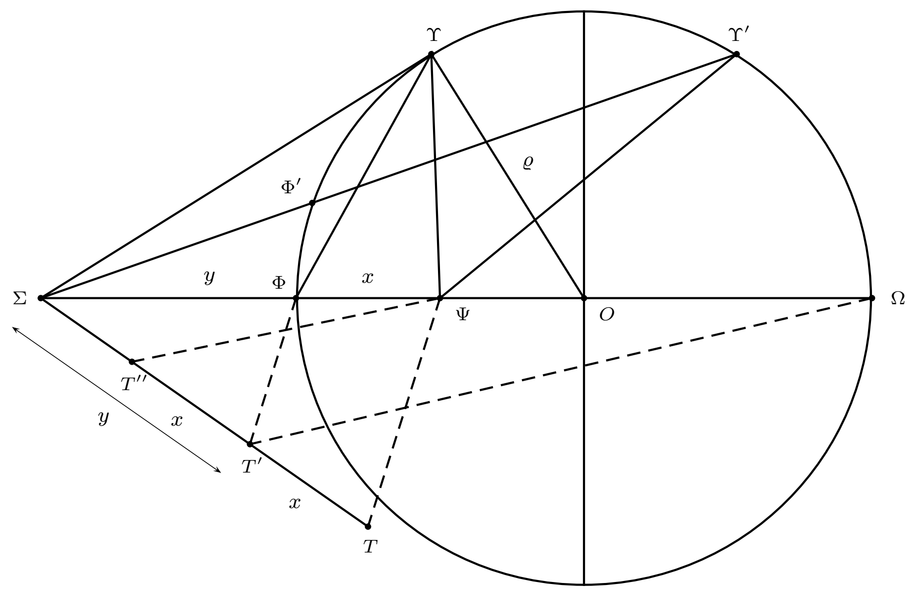

Figure 2.

Equating (13) and (14) results in the expression

$$\Psi\Upsilon^2 = \frac{x^2(y + 2\varrho)}{y}. \tag{15}$$

But, we should also have by construction that

$$\Sigma\Upsilon^2 = y(y + 2\varrho) = \pi\Psi\Upsilon^2 \implies \Psi\Upsilon^2 = \frac{y(y + 2\varrho)}{\pi}. \tag{16}$$

Combining (15) and (16) yields $y^2 = \pi x^2$, which is true. It could not be that $\Sigma\Omega = z$ and $\Psi\Omega = R$, because from Thales' intercept theorem,

$$\frac{\Sigma T''}{\Sigma\Psi} = \frac{y - x}{y + x} = \frac{T'T'' = x}{\Psi\Omega} \implies \Psi\Omega = x\frac{y + x}{y - x},$$

which in view of $y^2 = \pi x^2$, becomes

$$\Psi\Omega = x\frac{\sqrt{\pi}+1}{\sqrt{\pi}-1}$$

and hence, $\Sigma\Omega^2 = \pi\Psi\Omega^2 = \pi x^2 \frac{(\sqrt{\pi}+1)^2}{(\sqrt{\pi}-1)^2}$. $\Sigma\Omega$ is square and cube edge that does not double the cube with edge y and $\Psi\Omega$ is a circle radius of some horn torus that does not double torus $\mathbb{T}$. Consider finally a case like $\Sigma\Upsilon' = z$ and $\Psi\Upsilon' = R$. From the power of point theorem,

$$\Sigma\Phi'(\Sigma\Upsilon') = \Sigma\Phi(\Sigma\Omega) \implies \Sigma\Phi' z = y(y+2\varrho)$$

or setting $z = R\sqrt{\pi}$ and solving for R,

$$R = \frac{y(y+2\varrho)}{\Sigma\Phi'\sqrt{\pi}}.$$

That is, points in general like $\Upsilon' \neq \Upsilon$ do not solve the problem of the doubling of the cube either.

Construction: Given circle radius $x = \Phi\Psi$, construct square edge $y = \Sigma\Phi$ as in Soldatos [3], draw line segment $y + x = \Sigma\Phi + \Phi\Psi = \Sigma\Psi$, form the circle of Apollonius (O, ϱ) defined by the ratio of distances y/x with respect to points Σ and Ψ, draw from Σ tangent to this circle, and connect the tangency point Υ with point Ψ: The line segment $\Psi\Upsilon = R$ (while $\Sigma\Upsilon = z$).

Proof. The construction reproduces Figure 2. And, the proof reproduces the argument starting with relation (13), leading at the same time to the following corollary. □

Corollary: The line segment $\Sigma\Upsilon = z$.

References

[1] Eratosthenes, in T. L. Heath, *A History of Greek Mathematics*, vol. 1, Oxford University Press, 1951 [1921], 244–245.

[2] G. T. Soldatos, A toroidal approach to the Archimedian quadrature, *Forum Geom.*, 17 (2017) 17–19.

[3] P. L. Wantzel, Recherches sur les moyens de reconnaîtres un Problème de Géométrie peut se résoudre avec la règle et le compas, *Journal de Mathematiques Pures et Appliquees*, 1 (1837) 366–372.

Gerasimos T. Soldatos: Taburishanska 73, 27505 Svitlovodsk, Ukraine
E-mail address: soldgera@yahoo.com