Finite Volcano Potentials Admitting a Rational Eigenfunction

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Abstract

Starting with a general rational wave function, we search for potentials admitting it as a bound energy eigenfunction. We thus derive singular and regular potentials asymptotically decaying as the inverse of x squared, with the latter being simple or multiple volcanoes having a finite number of bound eigenstates. We present specific examples and examine the transition from singular to volcano potentials.

Keywords: volcano potentials, bound states, rational wave functions, nodes, poles

Contents

1. Introduction

Volcano potentials frequently appear in braneworld scenarios [1-4], while they've also been used in non-relativistic quantum mechanics [5].

A typical volcano potential includes a well, usually at the origin, having finite bottom and height on both sides, while at infinity, it can be unbounded from below (minus infinity) or it can tend to a constant, finite value (including zero) [6].

Herein, we examine volcano potentials admitting a rational bound energy eigenfunction. These volcano potentials, which include a simple or multiple well and decay asymptotically as $1/x^2$, have a finite number of bound eigenstates, with the highest of them having fixed energy that can be set to zero. Since they are finite at infinity, we'll refer to them as finite volcano potentials.

2. Rational eigenfunctions

We consider the function

$$\psi(x) = A\frac{P_n\left(\frac{x}{x_0}\right)}{Q_m\left(\frac{x}{x_0}\right)} \tag{1}$$

where P_n, Q_m are polynomials of degrees n, m, respectively, with $m \geq n+2$ [7], and x_0 is a positive length scale.

The polynomial Q_m has no zeros[1], i.e. it has constant sign in $\mathbb{R}$, so that $\psi(x)$ has no singularities. Then, m must be even[2].

Notes

1. By zero, we mean real zero.

2. Every odd-degree polynomial has at least one zero, since it is continuous and its values at minus and plus infinity have different signs.

A is the normalization constant with dimensions of wave function, i.e. $[A] = L^{-1/2}$.

Since x/x_0 is dimensionless, all coefficients of P_n have the same dimensions and, likewise, all coefficients of Q_m have also the same dimensions.

Then, since the normalization constant has dimensions of wave function, the dimensions of the coefficients of the two polynomials are the same and they are eliminated.

Thus, without loss of generality, we assume that the coefficients of the two polynomials are dimensionless.

Besides, if p_n and q_m are, respectively, the leading coefficients of P_n and Q_m, then

$$\frac{P_n\left(\frac{x}{x_0}\right)}{Q_m\left(\frac{x}{x_0}\right)}=\frac{p_n}{q_m}\frac{\tilde{P}_n\left(\frac{x}{x_0}\right)}{\tilde{Q}_m\left(\frac{x}{x_0}\right)}$$

where the polynomials $\tilde{P}_n$ and $\tilde{Q}_m$ are monic in x/x_0.

Incorporating the dimensionless factor p_n/q_m into the normalization constant, we make both polynomials monic, and thus, without loss of generality, we assume that both P_n and Q_m are monic.

Since Q_m has no zeros, $\psi(x)$ has no singularities.

Moreover, as we see from (1), $\psi(x)$ is infinitely many times differentiable on $\mathbb{R}$, i.e. it is $C^\infty(R)$.

Also, since $\deg(Q_m)\geq\deg(P_n)+2$, $\psi(x)$ tends to zero faster than $1/x^2$, as $|x|\to\infty$, and thus it is square integrable.

Therefore, $\psi(x)$, as given by (1), is eligible to describe a bound energy eigenstate of some potential.

We'll see that, although the wave function $\psi(x)$ is $C^\infty(R)$, the potential can have singularities, particularly simple poles.

3. The potential

Assuming that the wave function (1) is an eigenfunction of energy E_r, of some potential $V(x)$, it will satisfy the time-independent Schrödinger equation, i.e.

$$\psi''(x)+\frac{2m_0}{\hbar^2}\left(E_r-V(x)\right)\psi(x)=0$$

We denote the mass by m_0, so that we don't confuse it with the degree m of Q_m.

Solving the previous equation for the potential yields

$$V(x)=\frac{\hbar^2}{2m_0}\frac{\psi''(x)}{\psi(x)}+E_r$$

Since the potential depends on the ratio $\psi''(x)/\psi(x)$, the normalization constant A is eliminated and we can omit it in the calculation of the potential.
Introducing the dimensionless variable

$$\tilde{x}=\frac{x}{x_0} \tag{2}$$

we have

$$\frac{d^2}{dx^2}=\frac{1}{{x_0}^2}\frac{d^2}{d\tilde{x}^2}$$

and, in terms of $\tilde{x}$, the potential is written as

$$V(\tilde{x})=\frac{\hbar^2}{2m_0{x_0}^2}\frac{\psi''(\tilde{x})}{\psi(\tilde{x})}+E_r \tag{3}$$

where now the prime denotes differentiation with respect to $\tilde{x}$.
In terms of $\tilde{x}$, (1) is written as, omitting the normalization constant,

$$\psi(\tilde{x})=\frac{P_n(\tilde{x})}{Q_m(\tilde{x})} \tag{4}$$

Differentiating with respect to $\tilde{x}$, we obtain

$$\psi'(\tilde{x})=\frac{P_n'(\tilde{x})}{Q_m(\tilde{x})}-\frac{P_n(\tilde{x})Q_m'(\tilde{x})}{{Q_m}^2(\tilde{x})}$$

Differentiating once again yields

$$\psi''(\tilde{x})=\frac{P_n''(\tilde{x})}{Q_m(\tilde{x})}-\frac{P_n'(\tilde{x})Q_m'(\tilde{x})}{{Q_m}^2(\tilde{x})}-\frac{P_n'(\tilde{x})Q_m'(\tilde{x})+P_n(\tilde{x})Q_m''(\tilde{x})}{{Q_m}^2(\tilde{x})}+2\frac{P_n(\tilde{x}){Q_m'}^2(\tilde{x})}{{Q_m}^3(\tilde{x})}=$$
$$=\frac{P_n''(\tilde{x})}{Q_m(\tilde{x})}-\frac{2P_n'(\tilde{x})Q_m'(\tilde{x})+P_n(\tilde{x})Q_m''(\tilde{x})}{{Q_m}^2(\tilde{x})}+\frac{2P_n(\tilde{x}){Q_m'}^2(\tilde{x})}{{Q_m}^3(\tilde{x})}=$$

$$=\left(Q_m^2(\tilde{x})P_n''(\tilde{x})-Q_m(\tilde{x})\left(2P_n'(\tilde{x})Q_m'(\tilde{x})+P_n(\tilde{x})Q_m''(\tilde{x})\right)+2P_n(\tilde{x})Q_m'^2(\tilde{x})\right)\frac{1}{Q_m^3(\tilde{x})}=$$

$$=\left(Q_m^2(\tilde{x})P_n''(\tilde{x})-2Q_m(\tilde{x})Q_m'(\tilde{x})P_n'(\tilde{x})+\left(2Q_m'^2(\tilde{x})-Q_m(\tilde{x})Q_m''(\tilde{x})\right)P_n(\tilde{x})\right)\frac{1}{Q_m^3(\tilde{x})}=$$

$$=\left(\frac{Q_m^2(\tilde{x})P_n''(\tilde{x})-2Q_m(\tilde{x})Q_m'(\tilde{x})P_n'(\tilde{x})}{P_n(\tilde{x})}+2Q_m'^2(\tilde{x})-Q_m(\tilde{x})Q_m''(\tilde{x})\right)\frac{P_n(\tilde{x})}{Q_m^3(\tilde{x})}=$$

$$=\left(\frac{Q_m(\tilde{x})\left(Q_m(\tilde{x})P_n''(\tilde{x})-2Q_m'(\tilde{x})P_n'(\tilde{x})\right)}{P_n(\tilde{x})}+2Q_m'^2(\tilde{x})-Q_m(\tilde{x})Q_m''(\tilde{x})\right)\frac{\psi(\tilde{x})}{Q_m^2(\tilde{x})}$$

where in the last equality, we used (4).

Thus

$$\frac{\psi''(\tilde{x})}{\psi(\tilde{x})}=\frac{2Q_m'^2(\tilde{x})-Q_m(\tilde{x})Q_m''(\tilde{x})}{Q_m^2(\tilde{x})}+\frac{Q_m(\tilde{x})P_n''(\tilde{x})-2Q_m'(\tilde{x})P_n'(\tilde{x})}{Q_m(\tilde{x})P_n(\tilde{x})}$$

Substituting into (3) yields

$$V(\tilde{x})=\frac{\hbar^2}{2m_0x_0^2}\left(\frac{2Q_m'^2(\tilde{x})-Q_m(\tilde{x})Q_m''(\tilde{x})}{Q_m^2(\tilde{x})}+\frac{Q_m(\tilde{x})P_n''(\tilde{x})-2Q_m'(\tilde{x})P_n'(\tilde{x})}{Q_m(\tilde{x})P_n(\tilde{x})}\right)+E_r \quad (5)$$

In (5), the term $\left(2Q_m'^2(\tilde{x})-Q_m(\tilde{x})Q_m''(\tilde{x})\right)/Q_m^2(\tilde{x})$ is regular and does not depend on $P_n(\tilde{x})$, while the term $\left(Q_m(\tilde{x})P_n''(\tilde{x})-2Q_m'(\tilde{x})P_n'(\tilde{x})\right)/\left(Q_m(\tilde{x})P_n(\tilde{x})\right)$ has singularities at the zeros of $P_n(\tilde{x})$ (if any).

We'll consider only removable singularities – which are practically no singularities – and simple poles in the potential (5).

If at the zeros of $P_n(\tilde{x})$, the potential (5) has removable singularities, it can be written as a regular function, which is actually $C^\infty(R)$. In this case, all zeros of $P_n(\tilde{x})$ are simple [8], i.e. of multiplicity 1.

Next, we'll derive the expression of the potential at long distances.

To this end, we observe that

$$\deg\left(Q_m'^2(\tilde{x})\right)=2(m-1)$$

$$\deg\left(Q_m(\tilde{x})Q_m''(\tilde{x})\right) = m + m - 2 = 2(m-1)$$

$$\deg\left(Q_m^{\;2}(\tilde{x})\right) = 2m$$

Since Q_m is monic, the leading term of the polynomial $2Q_m'^2(\tilde{x}) - Q_m(\tilde{x})Q_m''(\tilde{x})$ is $2m^2\tilde{x}^{2(m-1)} - m(m-1)\tilde{x}^{2(m-1)} = \left(2m^2 - m(m-1)\right)\tilde{x}^{2(m-1)} = \left(m^2 + m\right)\tilde{x}^{2(m-1)}$.

The leading term of the polynomial $Q_m^{\;2}(\tilde{x})$ is $\tilde{x}^{2m}$.

Thus, at long distances,

$$\frac{2Q_m'^2(\tilde{x}) - Q_m(\tilde{x})Q_m''(\tilde{x})}{Q_m^{\;2}(\tilde{x})} \sim \frac{\left(m^2+m\right)\tilde{x}^{2(m-1)}}{\tilde{x}^{2m}} = \frac{m^2+m}{\tilde{x}^2}$$

In the same way, we have

$$\deg\left(Q_m(\tilde{x})P_n''(\tilde{x})\right) \overbrace{=}^{*} m + n - 2 = m + n - 2$$

    * If $n = 0,1$, the polynomial $Q_m(\tilde{x})P_n''(\tilde{x})$ vanishes.

Also

$$\deg\left(Q_m'(\tilde{x})P_n'(\tilde{x})\right) \overbrace{=}^{**} m - 1 + n - 1 = m + n - 2$$

    ** If $n = 0$, the polynomial $Q_m'(\tilde{x})P_n'(\tilde{x})$ vanishes

Thus

$$\deg\left(Q_m(\tilde{x})P_n''(\tilde{x}) - 2Q_m'(\tilde{x})P_n'(\tilde{x})\right) = \begin{cases} m+n-2, & n \neq 0 \\ 0, & n = 0 \end{cases}$$

Also

$$\deg\left(Q_m(\tilde{x})P_n(\tilde{x})\right) = m + n$$

Since the polynomials P_n and Q_m are monic, the leading term of the polynomial $Q_m(\tilde{x})P_n''(\tilde{x}) - 2Q_m'(\tilde{x})P_n'(\tilde{x})$ is $\left(n(n-1)-2mn\right)\tilde{x}^{m+n-2}$, and vanishes if $n=0$, as it should, since then the previous polynomial is zero.

The leading term of the polynomial $Q_m(\tilde{x})P_n(\tilde{x})$ is $\tilde{x}^{m+n}$, and thus, at long distances,

$$\frac{Q_m(\tilde{x})P_n''(\tilde{x}) - 2Q_m'(\tilde{x})P_n'(\tilde{x})}{Q_m(\tilde{x})P_n(\tilde{x})} \sim \frac{\left(n(n-1)-2mn\right)\tilde{x}^{m+n-2}}{\tilde{x}^{m+n}} = \frac{n(n-1)-2mn}{\tilde{x}^2}$$

We observe that both terms of the potential (5) decay as $1/\tilde{x}^2$ and thus, at long distances,

$$V_\infty(\tilde{x}) = \frac{\hbar^2}{2m_0 {x_0}^2} \frac{m^2+m+n(n-1)-2mn}{\tilde{x}^2} + E_r$$

Since

$$m^2+m+n(n-1)-2mn = m^2-2mn+n^2+m-n = (m-n)^2+m-n = (m-n)(m-n+1)$$

we end up to

$$V_\infty(\tilde{x}) = \frac{(m-n)(m-n+1)\hbar^2}{2m_0 {x_0}^2} \frac{1}{\tilde{x}^2} + E_r \qquad (6)$$

We see that, at long distances, the potential (5) is symmetric and, since $(m-n)(m-n+1) > 0$ *, the term $(m-n)(m-n+1)\hbar^2/2m_0{x_0}^2\tilde{x}^2$ is repulsive.

We also see that, at long distances, the potential depends only on the difference $m-n$ of the degrees of the two polynomials $Q_m(\tilde{x})$ and $P_n(\tilde{x})$. It does not depend on the coefficients of the two polynomials, thus it does not depend on the number of the zeros of $P_n(\tilde{x})$ either.

* Actually, since $m \geq n+2$, $(m-n)(m-n+1) \geq 6$.

Therefore, all potentials (5) – regular or singular – with the same difference $m-n$ have the same symmetric asymptotic form $(m-n)(m-n+1)\hbar^2/2m_0{x_0}^2\tilde{x}^2$, plus a constant E_r, which can be set to zero choosing the infinity as reference point and $V(\pm\infty)=0$.

In any case, since $V(\pm\infty)=E_r$, the energy E_r is the highest bound energy of the potential (5).

Proof

Indeed, assuming that there exists a bound energy $E_r' > E_r$, then if $\varphi(x)$ is the respective eigenfunction

$$\varphi''(x)+\frac{2m_0}{\hbar^2}\left(E_r'-V(x)\right)\varphi(x)=0 \tag{7}$$

From (5), we see that if the polynomial $P_n(\tilde{x})$ has no zeros, the potential is continuous everywhere, while if it has, then if $x_{\max}=\max\{|x_i|\,|\,P_n(x_i)=0\}$, the potential is continuous for $|x|>x_{\max}$.

In any case, the potential is continuous at long distances, and then $V(x)\simeq V(\pm\infty)=E_r$, and thus from (7) we obtain

$$\varphi_\infty''(x)+\frac{2m_0}{\hbar^2}\left(E_r'-E_r\right)\varphi_\infty(x)=0$$

Then, since $E_r'-E_r>0$, $\varphi_\infty(x)\sim\exp\left(\pm i\sqrt{\frac{2m_0}{\hbar^2}\left(E_r'-E_r\right)}x\right)$.

Thus, the probability density $|\varphi_\infty(x)|^2$ is either a non-zero constant or it oscillates.

In either case, the probability density does not tend to zero, and thus the eigenstate $|\varphi\rangle$ is not bound.

Therefore, the wave function (4) describes the highest bound eigenstate and the energy E_r is the highest bound energy of the potential (5).

4. Distributing the zeros of the wave function as typical nodes or simple poles in the potential

From (3), we see that at each simple zero of the wave function $\psi(\tilde{x})$ which is not a zero of its second derivative, the potential has a simple pole, or equivalently, at each simple zero of the polynomial $P_n(\tilde{x})$ which is not a zero of the polynomial $Q_m(\tilde{x})P_n''(\tilde{x})-2Q_m'(\tilde{x})P_n'(\tilde{x})$, the potential (5) has a simple pole.

If at a zero of $\psi(\tilde{x})$ – equivalently of $P_n(\tilde{x})$ – the potential (5) has a simple pole, that zero is considered as a special zero, not as a typical node [8].

If $\psi(\tilde{x})$ – equivalently $P_n(\tilde{x})$ – has no zeros, then $\psi(\tilde{x})$ is the ground-state wave function [8, 9] of the potential (5), which is then regular and has only one bound state, of energy E_r. The potential in this case is a simple or multiple finite volcano.

If $\psi(\tilde{x})$ – equivalently $P_n(\tilde{x})$ – has one zero which is a simple pole of the potential (5), then $\psi(\tilde{x})$ is the ground-state wave function, while if the zero is not pole of the potential (5), $\psi(\tilde{x})$ is the first-excited-state wave function of the potential (5). In the first case, the potential is singular, as it has a simple pole, and it has only one bound state, of energy E_r, while in the second case, the potential is regular and has two bound eigenstates, the ground state and the first-excited state, with the energy of the second being E_r, and the potential in this case is a simple or multiple finite volcano.

In the general case, where $\psi(\tilde{x})$ – equivalently $P_n(\tilde{x})$ – has r zeros, with $r=0,1,...,n$, some of them may be typical nodes in the wave function and the others may be simple poles in the potential.

5. Examples of volcano potentials

The case m=4, n=1

We'll examine the case where $m=4$ and $n=1$. We remind that $m\geq n+2$ and even. For simplicity, we'll assume that $Q_4(\tilde{x})$ is symmetric, i.e.

$$Q_4(\tilde{x})=\tilde{x}^4+q_2\tilde{x}^2+q_0 \tag{8}$$

Since $n=1$,

$$P_1(\tilde{x})=\tilde{x}+p_0 \tag{9}$$

The polynomial (8) must have no zeros, and since it is monic, it must be positive for every $\tilde{x}\in\mathbb{R}$.

The non-normalized wave function (4) is then written as

$$\psi(\tilde{x}) = \frac{\tilde{x} + p_0}{\tilde{x}^4 + q_2\tilde{x}^2 + q_0} \tag{10}$$

Assuming that the potential vanishes at infinity, i.e. $V(\pm\infty) = 0$, yields $E_r = 0$.

The wave function has a zero, at $\tilde{x} = -p_0$, which can be a typical node in the wave function or a simple pole in the potential.

For $-p_0$ to be a typical node, it must be a zero of the polynomial $Q_4(\tilde{x})P_1''(\tilde{x}) - 2Q_4'(\tilde{x})P_1'(\tilde{x})$.

Since $P_1'(\tilde{x}) = 1$ and $P_1''(\tilde{x}) = 0$, the previous polynomial equals $-2Q_4'(\tilde{x})$, and thus $-p_0$ must be a zero of $Q_4'(\tilde{x})$, i.e. $Q_4'(-p_0) = 0$.

Using that $Q_4'(\tilde{x}) = 4\tilde{x}^3 + 2q_2\tilde{x} = 2\tilde{x}(2\tilde{x}^2 + q_2)$, we obtain

$$p_0(2p_0^2 + q_2) = 0$$

and thus

$$p_0 = 0 \text{ or } q_2 = -2p_0^2 \le 0$$

Every even-degree polynomial $R_{2n}(\tilde{x})$ with positive leading coefficient has a global minimum, i.e. $R_{2n}(\tilde{x}) \ge R_0$, for every $\tilde{x} \in \mathbb{R}$.

This happens because $R_{2n}(\tilde{x})$ is continuous and finite for every finite $\tilde{x}$, and it tends to plus infinity as $\tilde{x}$ tends to minus or plus infinity.

Then, if $\varepsilon > 0$, the polynomial $R_{2n}(\tilde{x}) - R_0 + \varepsilon$, which differs from $R_{2n}(\tilde{x})$ only in the constant term, is positive for every $\tilde{x} \in \mathbb{R}$.

Thus, if the constant term of an even-degree polynomial with positive leading coefficient is big enough, the polynomial is everywhere positive, no matter what its intermediate coefficients are.

Therefore, $Q_4(\tilde{x})$ can be positive even if q_2 is negative.

For $p_0 = 0$, the wave function (10) becomes

$$\psi(\tilde{x}) = \frac{\tilde{x}}{\tilde{x}^4 + q_2\tilde{x}^2 + q_0}$$

For $q_2 = -2p_0^{\,2}$, the wave function (10) becomes

$$\psi(\tilde{x}) = \frac{\tilde{x} + p_0}{\tilde{x}^4 - 2p_0^{\,2}\tilde{x}^2 + q_0}$$

Both wave functions describe the first-excited state of a respective finite volcano potential having only two bound eigenstates, the highest of which has zero energy. Observe that if $p_0 = 0$ and $q_2 = 0$, the two previous cases are merged.

In any other case, i.e. if $p_0 \neq 0$ <u>and</u> $q_2 \neq -2p_0^{\,2}$, the wave function (10) describes the ground state of a singular potential with only one bound state having zero energy.

Next, we'll examine the case where $p_0 = 0$ and $q_2 = 0$.

The wave function (10) then becomes

$$\psi(\tilde{x}) = \frac{\tilde{x}}{\tilde{x}^4 + q_0} \qquad (11)$$

with $q_0 > 0$, so that the denominator does not vanish.

For $p_0 = 0$, $P_1(\tilde{x}) = \tilde{x}$ and thus $P_1'(\tilde{x}) = 1$, $P_1''(\tilde{x}) = 0$.

For $q_2 = 0$, $Q_4(\tilde{x}) = \tilde{x}^4 + q_0$ and thus $Q_4'(\tilde{x}) = 4\tilde{x}^3$, $Q_4''(\tilde{x}) = 12\tilde{x}^2$.

Plugging into (5), we obtain, for $E_r = 0$,

$$V(\tilde{x}) = \frac{\hbar^2}{2m_0 x_0^{\,2}}\left(\frac{2\left(4\tilde{x}^3\right)^2 - \left(\tilde{x}^4 + q_0\right)12\tilde{x}^2}{\left(\tilde{x}^4 + q_0\right)^2} + \frac{-8\tilde{x}^3}{\left(\tilde{x}^4 + q_0\right)\tilde{x}}\right) =$$

$$= \frac{\hbar^2}{m_0 x_0^{\,2}}\left(\frac{\left(4\tilde{x}^3\right)^2 - \left(\tilde{x}^4 + q_0\right)6\tilde{x}^2}{\left(\tilde{x}^4 + q_0\right)^2} - \frac{4\tilde{x}^3}{\left(\tilde{x}^4 + q_0\right)\tilde{x}}\right) = \frac{\hbar^2}{m_0 x_0^{\,2}}\left(\frac{16\tilde{x}^6 - 6\tilde{x}^6 - 6q_0\tilde{x}^2}{\left(\tilde{x}^4 + q_0\right)^2} - \frac{4\tilde{x}^2}{\tilde{x}^4 + q_0}\right) =$$

$$= \frac{\hbar^2}{m_0 x_0^{\,2}}\left(\frac{10\tilde{x}^6 - 6q_0\tilde{x}^2}{\left(\tilde{x}^4 + q_0\right)^2} - \frac{4\tilde{x}^2}{\tilde{x}^4 + q_0}\right) = \frac{2\hbar^2}{m_0 x_0^{\,2}}\left(\frac{5\tilde{x}^6 - 3q_0\tilde{x}^2}{\left(\tilde{x}^4 + q_0\right)^2} - \frac{2\tilde{x}^2}{\tilde{x}^4 + q_0}\right) =$$

$$= \frac{2\hbar^2}{m_0 x_0^{\,2}}\frac{5\tilde{x}^6 - 3q_0\tilde{x}^2 - 2\tilde{x}^2\left(\tilde{x}^4 + q_0\right)}{\left(\tilde{x}^4 + q_0\right)^2} = \frac{2\hbar^2}{m_0 x_0^{\,2}}\frac{3\tilde{x}^6 - 5q_0\tilde{x}^2}{\left(\tilde{x}^4 + q_0\right)^2}$$

That is

$$V(\tilde{x}) = \frac{2\hbar^2}{m_0 x_0^{\,2}}\frac{3\tilde{x}^6 - 5q_0\tilde{x}^2}{\left(\tilde{x}^4 + q_0\right)^2} \qquad (12)$$

The potential (12) is symmetric as a result of the wave function (11) having definite (odd) parity.

> If an eigenfunction has definite parity, its second derivative has the same parity, then the ratio ψ''/ψ is an even-parity function, and thus the resulting potential is also of even parity, i.e. symmetric.

At long distances, the potential (12) becomes $V_\infty\left(\tilde{x}\right)=6\hbar^2/m_0{x_0}^2\tilde{x}^2$, in agreement with (6) for $m-n=3$ and $E_r=0$.

The potential (12) vanishes at

$$3\tilde{x}^6-5q_0\tilde{x}^2=0$$

Thus

$$\tilde{x}=0 \text{ (double zero)}, \ \tilde{x}=\pm\sqrt[4]{\frac{5q_0}{3}}$$

The derivative of the potential (12) is – omitting the factor $2\hbar^2/m_0{x_0}^2$ which does not affect the sign of the derivative –

$$V'\left(\tilde{x}\right)=\frac{18\tilde{x}^5-10q_0\tilde{x}}{\left(\tilde{x}^4+q_0\right)^2}-2\frac{\left(3\tilde{x}^6-5q_0\tilde{x}^2\right)4\tilde{x}^3}{\left(\tilde{x}^4+q_0\right)^3}=\frac{2}{\left(\tilde{x}^4+q_0\right)^2}\left(9\tilde{x}^5-5q_0\tilde{x}-\frac{12\tilde{x}^9-20q_0\tilde{x}^5}{\tilde{x}^4+q_0}\right)=$$

$$=\frac{2}{\left(\tilde{x}^4+q_0\right)^3}\left(\left(9\tilde{x}^5-5q_0\tilde{x}\right)\left(\tilde{x}^4+q_0\right)-12\tilde{x}^9+20q_0\tilde{x}^5\right)=$$

$$=\frac{2}{\left(\tilde{x}^4+q_0\right)^3}\left(9\tilde{x}^9+9q_0\tilde{x}^5-5q_0\tilde{x}^5-5{q_0}^2\tilde{x}-12\tilde{x}^9+20q_0\tilde{x}^5\right)=$$

$$=\frac{2}{\left(\tilde{x}^4+q_0\right)^3}\left(-3\tilde{x}^9+24q_0\tilde{x}^5-5{q_0}^2\tilde{x}\right)=\frac{2}{\left(\tilde{x}^4+q_0\right)^3}\tilde{x}\left(-3\tilde{x}^8+24q_0\tilde{x}^4-5{q_0}^2\right)=$$

$$=-\frac{2\tilde{x}}{\left(\tilde{x}^4+q_0\right)^3}\left(3\tilde{x}^8-24q_0\tilde{x}^4+5{q_0}^2\right)$$

That is

$$V'\left(\tilde{x}\right)=-\frac{2\tilde{x}}{\left(\tilde{x}^4+q_0\right)^3}\left(3\tilde{x}^8-24q_0\tilde{x}^4+5{q_0}^2\right)$$

Setting $y=\tilde{x}^4$, the trinomial $3\tilde{x}^8-24q_0\tilde{x}^4+5{q_0}^2$ is written as

$3y^2 - 24q_0 y + 5q_0{}^2$

Its discriminant is $576q_0{}^2 - 60q_0{}^2 = 516q_0{}^2 > 0$, thus it has two zeros, at

$$y_{1,2} = \frac{24q_0 \pm \sqrt{516q_0{}^2}}{6} \simeq \frac{24q_0 \pm 22.72q_0}{6} \simeq 4q_0 \pm 3.79q_0 = 0.21q_0,\ 7.79q_0$$

Since both zeros are positive,

$$\tilde{x}^4 = y_{1,2} \Rightarrow \tilde{x}^2 = \sqrt{y_{1,2}} \Rightarrow \tilde{x} = \pm\sqrt[4]{y_{1,2}},\ y_{1,2} \simeq 0.21q_0,\ 7.79q_0$$

The trinomial $3y^2 - 24q_0 y + 5q_0{}^2$ is then positive for $y < y_1$ or $y > y_2$, and negative for $y_1 < y < y_2$.

Thus, the trinomial $3\tilde{x}^8 - 24q_0\tilde{x}^4 + 5q_0{}^2$ is positive for

$$\tilde{x}^4 < y_1 \Rightarrow \tilde{x}^2 < \sqrt{y_1} \Rightarrow |\tilde{x}| < \sqrt[4]{y_1} \Rightarrow -\sqrt[4]{y_1} < \tilde{x} < \sqrt[4]{y_1}$$

or

$$\tilde{x}^4 > y_2 \Rightarrow \tilde{x}^2 > \sqrt{y_2} \Rightarrow |\tilde{x}| > \sqrt[4]{y_2} \Rightarrow \tilde{x} < -\sqrt[4]{y_2} \text{ or } \tilde{x} > \sqrt[4]{y_2}$$

and it is negative for

$$-\sqrt[4]{y_2} < \tilde{x} < -\sqrt[4]{y_1} \text{ or } \sqrt[4]{y_1} < \tilde{x} < \sqrt[4]{y_2}$$

Then, we have

For $\tilde{x} < -\sqrt[4]{y_2}$, $V'(\tilde{x}) > 0$ and the potential (12) is strictly increasing.

For $-\sqrt[4]{y_2} < \tilde{x} < -\sqrt[4]{y_1}$, $V'(\tilde{x}) < 0$ and the potential (12) is strictly decreasing.

For $-\sqrt[4]{y_1} < \tilde{x} < 0$, $V'(\tilde{x}) > 0$ and the potential (12) is strictly increasing.

For $0 < \tilde{x} < \sqrt[4]{y_1}$, $V'(\tilde{x}) < 0$ and the potential (12) is strictly decreasing.

For $\sqrt[4]{y_1} < \tilde{x} < \sqrt[4]{y_2}$, $V'(\tilde{x}) > 0$ and the potential (12) is strictly increasing.

For $\tilde{x} > \sqrt[4]{y_2}$, $V'(\tilde{x}) < 0$ and the potential (12) is strictly decreasing.

Finally, at $\pm\sqrt[4]{y_{1,2}}$, $V'(\tilde{x}) = 0$.

Therefore, at $-\sqrt[4]{y_2}, 0, \sqrt[4]{y_2}$ the potential has local maxima, while at $-\sqrt[4]{y_1}, \sqrt[4]{y_1}$ it has local minima.

Since the potential is symmetric, its values at the two symmetric minima are equal, as they are at the two non-zero symmetric maxima too.

The potential (12) is then a symmetric double finite volcano that, at long distances, decays as $1/\tilde{x}^2$. It has only two bound eigenstates, a ground state and a first-excited state, with the energy of the second being zero.

In Figure 1, the potential (12) is plotted for $q_0 = 0.1$ (red line), $q_0 = 1$ (blue line), and $q_0 = 10$ (green line), in units $2\hbar^2/m_0 x_0{}^2 = 1$.

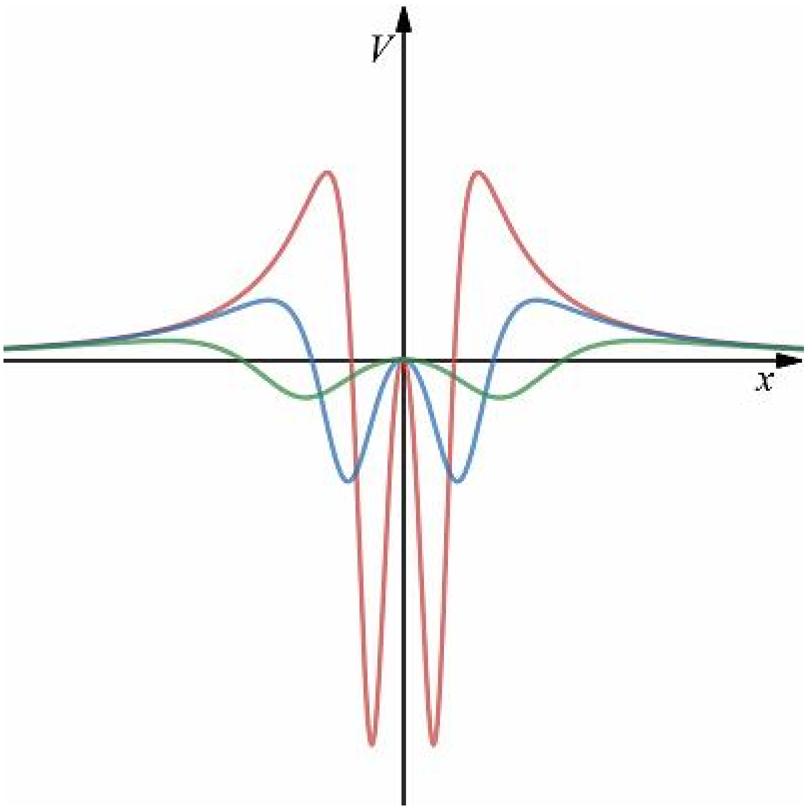

Figure 1

We see that, as q_0 increases, the minimum value of the potential increases, approaching zero from below.

The value of the potential at the two minima is

$$V\left(\pm\sqrt[4]{y_1}\right) = \frac{2\hbar^2}{m_0 x_0{}^2} \frac{3\left(\pm\sqrt[4]{y_1}\right)^6 - 5q_0\left(\pm\sqrt[4]{y_1}\right)^2}{\left(\left(\pm\sqrt[4]{y_1}\right)^4 + q_0\right)^2} = \frac{2\hbar^2}{m_0 x_0{}^2} \frac{3\left(\sqrt{y_1}\right)^3 - 5q_0\sqrt{y_1}}{\left(y_1 + q_0\right)^2} =$$

$$= \frac{2\hbar^2}{m_0 x_0{}^2} \frac{3y_1\sqrt{y_1} - 5q_0\sqrt{y_1}}{\left(y_1 + q_0\right)^2} = \frac{2\hbar^2}{m_0 x_0{}^2} \frac{\left(3y_1 - 5q_0\right)\sqrt{y_1}}{\left(y_1 + q_0\right)^2}$$

Thus

$$V\left(\pm\sqrt[4]{y_1}\right)=\frac{2\hbar^2}{m_0{x_0}^2}\frac{\left(3y_1-5q_0\right)\sqrt{y_1}}{\left(y_1+q_0\right)^2} \tag{13}$$

Substituting $y_1 \simeq 0.21q_0$, we obtain

$$V\left(\pm\sqrt[4]{0.21q_0}\right)\simeq\frac{2\hbar^2}{m_0{x_0}^2}\frac{\left(0.63q_0-5q_0\right)\sqrt{0.21q_0}}{\left(1.21q_0\right)^2}\simeq-\frac{2\hbar^2}{m_0{x_0}^2}\frac{4.37q_0 0.46\sqrt{q_0}}{1.46{q_0}^2}\simeq-\frac{2.75\hbar^2}{m_0{x_0}^2\sqrt{q_0}}$$

The minimum value of the potential is then $-2.75\hbar^2\big/m_0{x_0}^2\sqrt{q_0}$, and it increases as $-1\big/\sqrt{q_0}$, as q_0 increases.

Since the ground-state energy must exceed the minimum value of the potential [10], it will also approach zero from below, as q_0 increases.

We expect that, as $q_0 \to \infty$, the ground-state energy will approach zero as $-1\big/\sqrt{q_0}$ or faster, otherwise the minimum value of the potential will exceed the ground-state energy.

On the other hand, the first-excited-state energy does not depend on q_0, it is always zero.

Then, as q_0 increases, the difference between the two bound energies decreases, the ground-state energy comes closer to the first-excited-state energy.

Similarly to (13), the value of the potential at the two local maxima $\pm\sqrt[4]{y_2}$ is

$$V\left(\pm\sqrt[4]{y_2}\right)=\frac{2\hbar^2}{m_0{x_0}^2}\frac{\left(3y_2-5q_0\right)\sqrt{y_2}}{\left(y_2+q_0\right)^2}$$

Substituting $y_2 \simeq 7.79q_0$, we obtain

$$V\left(\pm\sqrt[4]{7.79q_0}\right)\simeq\frac{2\hbar^2}{m_0{x_0}^2}\frac{18.37q_0 2.79\sqrt{q_0}}{77.26{q_0}^2}\simeq\frac{1.33\hbar^2}{m_0{x_0}^2\sqrt{q_0}}$$

Since $1.33\hbar^2\big/m_0{x_0}^2\sqrt{q_0}>0$, $1.33\hbar^2\big/m_0{x_0}^2\sqrt{q_0}$ is the global maximum of the potential, and thus it is the height of the two volcanoes.

The depth of each of the two volcanoes is then

$$\frac{1.33\hbar^2}{m_0 x_0{}^2 \sqrt{q_0}} - \left(-\frac{2.75\hbar^2}{m_0 x_0{}^2 \sqrt{q_0}} \right) = \frac{4.08\hbar^2}{m_0 x_0{}^2 \sqrt{q_0}}$$

and it goes to zero as $1/\sqrt{q_0}$, when $q_0 \to \infty$.

The width of each volcano is $\sqrt[4]{7.79 q_0}$ (see Figure 1), thus it increases and goes to infinity as $\sqrt[4]{q_0}$, when $q_0 \to \infty$.

As q_0 increases, the two volcanoes become shallower and wider and as $q_0 \to \infty$, the height of the two volcanoes vanishes, while their width becomes infinite, i.e. the volcanoes decay.

Transition from a singular to a regular potential, in a m=4, n=1 case

We showed that if $p_0 \neq 0$ and $q_2 \neq -2p_0{}^2$, the resulting potential has a simple pole at p_0.

For $p_0 \neq 0$ and $q_2 = 0$, the two previous conditions are satisfied and we have the case of a singular potential, with a simple pole at p_0.

In this case, we have

$$P_1(\tilde{x}) = \tilde{x} + p_0$$

$$Q_4(\tilde{x}) = \tilde{x}^4 + q_0$$

with $q_0 > 0$, so that $Q_4(\tilde{x}) \neq 0$.

For simplicity, we'll further assume that $q_0 = 1$.

Then

$$P_1'(\tilde{x}) = 1,\ P_1''(\tilde{x}) = 0$$

and

$$Q_4'(\tilde{x}) = 4\tilde{x}^3,\ Q_4''(\tilde{x}) = 12\tilde{x}^2$$

Plugging into (5), we obtain, for $E_r = 0$,

$$V(\tilde{x}) = \frac{\hbar^2}{2m_0 x_0{}^2}\left(\frac{2\left(4\tilde{x}^3\right)^2 - \left(\tilde{x}^4+1\right)12\tilde{x}^2}{\left(\tilde{x}^4+1\right)^2} + \frac{-8\tilde{x}^3}{\left(\tilde{x}^4+1\right)\left(\tilde{x}+p_0\right)}\right) =$$

$$= \frac{\hbar^2}{2m_0 x_0{}^2}\left(\frac{32\tilde{x}^6 - 12\tilde{x}^6 - 12\tilde{x}^2}{\left(\tilde{x}^4+1\right)^2} + \frac{-8\tilde{x}^3}{\left(\tilde{x}^4+1\right)\left(\tilde{x}+p_0\right)}\right) = \frac{\hbar^2}{2m_0 x_0{}^2}\left(\frac{20\tilde{x}^6 - 12\tilde{x}^2}{\left(\tilde{x}^4+1\right)^2} - \frac{8\tilde{x}^3}{\left(\tilde{x}^4+1\right)\left(\tilde{x}+p_0\right)}\right) =$$

$$= \frac{\hbar^2}{m_0 x_0{}^2}\left(\frac{10\tilde{x}^6 - 6\tilde{x}^2}{\left(\tilde{x}^4+1\right)^2} - \frac{4\tilde{x}^3}{\left(\tilde{x}^4+1\right)\left(\tilde{x}+p_0\right)}\right) = \frac{2\hbar^2}{m_0 x_0{}^2}\left(\frac{5\tilde{x}^6 - 3\tilde{x}^2}{\left(\tilde{x}^4+1\right)^2} - \frac{2\tilde{x}^3}{\left(\tilde{x}^4+1\right)\left(\tilde{x}+p_0\right)}\right) =$$

$$= \frac{2\hbar^2}{m_0 x_0{}^2}\frac{\left(5\tilde{x}^6 - 3\tilde{x}^2\right)\left(\tilde{x}+p_0\right) - 2\tilde{x}^3\left(\tilde{x}^4+1\right)}{\left(\tilde{x}^4+1\right)^2\left(\tilde{x}+p_0\right)} = \frac{2\hbar^2}{m_0 x_0{}^2}\frac{5\tilde{x}^7 + 5p_0\tilde{x}^6 - 3\tilde{x}^3 - 3p_0\tilde{x}^2 - 2\tilde{x}^7 - 2\tilde{x}^3}{\left(\tilde{x}^4+1\right)^2\left(\tilde{x}+p_0\right)} =$$

$$= \frac{2\hbar^2}{m_0 x_0{}^2}\frac{3\tilde{x}^7 + 5p_0\tilde{x}^6 - 5\tilde{x}^3 - 3p_0\tilde{x}^2}{\left(\tilde{x}^4+1\right)^2\left(\tilde{x}+p_0\right)}$$

That is, in units $2\hbar^2 / m_0 x_0{}^2 = 1$, the potential is

$$V(\tilde{x}) = \frac{3\tilde{x}^7 + 5p_0\tilde{x}^6 - 5\tilde{x}^3 - 3p_0\tilde{x}^2}{\left(\tilde{x}^4+1\right)^2\left(\tilde{x}+p_0\right)} \qquad (14)$$

In Figure 2, the potential (14) is plotted for $p_0 = -1$ (red line), $p_0 = 0$ (blue line), and $p_0 = 1$ (green line).

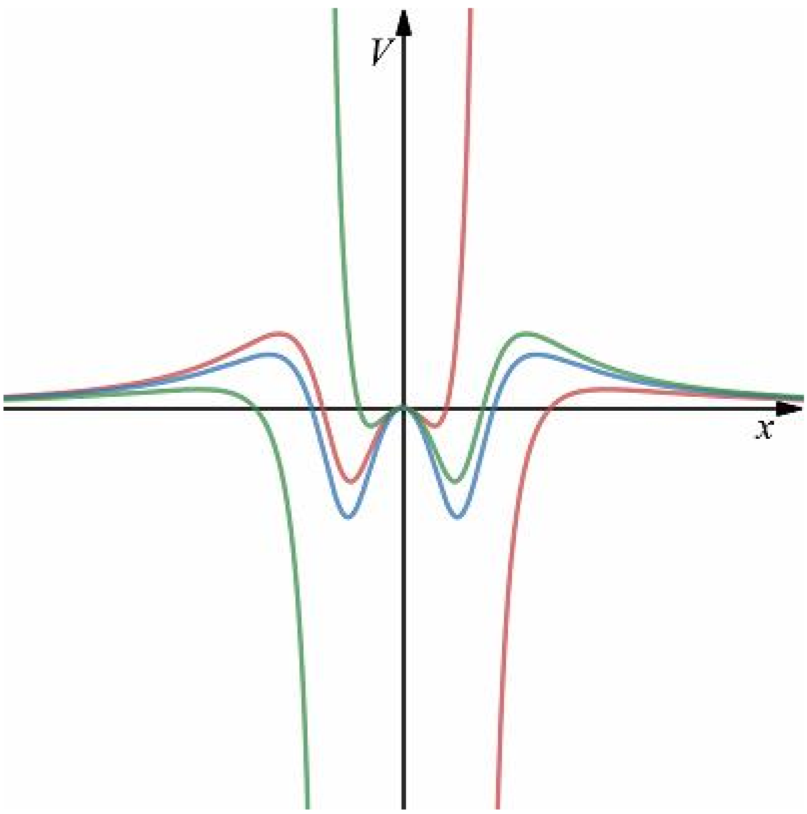

Figure 2

When the parameter p_0 takes the critical value $p_0 = 0$, the two open edges of the potential pole, one at plus/minus infinity and the other at minus/plus, are "glued" together forming a volcano which is added to the existing volcano, resulting in a symmetric double volcano, and at the same time, one more bound eigenstate is created, with zero energy.

A similar formation of a volcano takes place at every value of $n \le m-2$, with $m \ge 2$ and even, when the coefficients of the polynomial $Q_m(\tilde{x})P_n''(\tilde{x}) - 2Q_m'(\tilde{x})P_n'(\tilde{x})$ take such values that one of the zeros of the previous polynomial becomes equal to one of the zeros of the polynomial $P_n(\tilde{x})$, but the correspondence between the zeros of $P_n(\tilde{x})$ and the finally formed volcanoes is not one-to-one, as we'll show below.

If the polynomial $P_n(\tilde{x})$ has n zeros and the coefficients of the polynomial $Q_m(\tilde{x})P_n''(\tilde{x}) - 2Q_m'(\tilde{x})P_n'(\tilde{x})$ can take such values that there exists an $(m-2)$-degree polynomial $R_{m-2}(\tilde{x})$ such that

$$Q_m(\tilde{x})P_n''(\tilde{x}) - 2Q_m'(\tilde{x})P_n'(\tilde{x}) = R_{m-2}(\tilde{x})P_n(\tilde{x}) \tag{15}$$

with $Q_m(\tilde{x}) \ne 0$, then we can transform the n simple poles of the initial potential, which has only one bound eigenstate, with zero energy, into a multiple finite volcano with n bound eigenstates, with the highest of them, i.e. the nth-excited state, having zero energy.

By means of (15), the potential (5) is written as, for $E_r = 0$,

$$V(\tilde{x}) = \frac{\hbar^2}{2m_0 x_0^{\,2}} \left(\frac{2Q_m'^2(\tilde{x}) - Q_m(\tilde{x})Q_m''(\tilde{x})}{Q_m^{\,2}(\tilde{x})} + \frac{R_{m-2}(\tilde{x})}{Q_m(\tilde{x})} \right) \tag{16}$$

with $Q_m(\tilde{x}) \ne 0$.

A double finite volcano with four bound eigenstates, in a m=6, n=3 case

We'll consider a case where $P_3(\tilde{x})$ is of odd parity and $Q_6(\tilde{x})$ is of even parity, i.e.

$P_3(\tilde{x}) = \tilde{x}^3 - \tilde{x}$

$$Q_6(\tilde{x}) = \tilde{x}^6 + q_4\tilde{x}^4 + q_2\tilde{x}^2 + q_0$$

with $q_0 > 0$, otherwise $Q_6(\tilde{x})$ has zeros.

Then, $P_3''(\tilde{x})$ is of odd parity, thus $Q_6(\tilde{x})P_3''(\tilde{x})$ is also of odd parity, while $P_3'(\tilde{x})$ is of even parity and $Q_6'(\tilde{x})$ is of odd parity, thus $Q_6'(\tilde{x})P_3'(\tilde{x})$ is again of odd parity, and finally, $Q_6(\tilde{x})P_3''(\tilde{x}) - 2Q_6'(\tilde{x})P_3'(\tilde{x})$ is of odd parity.

Then, since $P_3(\tilde{x})$ is of odd parity, $R_4(\tilde{x})$ in (15) must be of even parity, and thus it has the form

$$R_4(\tilde{x}) = r_4\tilde{x}^4 + r_2\tilde{x}^2 + r_0$$

Plugging into (15), we obtain

$$\begin{aligned}
&\left(\tilde{x}^6 + q_4\tilde{x}^4 + q_2\tilde{x}^2 + q_0\right)6\tilde{x} - 2\left(6\tilde{x}^5 + 4q_4\tilde{x}^3 + 2q_2\tilde{x}\right)\left(3\tilde{x}^2 - 1\right) = \left(r_4\tilde{x}^4 + r_2\tilde{x}^2 + r_0\right)\left(\tilde{x}^3 - \tilde{x}\right) \Rightarrow \\
&\Rightarrow 6\tilde{x}^7 + 6q_4\tilde{x}^5 + 6q_2\tilde{x}^3 + 6q_0\tilde{x} - 2\left(18\tilde{x}^7 - 6\tilde{x}^5 + 12q_4\tilde{x}^5 - 4q_4\tilde{x}^3 + 6q_2\tilde{x}^3 - 2q_2\tilde{x}\right) = \\
&= r_4\tilde{x}^7 - r_4\tilde{x}^5 + r_2\tilde{x}^5 - r_2\tilde{x}^3 + r_0\tilde{x}^3 - r_0\tilde{x} \Rightarrow \\
&\Rightarrow 6\tilde{x}^7 + 6q_4\tilde{x}^5 + 6q_2\tilde{x}^3 + 6q_0\tilde{x} - 36\tilde{x}^7 + 12\tilde{x}^5 - 24q_4\tilde{x}^5 + 8q_4\tilde{x}^3 - 12q_2\tilde{x}^3 + 4q_2\tilde{x} = \\
&= r_4\tilde{x}^7 - r_4\tilde{x}^5 + r_2\tilde{x}^5 - r_2\tilde{x}^3 + r_0\tilde{x}^3 - r_0\tilde{x} \Rightarrow \\
&\Rightarrow -30\tilde{x}^7 - 18q_4\tilde{x}^5 - 6q_2\tilde{x}^3 + 6q_0\tilde{x} + 12\tilde{x}^5 + 8q_4\tilde{x}^3 + 4q_2\tilde{x} = r_4\tilde{x}^7 + \left(r_2 - r_4\right)\tilde{x}^5 + \left(r_0 - r_2\right)\tilde{x}^3 - r_0\tilde{x} \Rightarrow \\
&\Rightarrow -30\tilde{x}^7 + 6\left(2 - 3q_4\right)\tilde{x}^5 + 2\left(4q_4 - 3q_2\right)\tilde{x}^3 + 2\left(3q_0 + 2q_2\right)\tilde{x} = r_4\tilde{x}^7 + \left(r_2 - r_4\right)\tilde{x}^5 + \left(r_0 - r_2\right)\tilde{x}^3 - r_0\tilde{x}
\end{aligned}$$

Comparing the same-degree terms in both sides of the last equation, we obtain

$$r_4 = -30 \tag{17}$$

$$r_2 - r_4 = 6\left(2 - 3q_4\right) \tag{18}$$

$$r_0 - r_2 = 2\left(4q_4 - 3q_2\right) \tag{19}$$

$$-r_0 = 2\left(3q_0 + 2q_2\right) \tag{20}$$

Substituting (17) into (18) yields

$$r_2 + 30 = 6\left(2 - 3q_4\right) = 12 - 18q_4 \Rightarrow 18q_4 = -18 - r_2 \Rightarrow q_4 = -1 - \frac{r_2}{18}$$

Substituting q_4 into (19) yields

$$r_0 - r_2 = 2\left(4\left(-1-\frac{r_2}{18}\right)-3q_2\right) = 2\left(-4-\frac{2r_2}{9}-3q_2\right) = -8-\frac{4r_2}{9}-6q_2 \Rightarrow$$

$$\Rightarrow 6q_2 = -8-\frac{4r_2}{9}-r_0+r_2 = -8+\frac{5r_2}{9}-r_0 \Rightarrow q_2 = -\frac{4}{3}+\frac{5r_2}{54}-\frac{r_0}{6}$$

Substituting q_2 into (20) yields

$$-r_0 = 2\left(3q_0+2\left(-\frac{4}{3}+\frac{5r_2}{54}-\frac{r_0}{6}\right)\right) = 2\left(3q_0-\frac{8}{3}+\frac{5r_2}{27}-\frac{r_0}{3}\right) = 6q_0-\frac{16}{3}+\frac{10r_2}{27}-\frac{2r_0}{3} \Rightarrow$$

$$\Rightarrow 6q_0 = -r_0+\frac{16}{3}-\frac{10r_2}{27}+\frac{2r_0}{3} = \frac{16}{3}-\frac{10r_2}{27}-\frac{r_0}{3} \Rightarrow q_0 = \frac{8}{9}-\frac{5r_2}{81}-\frac{r_0}{18}$$

If the intermediate coefficients q_2 and q_4 of $Q_6\left(\tilde{x}\right)$ are non-negative, and the constant term q_0 is positive, $Q_6\left(\tilde{x}\right)$ has no zeros.

For q_4 to be non-negative,

$$-1-\frac{r_2}{18} \geq 0 \Rightarrow \frac{r_2}{18} \leq -1 \Rightarrow r_2 \leq -18$$

For q_2 to be non-negative,

$$-\frac{4}{3}+\frac{5r_2}{54}-\frac{r_0}{6} \geq 0 \Rightarrow \frac{r_0}{6} \leq -\frac{4}{3}+\frac{5r_2}{54} \Rightarrow r_0 \leq -8+\frac{5r_2}{9}$$

For q_0 to be positive,

$$\frac{8}{9}-\frac{5r_2}{81}-\frac{r_0}{18} > 0 \Rightarrow \frac{r_0}{18} < \frac{8}{9}-\frac{5r_2}{81} \Rightarrow r_0 < 16-\frac{10r_2}{9}$$

We observe that if r_0 is enough negative and $r_2 \leq -18$, all three inequalities are satisfied.

In the limiting case, where

$$r_2 = -18 \text{ and } r_0 = -8+\frac{5(-18)}{9} = -8-10 = -18,$$

the third inequality holds, since

$$-18 < 16-\frac{10(-18)}{9} \Rightarrow -18 < 16+20 = 36 \text{ (it holds)}$$

Then

$$q_0 = \frac{8}{9} - \frac{5\left(-18\right)}{81} - \frac{-18}{18} = \frac{8}{9} + \frac{10}{9} + 1 = 3$$

and

$$q_2 = q_4 = 0$$

Thus

$$Q_6\left(\tilde{x}\right) = \tilde{x}^6 + 3$$

and

$$R_4\left(\tilde{x}\right) = -30\tilde{x}^4 - 18\tilde{x}^2 - 18$$

Substituting into (16) yields

$$V\left(\tilde{x}\right) = \frac{\hbar^2}{2m_0{x_0}^2}\left(\frac{2\left(6\tilde{x}^5\right)^2 - \left(\tilde{x}^6+3\right)30\tilde{x}^4}{\left(\tilde{x}^6+3\right)^2} + \frac{-30\tilde{x}^4 - 18\tilde{x}^2 - 18}{\left(\tilde{x}^6+3\right)}\right) =$$

$$= \frac{\hbar^2}{m_0{x_0}^2}\left(\frac{\left(6\tilde{x}^5\right)^2 - \left(\tilde{x}^6+3\right)15\tilde{x}^4}{\left(\tilde{x}^6+3\right)^2} - \frac{15\tilde{x}^4 + 9\tilde{x}^2 + 9}{\left(\tilde{x}^6+3\right)}\right) =$$

$$= \frac{\hbar^2}{m_0{x_0}^2}\frac{36\tilde{x}^{10} - 15\tilde{x}^{10} - 45\tilde{x}^4 - \left(15\tilde{x}^4 + 9\tilde{x}^2 + 9\right)\left(\tilde{x}^6+3\right)}{\left(\tilde{x}^6+3\right)^2} =$$

$$= \frac{\hbar^2}{m_0{x_0}^2}\frac{36\tilde{x}^{10} - 15\tilde{x}^{10} - 45\tilde{x}^4 - 15\tilde{x}^{10} - 45\tilde{x}^4 - 9\tilde{x}^8 - 27\tilde{x}^2 - 9\tilde{x}^6 - 27}{\left(\tilde{x}^6+3\right)^2} =$$

$$= \frac{\hbar^2}{m_0{x_0}^2}\frac{6\tilde{x}^{10} - 90\tilde{x}^4 - 9\tilde{x}^8 - 27\tilde{x}^2 - 9\tilde{x}^6 - 27}{\left(\tilde{x}^6+3\right)^2} =$$

$$= \frac{\hbar^2}{m_0{x_0}^2}\frac{6\tilde{x}^{10} - 9\tilde{x}^8 - 9\tilde{x}^6 - 90\tilde{x}^4 - 27\tilde{x}^2 - 27}{\left(\tilde{x}^6+3\right)^2}$$

That is

$$V\left(\tilde{x}\right) = \frac{3\hbar^2}{m_0{x_0}^2}\frac{2\tilde{x}^{10} - 3\tilde{x}^8 - 3\tilde{x}^6 - 30\tilde{x}^4 - 9\tilde{x}^2 - 9}{\left(\tilde{x}^6+3\right)^2} \qquad (21)$$

At long distances, the potential (21) takes the form

$$V_\infty\left(\tilde{x}\right) = \frac{3\hbar^2}{m_0{x_0}^2}\frac{2\tilde{x}^{10}}{\tilde{x}^{12}} = \frac{6\hbar^2}{m_0{x_0}^2}\frac{1}{\tilde{x}^2}$$

in agreement with the general expression (6) for $m-n=3$ and $E_r=0$.

In Figure 3, the potential (21) is plotted in units $3\hbar^2/m_0 x_0^{\ 2}=1$. It is a symmetric double finite volcano potential with four bound eigenstates, the highest of which, i.e. the third-excited state, has zero energy and is described by the wave function $\psi(\tilde{x})=A\,\tilde{x}(\tilde{x}^2-1)/(\tilde{x}^6+3)$, where A is the normalization constant.

We see that, although the polynomial $P_3(\tilde{x})=\tilde{x}^3-\tilde{x}$ has three zeros, the potential is still a double and not a triple volcano.

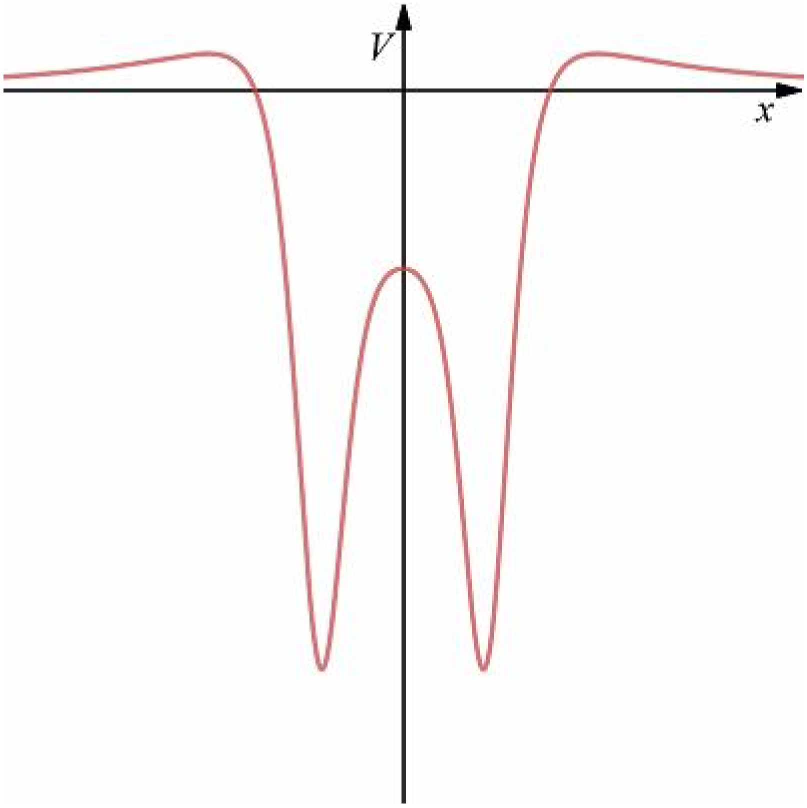

Figure 3

A triple finite volcano with five bound eigenstates, in a m=6, n=4 case

We'll consider a case where

$$P_4(\tilde{x})=(\tilde{x}^2-1)(\tilde{x}^2-2)=\tilde{x}^4-3\tilde{x}^2+2$$

$$Q_6(\tilde{x})=\tilde{x}^6+q_4\tilde{x}^4+q_2\tilde{x}^2+q_0$$

with $q_0>0$, otherwise $Q_6(\tilde{x})$ has zeros.

Both polynomials are of even parity, thus $Q_6\left(\tilde{x}\right)P_4''\left(\tilde{x}\right)-2Q_6'\left(\tilde{x}\right)P_4'\left(\tilde{x}\right)$ is now of even parity, and since $P_4\left(\tilde{x}\right)$ is also of even parity, $R_4\left(\tilde{x}\right)$ must be of even parity too, and thus it has the form

$$R_4\left(\tilde{x}\right)=r_4\tilde{x}^4+r_2\tilde{x}^2+r_0$$

> If $Q_m\left(\tilde{x}\right)$ is of even parity and $P_n\left(\tilde{x}\right)$ has definite parity, i.e. it is of even or odd parity, then $P_n''\left(\tilde{x}\right)$ has the same parity as $P_n\left(\tilde{x}\right)$, and thus $Q_m\left(\tilde{x}\right)P_n''\left(\tilde{x}\right)$ has also the same parity as $P_n\left(\tilde{x}\right)$, while $P_n'\left(\tilde{x}\right)$ has different parity from that of $P_n\left(\tilde{x}\right)$, and since $Q_m'\left(\tilde{x}\right)$ is of odd parity, $Q_m'\left(\tilde{x}\right)P_n'\left(\tilde{x}\right)$ has the same parity as $P_n\left(\tilde{x}\right)$, and thus the polynomial $Q_m\left(\tilde{x}\right)P_n''\left(\tilde{x}\right)-2Q_m'\left(\tilde{x}\right)P_n'\left(\tilde{x}\right)$ has the same parity as $P_n\left(\tilde{x}\right)$, and then from (15) we derive that $R_{m-2}\left(\tilde{x}\right)$ must be of even parity.
> That is, if $Q_m\left(\tilde{x}\right)$ is of even parity and $P_n\left(\tilde{x}\right)$ has definite parity, $R_{m-2}\left(\tilde{x}\right)$ is of even parity.

Plugging into (15), we obtain

$$\begin{aligned}
&\left(\tilde{x}^6+q_4\tilde{x}^4+q_2\tilde{x}^2+q_0\right)\left(12\tilde{x}^2-6\right)-2\left(6\tilde{x}^5+4q_4\tilde{x}^3+2q_2\tilde{x}\right)\left(4\tilde{x}^3-6\tilde{x}\right)=\\
&=\left(r_4\tilde{x}^4+r_2\tilde{x}^2+r_0\right)\left(\tilde{x}^4-3\tilde{x}^2+2\right)\Rightarrow\\
&\Rightarrow 12\tilde{x}^8-6\tilde{x}^6+12q_4\tilde{x}^6-6q_4\tilde{x}^4+12q_2\tilde{x}^4-6q_2\tilde{x}^2+12q_0\tilde{x}^2-6q_0-\\
&-2\left(24\tilde{x}^8-36\tilde{x}^6+16q_4\tilde{x}^6-24q_4\tilde{x}^4+8q_2\tilde{x}^4-12q_2\tilde{x}^2\right)=r_4\tilde{x}^8-3r_4\tilde{x}^6+2r_4\tilde{x}^4+r_2\tilde{x}^6-3r_2\tilde{x}^4+2r_2\tilde{x}^2+\\
&+r_0\tilde{x}^4-3r_0\tilde{x}^2+2r_0\Rightarrow 12\tilde{x}^8-6\tilde{x}^6+12q_4\tilde{x}^6-6q_4\tilde{x}^4+12q_2\tilde{x}^4-6q_2\tilde{x}^2+12q_0\tilde{x}^2-6q_0-\\
&-48\tilde{x}^8+72\tilde{x}^6-32q_4\tilde{x}^6+48q_4\tilde{x}^4-16q_2\tilde{x}^4+24q_2\tilde{x}^2=r_4\tilde{x}^8+\left(r_2-3r_4\right)\tilde{x}^6+\left(2r_4-3r_2+r_0\right)\tilde{x}^4+\\
&+\left(2r_2-3r_0\right)\tilde{x}^2+2r_0\Rightarrow\\
&\Rightarrow -36\tilde{x}^8+66\tilde{x}^6-20q_4\tilde{x}^6+42q_4\tilde{x}^4-4q_2\tilde{x}^4+18q_2\tilde{x}^2+12q_0\tilde{x}^2-6q_0=\\
&=r_4\tilde{x}^8+\left(r_2-3r_4\right)\tilde{x}^6+\left(2r_4-3r_2+r_0\right)\tilde{x}^4+\left(2r_2-3r_0\right)\tilde{x}^2+2r_0\Rightarrow\\
&\Rightarrow -36\tilde{x}^8+2\left(33-10q_4\right)\tilde{x}^6+2\left(21q_4-2q_2\right)\tilde{x}^4+6\left(3q_2+2q_0\right)\tilde{x}^2-6q_0=\\
&=r_4\tilde{x}^8+\left(r_2-3r_4\right)\tilde{x}^6+\left(2r_4-3r_2+r_0\right)\tilde{x}^4+\left(2r_2-3r_0\right)\tilde{x}^2+2r_0
\end{aligned}$$

Comparing the same-degree terms in both sides of the last equation, we obtain

$$r_4=-36 \tag{22}$$

$$r_2 - 3r_4 = 2(33 - 10q_4) \quad (23)$$

$$2r_4 - 3r_2 + r_0 = 2(21q_4 - 2q_2) \quad (24)$$

$$2r_2 - 3r_0 = 6(3q_2 + 2q_0) \quad (25)$$

$$2r_0 = -6q_0 \quad (26)$$

Substituting (22) into (23) yields

$$r_2 + 108 = 2(33 - 10q_4) \Rightarrow r_2 + 108 = 66 - 20q_4 \Rightarrow 20q_4 = -42 - r_2$$

Thus

$$q_4 = -\frac{21}{10} - \frac{r_2}{20} \quad (27)$$

Substituting (22) and (27) into (24) yields

$$-72 - 3r_2 + r_0 = 2\left(21\left(-\frac{21}{10} - \frac{r_2}{20}\right) - 2q_2\right) = 2\left(-\frac{21*21}{10} - \frac{21r_2}{20} - 2q_2\right) = -\frac{21*21}{5} - \frac{21r_2}{10} - 4q_2 \Rightarrow$$
$$\Rightarrow 4q_2 = -\frac{441}{5} - \frac{21r_2}{10} + 72 + 3r_2 - r_0 = -\frac{81}{5} + \frac{9r_2}{10} - r_0$$

Thus

$$q_2 = -\frac{81}{20} + \frac{9r_2}{40} - \frac{r_0}{4} \quad (28)$$

Substituting (28) into (25) yields

$$2r_2 - 3r_0 = 6\left(3\left(-\frac{81}{20} + \frac{9r_2}{40} - \frac{r_0}{4}\right) + 2q_0\right) = 6\left(-\frac{243}{20} + \frac{27r_2}{40} - \frac{3r_0}{4} + 2q_0\right) =$$
$$= -\frac{3*243}{10} + \frac{3*27r_2}{20} - \frac{9r_0}{2} + 12q_0 \Rightarrow$$
$$\Rightarrow 12q_0 = \frac{3*243}{10} - \frac{3*27r_2}{20} + \frac{9r_0}{2} + 2r_2 - 3r_0 = \frac{729}{10} - \frac{41r_2}{20} + \frac{3r_0}{2}$$

Thus

$$q_0 = \frac{729}{120} - \frac{41r_2}{240} + \frac{3r_0}{24} \quad (29)$$

Substituting (29) into (26) yields

$$2r_0 = -6\left(\frac{729}{120} - \frac{41r_2}{240} + \frac{3r_0}{24}\right) = -\frac{729}{20} + \frac{41r_2}{40} - \frac{3r_0}{4} \Rightarrow \frac{41r_2}{40} = \frac{729}{20} + \frac{11r_0}{4} \Rightarrow \frac{41r_2}{10} = \frac{729}{5} + 11r_0 \Rightarrow$$
$$\Rightarrow 11r_0 = -\frac{729}{5} + \frac{41r_2}{10}$$

Thus

$$r_0 = -\frac{729}{55} + \frac{41r_2}{110} \qquad (30)$$

Substituting (30) into (29) yields

$$q_0 = \frac{729}{120} - \frac{41r_2}{240} + \frac{3\left(-\frac{729}{55} + \frac{41r_2}{110}\right)}{24} = \frac{729}{120} - \frac{41r_2}{240} - \frac{2187}{1320} + \frac{123r_2}{2640} =$$
$$= \frac{729}{120} - \frac{2187}{120*11} + \frac{123r_2}{240*11} - \frac{41r_2}{240} = \frac{5832}{1320} - \frac{328r_2}{2640} = \frac{729}{165} - \frac{41r_2}{330}$$

That is

$$q_0 = \frac{729}{165} - \frac{41r_2}{330} \qquad (31)$$

Substituting (30) into (28) yields

$$q_2 = -\frac{81}{20} + \frac{9r_2}{40} - \frac{-\frac{729}{55} + \frac{41r_2}{110}}{4} = -\frac{81}{20} + \frac{9r_2}{40} + \frac{729}{220} - \frac{41r_2}{440} = \frac{729}{20*11} - \frac{81}{20} + \frac{9r_2}{40} - \frac{41r_2}{40*11} =$$
$$= -\frac{162}{220} + \frac{58r_2}{440} = -\frac{81}{110} + \frac{29r_2}{220}$$

That is

$$q_2 = -\frac{81}{110} + \frac{29r_2}{220} \qquad (32)$$

Since q_0 must be positive, from (31) we obtain

$$\frac{729}{165} - \frac{41r_2}{330} > 0 \Rightarrow \frac{41r_2}{330} < \frac{729}{165} \Rightarrow r_2 < \frac{240570}{6765} \simeq 35.56$$

To simplify things, we consider the case where r_2 vanishes.

Then, (31), (32), and (27) give, respectively,

$$q_0 = \frac{729}{165} > 0$$

$$q_2 = -\frac{81}{110}$$

$$q_4 = -\frac{21}{10}$$

Thus, the polynomial $Q_6(\tilde{x})$ is

$$Q_6(\tilde{x}) = \tilde{x}^6 - \frac{21}{10}\tilde{x}^4 - \frac{81}{110}\tilde{x}^2 + \frac{729}{165}$$

The polynomial $Q_6(\tilde{x})$ is plotted in Figure 4 and, as we see, it is everywhere positive, i.e. it has no zeros.

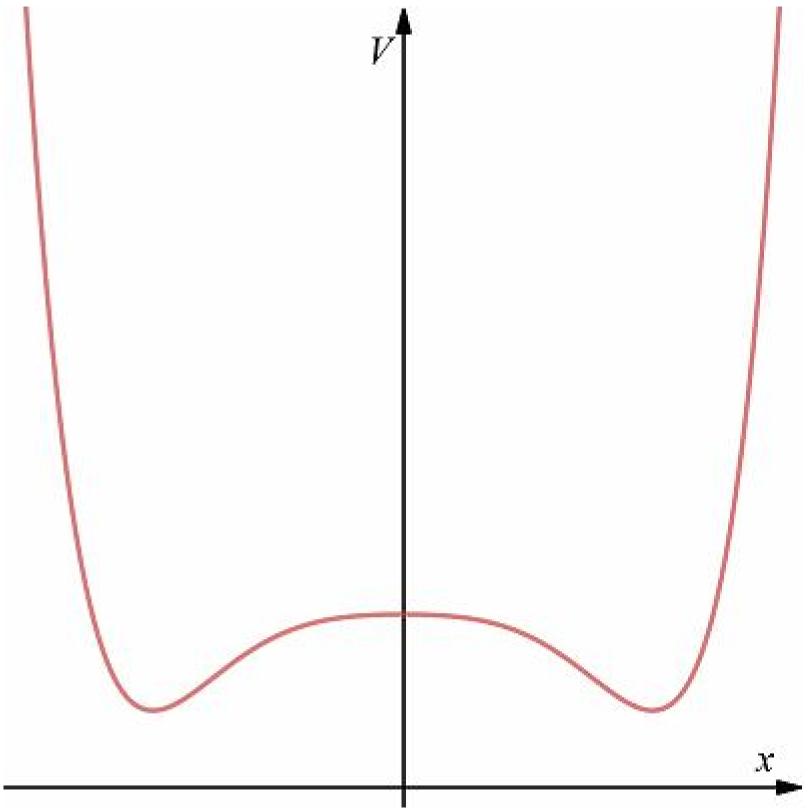

Figure 4

For $r_2 = 0$, (30) gives

$$r_0 = -\frac{729}{55}$$

Then, since $r_4 = -36$ (eq. (22)), the polynomial $R_4(\tilde{x})$ is

$$R_4(\tilde{x}) = -36\tilde{x}^4 - \frac{729}{55}$$

Substituting into (16), we obtain, in units $\hbar^2/2m_0x_0{}^2$,

$$V(\tilde{x})=\frac{2\left(6\tilde{x}^5-\frac{42}{5}\tilde{x}^3-\frac{81}{55}\tilde{x}\right)^2-\left(\tilde{x}^6-\frac{21}{10}\tilde{x}^4-\frac{81}{110}\tilde{x}^2+\frac{729}{165}\right)\left(30\tilde{x}^4-\frac{126}{5}\tilde{x}^2-\frac{81}{55}\right)}{\left(\tilde{x}^6-\frac{21}{10}\tilde{x}^4-\frac{81}{110}\tilde{x}^2+\frac{729}{165}\right)^2}+$$

$$+\frac{-36\tilde{x}^4-\frac{729}{55}}{\tilde{x}^6-\frac{21}{10}\tilde{x}^4-\frac{81}{110}\tilde{x}^2+\frac{729}{165}}$$

The previous potential is plotted in Figure 5. It is a symmetric triple finite volcano potential consisting of two deep, symmetric-around-zero volcanoes and one shallow volcano which is also symmetric around zero.

The potential has five bound eigenstates, the highest of which, i.e. the fourth-excited state, is described by the wave function

$$\psi(\tilde{x})=\frac{A\left(\tilde{x}^2-1\right)\left(\tilde{x}^2-2\right)}{\tilde{x}^6-\frac{21}{10}\tilde{x}^4-\frac{81}{110}\tilde{x}^2+\frac{729}{165}}$$, where A is the normalization constant,

and it has zero energy.

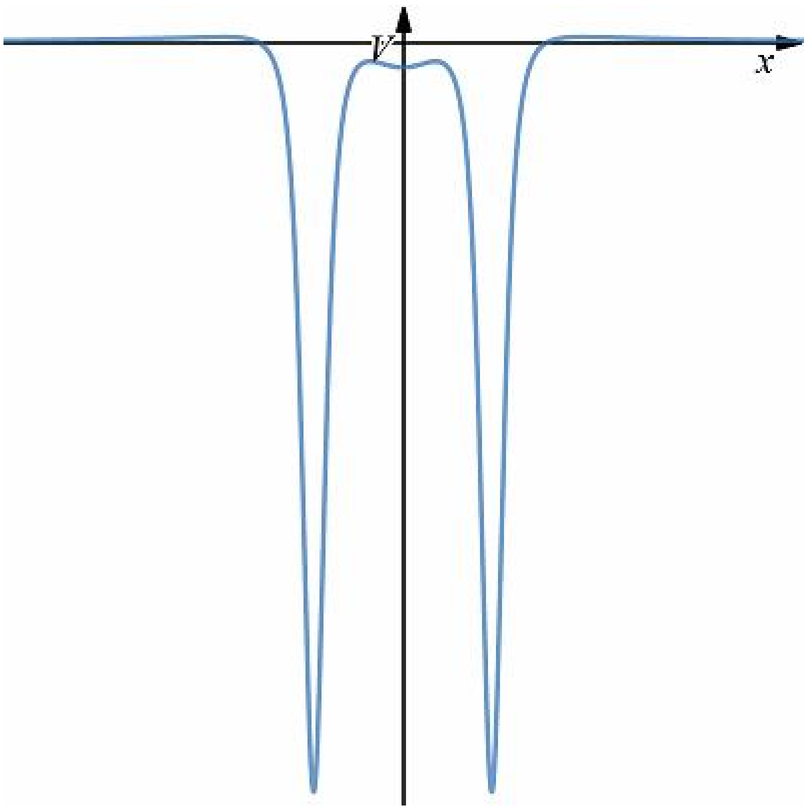

Figure 5

The case n=0

Since $P_0(\tilde{x})=1$, the wave function (4) is, including the normalization constant A,

$$\psi(\tilde{x})=\frac{A}{Q_m(\tilde{x})} \tag{33}$$

with $m\geq 2$ and even.

Then, the potential (5) becomes

$$V(\tilde{x})=\frac{\hbar^2}{2m_0x_0^{\,2}}\frac{2Q_m'^2(\tilde{x})-Q_m(\tilde{x})Q_m''(\tilde{x})}{Q_m^{\,2}(\tilde{x})}+E_r$$

Since $Q_m(\tilde{x})$ has no zeros, the potential has no singularities. Actually it is $C^\infty(R)$.

The wave function (33) has no zeros, thus it describes the ground state of the previous regular potential [8, 9], which has only one bound state, of energy E_r.

Using that $n=0$, from (6) we derive that, at long distances, the previous potential takes the form

$$V_\infty(\tilde{x})=\frac{m(m+1)\hbar^2}{2m_0x_0^{\,2}}\frac{1}{\tilde{x}^2}+E_r \tag{34}$$

Assuming that the potential vanishes at infinity, $E_r=0$, and then it is written as

$$V(\tilde{x})=\frac{\hbar^2}{2m_0x_0^{\,2}}\frac{2Q_m'^2(\tilde{x})-Q_m(\tilde{x})Q_m''(\tilde{x})}{Q_m^{\,2}(\tilde{x})} \tag{35}$$

As an example, we'll examine the case where $Q_m(\tilde{x})=\tilde{x}^{2m}+q_0$, with $m=1,2,...$, and $q_0>0$, so that $Q_m(\tilde{x})\neq 0$. In this case, the wave function has definite (even) parity.

We have

$$Q_m'(\tilde{x})=2m\tilde{x}^{2m-1},\; Q_m''(\tilde{x})=2m(2m-1)\tilde{x}^{2m-2}$$

Plugging into (35) yields

$$V(\tilde{x}) = \frac{\hbar^2}{2m_0 x_0{}^2} \frac{2\left(2m\tilde{x}^{2m-1}\right)^2 - \left(\tilde{x}^{2m} + q_0\right) 2m(2m-1)\tilde{x}^{2m-2}}{\left(\tilde{x}^{2m} + q_0\right)^2} =$$

$$= \frac{\hbar^2}{2m_0 x_0{}^2} \frac{8m^2 \tilde{x}^{4m-2} - 2m(2m-1)\tilde{x}^{4m-2} - 2m(2m-1) q_0 \tilde{x}^{2m-2}}{\left(\tilde{x}^{2m} + q_0\right)^2} =$$

$$= \frac{\hbar^2}{2m_0 x_0{}^2} \frac{\left(8m^2 - 2m(2m-1)\right)\tilde{x}^{4m-2} - 2m(2m-1) q_0 \tilde{x}^{2m-2}}{\left(\tilde{x}^{2m} + q_0\right)^2} =$$

$$= \frac{\hbar^2}{2m_0 x_0{}^2} \frac{\left(4m^2 + 2m\right)\tilde{x}^{4m-2} - 2m(2m-1) q_0 \tilde{x}^{2m-2}}{\left(\tilde{x}^{2m} + q_0\right)^2} = \frac{\hbar^2}{2m_0 x_0{}^2} \frac{2m\left((2m+1)\tilde{x}^{2m} - (2m-1) q_0\right)\tilde{x}^{2m-2}}{\left(\tilde{x}^{2m} + q_0\right)^2} =$$

$$= \frac{\hbar^2}{m_0 x_0{}^2} \frac{m\left((2m+1)\tilde{x}^{2m} - (2m-1) q_0\right)\tilde{x}^{2m-2}}{\left(\tilde{x}^{2m} + q_0\right)^2}$$

That is

$$V(\tilde{x}) = \frac{\hbar^2}{m_0 x_0{}^2} \frac{m\left((2m+1)\tilde{x}^{2m} - (2m-1) q_0\right)\tilde{x}^{2m-2}}{\left(\tilde{x}^{2m} + q_0\right)^2} \qquad (36)$$

The potential (36) is symmetric, as a result of the wave function having definite parity.

At long distances, (36) becomes

$$V_\infty(\tilde{x}) = \frac{\hbar^2}{m_0 x_0{}^2} \frac{m(2m+1)\tilde{x}^{4m-2}}{\tilde{x}^{4m}} = \frac{\hbar^2}{2m_0 x_0{}^2} \frac{2m(2m+1)}{\tilde{x}^2} = \frac{2m(2m+1)\hbar^2}{2m_0 x_0{}^2} \frac{1}{\tilde{x}^2}$$

in agreement with (34) for $E_r = 0$ and $m \to 2m$.

The potential (36) vanishes at

$$(2m+1)\tilde{x}^{2m} - (2m-1) q_0 = 0 \Rightarrow \tilde{x}^{2m} = \frac{2m-1}{2m+1} q_0 \underbrace{\Rightarrow}^{\frac{2m-1}{2m+1} q_0 > 0} \tilde{x}^2 = \left(\frac{2m-1}{2m+1} q_0\right)^{\frac{1}{m}} \Rightarrow$$

$$\Rightarrow \tilde{x} = \pm \left(\frac{2m-1}{2m+1} q_0\right)^{\frac{1}{2m}}$$

and at 0 if $m > 1$.

Since $\tilde{x}^{2m-2} \geq 0$ and the denominator of (36) is positive, we have

$$V(\tilde{x}) > 0 \text{ for } |\tilde{x}| > \left(\frac{2m-1}{2m+1} q_0\right)^{\frac{1}{2m}}$$

$$V(\tilde{x}) < 0 \text{ for } |\tilde{x}| < \left(\frac{2m-1}{2m+1} q_0\right)^{\frac{1}{2m}} \text{ and } \tilde{x} \neq 0 \text{ (if } m > 1)$$

Thus, if $m > 1$, the potential (36) is a symmetric double finite volcano, while if $m = 1$, it is a symmetric simple finite volcano, since then it does not vanish at 0.

In both cases, $m = 1$ and $m > 1$, the potential (36) has only one bound state, its ground state, with zero energy, which is described by the wave function $\psi(\tilde{x}) = A/(\tilde{x}^{2m} + q_0)$.

In Figure 6, the potential (36) is plotted in units $\hbar^2/m_0 x_0^{\ 2} = 1$, for $q_0 = 1$ and $m = 1$ (red line), $m = 2$ (blue line), and $m = 3$ (green line).

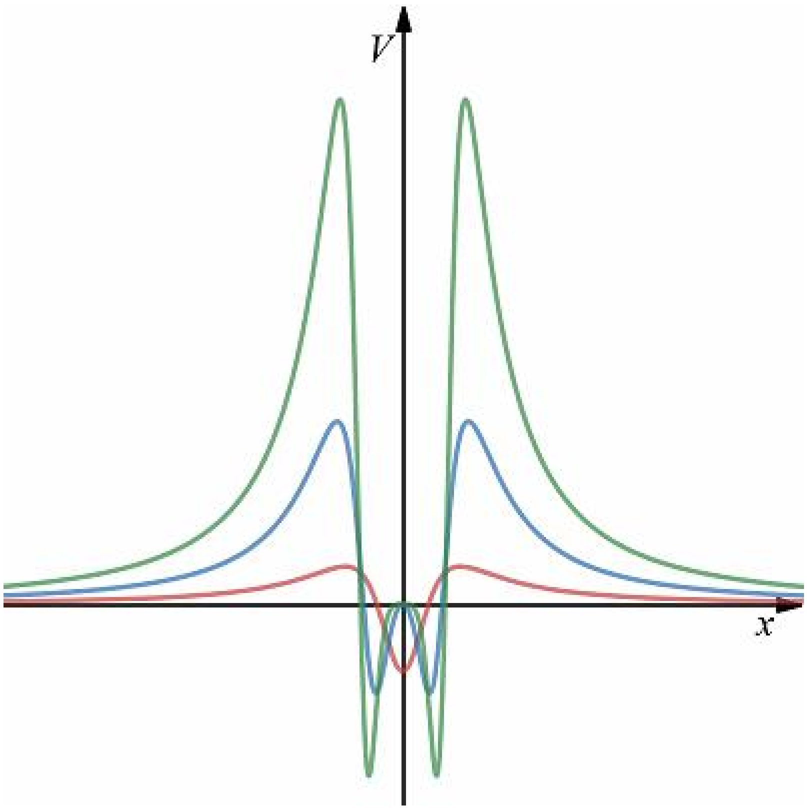

Figure 6

The case m=4, n=2

For simplicity, we'll assume that both $P_2(\tilde{x})$ and $Q_4(\tilde{x})$ are symmetric.

Then, since both are also monic, they have the form

$$P_2(\tilde{x}) = \tilde{x}^2 + p_0$$

$$Q_4(\tilde{x}) = \tilde{x}^4 + q_2\tilde{x}^2 + q_0$$

with $Q_4(\tilde{x}) > 0$.

As explained, for every value of q_2, there exists a q_0 such that the global minimum of $Q_4(\tilde{x})$ is positive.

Also, $p_0 \neq 0$, so that the zeros of $P_2(\tilde{x})$ (if any) are simple zeros.

The wave function (4) is, including the normalization constant A,

$$\psi(\tilde{x}) = A\frac{\tilde{x}^2 + p_0}{\tilde{x}^4 + q_2\tilde{x}^2 + q_0} \tag{37}$$

The wave function (37) has definite parity (even), thus $\psi''(\tilde{x})$ has also definite parity (even), and then the potential (3) is of even parity (symmetric).

Assuming, as usual, that the potential vanishes at infinity, $E_r = 0$, i.e. the energy of the eigenstate described by the wave function (37) is zero.

I. If $p_0 > 0$, $P_2(\tilde{x}) > 0$ and the wave function (37) has no zeros, and thus it describes the ground state of the potential (5) (with $E_r = 0$), which is regular.

In this case, the potential is a symmetric double finite volcano.

II. If $p_0 < 0$, $P_2(\tilde{x})$ has two zeros, at $\pm\sqrt{-p_0}$.

Then, the polynomial $Q_4(\tilde{x})P_2''(\tilde{x}) - 2Q_4'(\tilde{x})P_2'(\tilde{x})$ is

$$2\left(\tilde{x}^4 + q_2\tilde{x}^2 + q_0\right) - 2\left(4\tilde{x}^3 + 2q_2\tilde{x}\right)2\tilde{x} = -14\tilde{x}^4 - 6q_2\tilde{x}^2 + 2q_0$$

That is

$$Q_4(\tilde{x})P_2''(\tilde{x}) - 2Q_4'(\tilde{x})P_2'(\tilde{x}) = -2\left(7\tilde{x}^4 + 3q_2\tilde{x}^2 - q_0\right)$$

The previous polynomial is symmetric, thus its zeros (if any) will come in opposite pairs.

Then, since $P_2(\tilde{x})$ has two opposite zeros, either none or both of them will be zeros of $Q_4(\tilde{x})P_2''(\tilde{x}) - 2Q_4'(\tilde{x})P_2'(\tilde{x})$.

IIa. Substituting the two zeros of $P_2(\tilde{x})$ into the expression of $Q_4(\tilde{x})P_2''(\tilde{x}) - 2Q_4'(\tilde{x})P_2'(\tilde{x})$, we obtain

$$7\left(\pm\sqrt{-p_0}\right)^4 + 3q_2\left(\pm\sqrt{-p_0}\right)^2 - q_0 = 7p_0^{\,2} - 3q_2 p_0 - q_0$$

The discriminant of the previous trinomial is $9q_2^{\,2} + 28q_0$.

But $q_0 > 0$, otherwise $Q_4(\tilde{x})$ has zeros.

Thus, the discriminant is positive and the trinomial has two zeros, the product of which is $-q_0/7 < 0$, thus one zero is positive and the other is negative.

The negative zero is $p_0 = \left(3q_2 - \sqrt{9q_2^{\,2} + 28q_0}\right)/14$.

Therefore, for every pair of the parameters q_0, q_2, with $q_0 > 0$, there exists a $p_0 < 0$ such that the polynomial $Q_4(\tilde{x})P_2''(\tilde{x}) - 2Q_4'(\tilde{x})P_2'(\tilde{x})$ vanishes at $\pm\sqrt{-p_0}$, and the resulting potential is regular.

In this case, the potential is again a symmetric double finite volcano, but now it has three bound eigenstates, with the highest of them, i.e. the second-excited state, being described by the wave function (37), which takes the form

$$\psi(\tilde{x}) = A\frac{\tilde{x}^2 + 3q_2 - \sqrt{9q_2^{\,2} + 28q_0}}{\tilde{x}^4 + q_2\tilde{x}^2 + q_0} \qquad (38)$$

where we've incorporated into the normalization constant A the factor $1/14$ appearing in the expression of p_0.

The wave function (38) has two zeros, thus it is the second-excited-state wave function of the resulting potential, with energy $E_r = 0$.

Next, we'll calculate the potential for the case where $q_2 = 0$.

In this case, $p_0 = -\sqrt{\frac{q_0}{7}}$, and then

$$P_2(\tilde{x}) = \tilde{x}^2 - \sqrt{\frac{q_0}{7}}, \; P_2'(\tilde{x}) = 2\tilde{x}, \; P_2''(\tilde{x}) = 2$$

Also

$$Q_4(\tilde{x}) = \tilde{x}^4 + q_0, \; Q_4'(\tilde{x}) = 4\tilde{x}^3, \; Q_4''(\tilde{x}) = 12\tilde{x}^2$$

Then, we have

$$\frac{2Q_4'^2(\tilde{x})-Q_4(\tilde{x})Q_4''(\tilde{x})}{Q_4^2(\tilde{x})}=\frac{32\tilde{x}^6-12\tilde{x}^2\left(\tilde{x}^4+q_0\right)}{\left(\tilde{x}^4+q_0\right)^2}=\frac{20\tilde{x}^6-12q_0\tilde{x}^2}{\left(\tilde{x}^4+q_0\right)^2}$$

and

$$\frac{Q_4(\tilde{x})P_2''(\tilde{x})-2Q_4'(\tilde{x})P_2'(\tilde{x})}{Q_4(\tilde{x})P_2(\tilde{x})}=\frac{2\left(\tilde{x}^4+q_0\right)-16\tilde{x}^4}{\left(\tilde{x}^4+q_0\right)\left(\tilde{x}^2-\sqrt{\frac{q_0}{7}}\right)}=\frac{-14\tilde{x}^4+2q_0}{\left(\tilde{x}^4+q_0\right)\left(\tilde{x}^2-\sqrt{\frac{q_0}{7}}\right)}=$$

$$=\frac{-14\left(\tilde{x}^4-\frac{q_0}{7}\right)}{\left(\tilde{x}^4+q_0\right)\left(\tilde{x}^2-\sqrt{\frac{q_0}{7}}\right)}=\frac{-14\left(\tilde{x}^2-\sqrt{\frac{q_0}{7}}\right)\left(\tilde{x}^2+\sqrt{\frac{q_0}{7}}\right)}{\left(\tilde{x}^4+q_0\right)\left(\tilde{x}^2-\sqrt{\frac{q_0}{7}}\right)}=-\frac{14\left(\tilde{x}^2+\sqrt{\frac{q_0}{7}}\right)}{\left(\tilde{x}^4+q_0\right)}$$

Substituting into (5), with $E_r=0$, we obtain the potential

$$V(\tilde{x})=\frac{\hbar^2}{2m_0x_0^2}\left(\frac{20\tilde{x}^6-12q_0\tilde{x}^2}{\left(\tilde{x}^4+q_0\right)^2}-\frac{14\left(\tilde{x}^2+\sqrt{\frac{q_0}{7}}\right)}{\left(\tilde{x}^4+q_0\right)}\right)=\frac{\hbar^2}{m_0x_0^2}\left(\frac{10\tilde{x}^6-6q_0\tilde{x}^2}{\left(\tilde{x}^4+q_0\right)^2}-\frac{7\left(\tilde{x}^2+\sqrt{\frac{q_0}{7}}\right)}{\left(\tilde{x}^4+q_0\right)}\right)=$$

$$=\frac{\hbar^2}{m_0x_0^2}\left(\frac{10\tilde{x}^6-6q_0\tilde{x}^2}{\left(\tilde{x}^4+q_0\right)^2}-\frac{7\tilde{x}^2+\sqrt{7q_0}}{\left(\tilde{x}^4+q_0\right)}\right)=\frac{\hbar^2}{m_0x_0^2}\frac{10\tilde{x}^6-6q_0\tilde{x}^2-\left(7\tilde{x}^2+\sqrt{7q_0}\right)\left(\tilde{x}^4+q_0\right)}{\left(\tilde{x}^4+q_0\right)^2}=$$

$$=\frac{\hbar^2}{m_0x_0^2}\frac{10\tilde{x}^6-6q_0\tilde{x}^2-7\tilde{x}^6-7q_0\tilde{x}^2-\sqrt{7q_0}\tilde{x}^4-\sqrt{7q_0}q_0}{\left(\tilde{x}^4+q_0\right)^2}=\frac{\hbar^2}{m_0x_0^2}\frac{3\tilde{x}^6-\sqrt{7q_0}\tilde{x}^4-13q_0\tilde{x}^2-\sqrt{7q_0}q_0}{\left(\tilde{x}^4+q_0\right)^2}$$

That is

$$V(\tilde{x})=\frac{\hbar^2}{m_0x_0^2}\frac{3\tilde{x}^6-\sqrt{7q_0}\tilde{x}^4-13q_0\tilde{x}^2-\sqrt{7q_0}q_0}{\left(\tilde{x}^4+q_0\right)^2} \qquad (39)$$

At long distances, the potential (39) takes the form $V_\infty(\tilde{x})=3\hbar^2/m_0x_0^2\tilde{x}^2$, in agreement with (6), for $m-n=2$ and $E_r=0$.

In units $\hbar^2/m_0x_0^2=1$, the potential (39) becomes

$$V(\tilde{x})=\frac{3\tilde{x}^6-\sqrt{7q_0}\tilde{x}^4-13q_0\tilde{x}^2-\sqrt{7q_0}q_0}{\left(\tilde{x}^4+q_0\right)^2}$$

For $q_0 = 1/7$ and $q_0 = 4/7$ the previous potential is written as, respectively,

$$V(\tilde{x}) = \frac{3\tilde{x}^6 - \tilde{x}^4 - \frac{13}{7}\tilde{x}^2 - \frac{1}{7}}{\left(\tilde{x}^4 + \frac{1}{7}\right)^2} = \frac{3\tilde{x}^6 - \tilde{x}^4 - \frac{13}{7}\tilde{x}^2 - \frac{1}{7}}{\frac{1}{7^2}\left(7\tilde{x}^4 + 1\right)^2} = \frac{7^2 * 3\tilde{x}^6 - 7^2\tilde{x}^4 - 7*13\tilde{x}^2 - 7}{\left(7\tilde{x}^4 + 1\right)^2} =$$
$$= \frac{147\tilde{x}^6 - 49\tilde{x}^4 - 91\tilde{x}^2 - 7}{\left(7\tilde{x}^4 + 1\right)^2}$$

That is

$$V(\tilde{x}) = \frac{147\tilde{x}^6 - 49\tilde{x}^4 - 91\tilde{x}^2 - 7}{\left(7\tilde{x}^4 + 1\right)^2}$$

And

$$V(\tilde{x}) = \frac{3\tilde{x}^6 - 2\tilde{x}^4 - 13\frac{4}{7}\tilde{x}^2 - 2\frac{4}{7}}{\left(\tilde{x}^4 + \frac{4}{7}\right)^2} = \frac{3\tilde{x}^6 - 2\tilde{x}^4 - \frac{52}{7}\tilde{x}^2 - \frac{8}{7}}{\frac{1}{7^2}\left(7\tilde{x}^4 + 4\right)^2} = \frac{7^2 * 3\tilde{x}^6 - 7^2 * 2\tilde{x}^4 - 7*52\tilde{x}^2 - 7*8}{\left(7\tilde{x}^4 + 4\right)^2} =$$
$$= \frac{147\tilde{x}^6 - 98\tilde{x}^4 - 364\tilde{x}^2 - 56}{\left(7\tilde{x}^4 + 4\right)^2}$$

That is

$$V(\tilde{x}) = \frac{147\tilde{x}^6 - 98\tilde{x}^4 - 364\tilde{x}^2 - 56}{\left(7\tilde{x}^4 + 4\right)^2}$$

Both potentials are plotted in Figure 7 (red and blue line, respectively). Each of them has three bound eigenstates, with the highest of them, i.e. the second-excited state, being of zero energy.

We see that, as q_0 increases, the bottom of the symmetric double volcano approaches zero. Then, since the ground-state energy must exceed the minimum value of the potential [10], both the ground-state and the first-excited-state energies approach zero, as q_0 increases. In other words, as q_0 increases, the two lower bound energies of each volcano potential get closer to the third, which is fixed to zero.

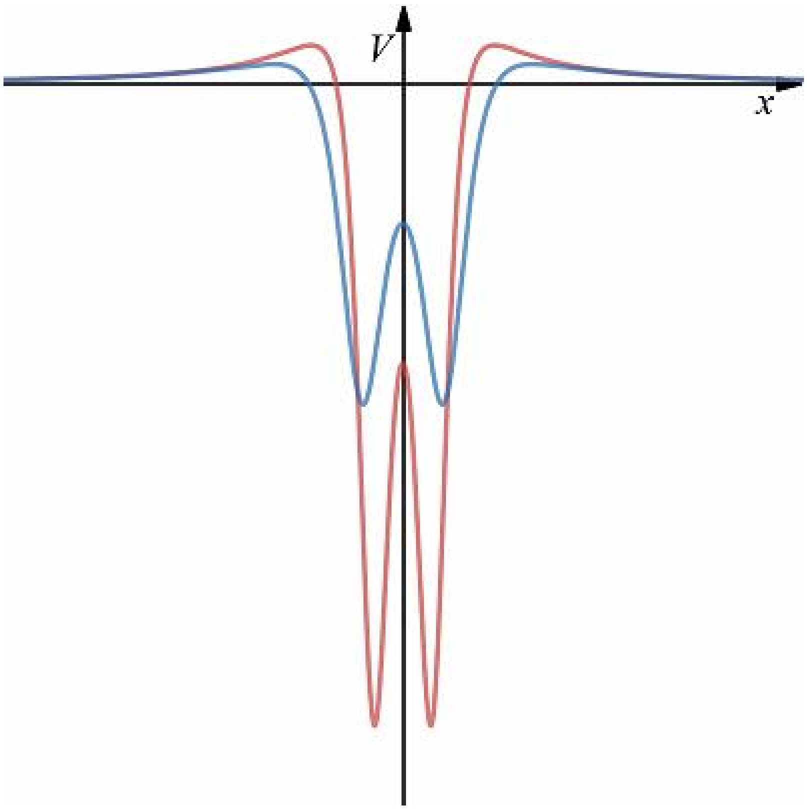

Figure 7

IIb. If $p_0 \neq \left(3q_2 - \sqrt{9q_2^{\,2} + 28q_0}\right)\big/14$, the polynomial $Q_4(\tilde{x})P_2''(\tilde{x}) - 2Q_4'(\tilde{x})P_2'(\tilde{x})$ does not vanish at the two simple zeros of $P_2(\tilde{x})$ and then the potential is singular, it has two simple poles, at $\pm\sqrt{-p_0}$.

In this case, the two zeros of the wave function (37) are not typical nodes, they are special zeros, and the wave function is the ground-state wave function of the potential (5) (with $E_r = 0$), which is a symmetric singular potential with only one bound state, its ground state, having zero energy.

As the coefficient p_0 reaches the critical value $\left(3q_2 - \sqrt{9q_2^{\,2} + 28q_0}\right)\big/14$, the two open edges at each of the two simple poles of the potential are "glued" together forming a volcano and, at the same time, two more bound eigenstates are created, with the highest of them, i.e. the second-excited state, having zero energy.

For the case where $q_2 = 0$, we saw that the critical value of p_0 is $-\sqrt{q_0/7}$.

We'll examine what happens if $p_0 = -(1+\varepsilon)\sqrt{q_0/7}$, where $\varepsilon > -1$, so that $p_0 < 0$.

For $\varepsilon = 0$, we obtain the critical value of p_0.

The polynomials $P_2(\tilde{x})$ and $Q_4(\tilde{x})$ are, respectively,

$$P_2(\tilde{x}) = \tilde{x}^2 - (1+\varepsilon)\sqrt{\frac{q_0}{7}}\,,\; Q_4(\tilde{x}) = \tilde{x}^4 + q_0$$

The regular term of the potential (5), i.e. the term $\left(2Q_4'^2(\tilde{x}) - Q_4(\tilde{x})Q_4''(\tilde{x})\right)\big/Q_4^2(\tilde{x})$, is the same as before, i.e.

$$\frac{2Q_4'^2(\tilde{x}) - Q_4(\tilde{x})Q_4''(\tilde{x})}{Q_4^2(\tilde{x})} = \frac{20\tilde{x}^6 - 12q_0\tilde{x}^2}{\left(\tilde{x}^4 + q_0\right)^2}$$

while the singular term $\left(Q_4(\tilde{x})P_2''(\tilde{x}) - 2Q_4'(\tilde{x})P_2'(\tilde{x})\right)\big/Q_4(\tilde{x})P_2(\tilde{x})$ is

$$\frac{Q_4(\tilde{x})P_2''(\tilde{x}) - 2Q_4'(\tilde{x})P_2'(\tilde{x})}{Q_4(\tilde{x})P_2(\tilde{x})} = -\frac{14\left(\tilde{x}^4 - \frac{q_0}{7}\right)}{\left(\tilde{x}^4 + q_0\right)\left(\tilde{x}^2 - (1+\varepsilon)\sqrt{\frac{q_0}{7}}\right)}$$

Substituting the two previous terms into (5), we obtain, assuming again that the potential vanishes at infinity, i.e. $E_r = 0$,

$$V(\tilde{x}) = \frac{\hbar^2}{2m_0x_0^2}\left(\frac{20\tilde{x}^6 - 12q_0\tilde{x}^2}{\left(\tilde{x}^4 + q_0\right)^2} - \frac{14\left(\tilde{x}^4 - \frac{q_0}{7}\right)}{\left(\tilde{x}^4 + q_0\right)\left(\tilde{x}^2 - (1+\varepsilon)\sqrt{\frac{q_0}{7}}\right)}\right)$$

To simplify the calculation, we further assume that $q_0 = 7$.

Then, in units $\hbar^2/m_0x_0^2 = 1$, the previous potential is written as

$$V(\tilde{x}) = \frac{1}{2}\left(\frac{20\tilde{x}^6 - 12*7\tilde{x}^2}{\left(\tilde{x}^4+7\right)^2} - \frac{14\left(\tilde{x}^4 - 1\right)}{\left(\tilde{x}^4+7\right)\left(\tilde{x}^2 - (1+\varepsilon)\right)}\right) = \frac{10\tilde{x}^6 - 6*7\tilde{x}^2}{\left(\tilde{x}^4+7\right)^2} - \frac{7\left(\tilde{x}^4-1\right)}{\left(\tilde{x}^4+7\right)\left(\tilde{x}^2-(1+\varepsilon)\right)} =$$
$$= \frac{\left(10\tilde{x}^6 - 42\tilde{x}^2\right)\left(\tilde{x}^2 - (1+\varepsilon)\right) - 7\left(\tilde{x}^4-1\right)\left(\tilde{x}^4+7\right)}{\left(\tilde{x}^4+7\right)^2\left(\tilde{x}^2-(1+\varepsilon)\right)}$$

But

$$\left(10\tilde{x}^6-42\tilde{x}^2\right)\left(\tilde{x}^2-(1+\varepsilon)\right)-7\left(\tilde{x}^4-1\right)\left(\tilde{x}^4+7\right)=10\tilde{x}^8-10(1+\varepsilon)\tilde{x}^6-42\tilde{x}^4+42(1+\varepsilon)\tilde{x}^2-$$
$$-7\left(\tilde{x}^8+6\tilde{x}^4-7\right)=10\tilde{x}^8-10(1+\varepsilon)\tilde{x}^6-42\tilde{x}^4+42(1+\varepsilon)\tilde{x}^2-7\tilde{x}^8-42\tilde{x}^4+49=$$
$$=3\tilde{x}^8-10(1+\varepsilon)\tilde{x}^6-84\tilde{x}^4+42(1+\varepsilon)\tilde{x}^2+49$$

Thus, we end up to the potential

$$V(\tilde{x})=\frac{3\tilde{x}^8-10(1+\varepsilon)\tilde{x}^6-84\tilde{x}^4+42(1+\varepsilon)\tilde{x}^2+49}{\left(\tilde{x}^4+7\right)^2\left(\tilde{x}^2-(1+\varepsilon)\right)} \tag{40}$$

For $\varepsilon \neq 0$, the potential (40) has two simple poles, at $\pm\sqrt{1+\varepsilon}$ (with $\varepsilon > -1$).

For $\varepsilon = 0$, a volcano is formed at each pole and the potential then becomes a double volcano, with three bound eigenstates, the highest of which, i.e. the second-excited state, has zero energy.

In Figures 8-10, the potential (40) is plotted for $\varepsilon = -0.01$, $\varepsilon = 0$, and $\varepsilon = 0.01$, respectively.

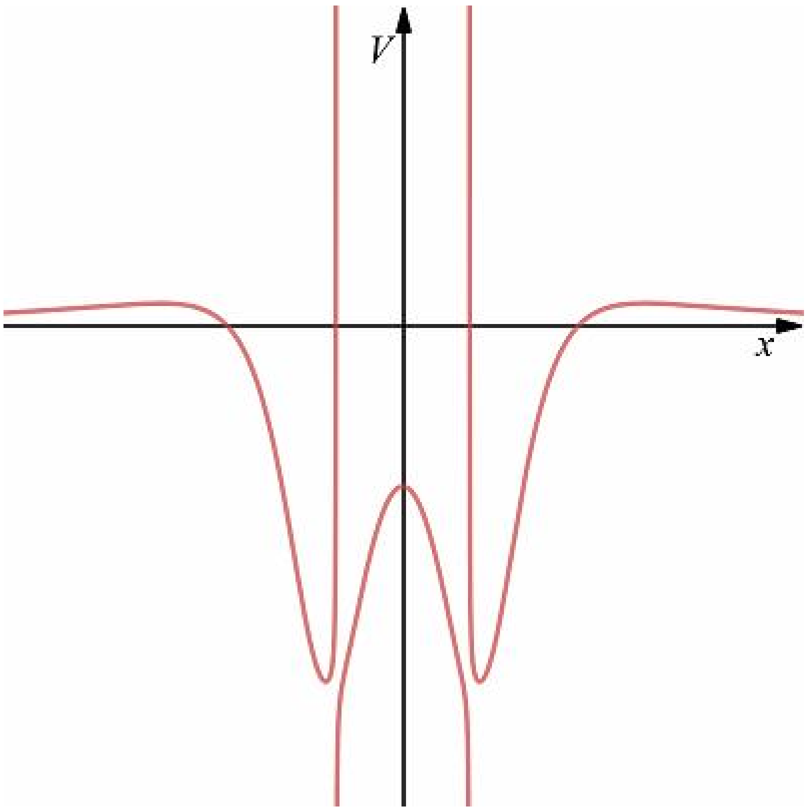

Figure 8

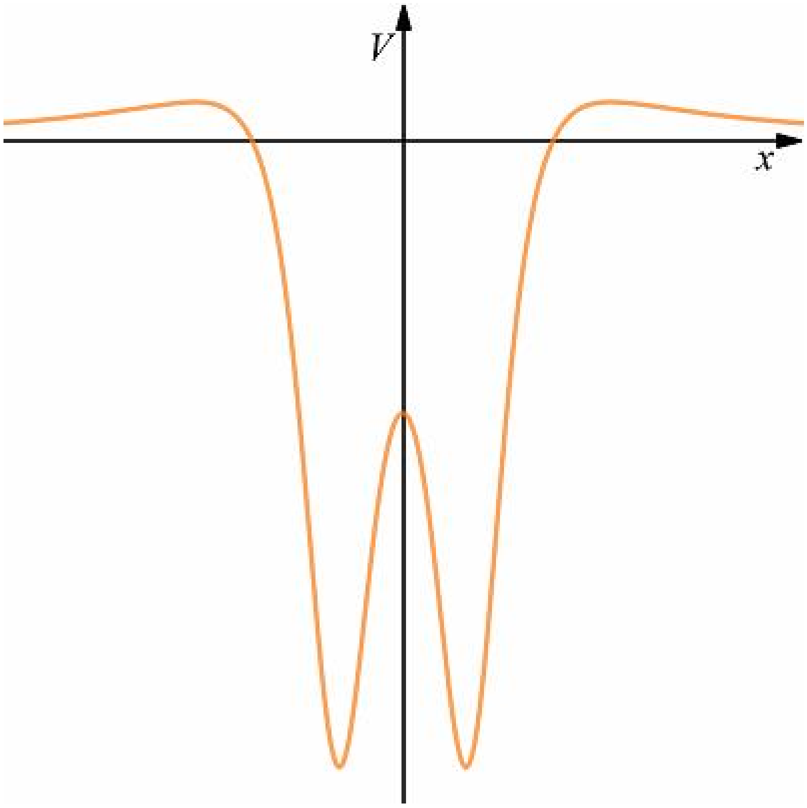

Figure 9

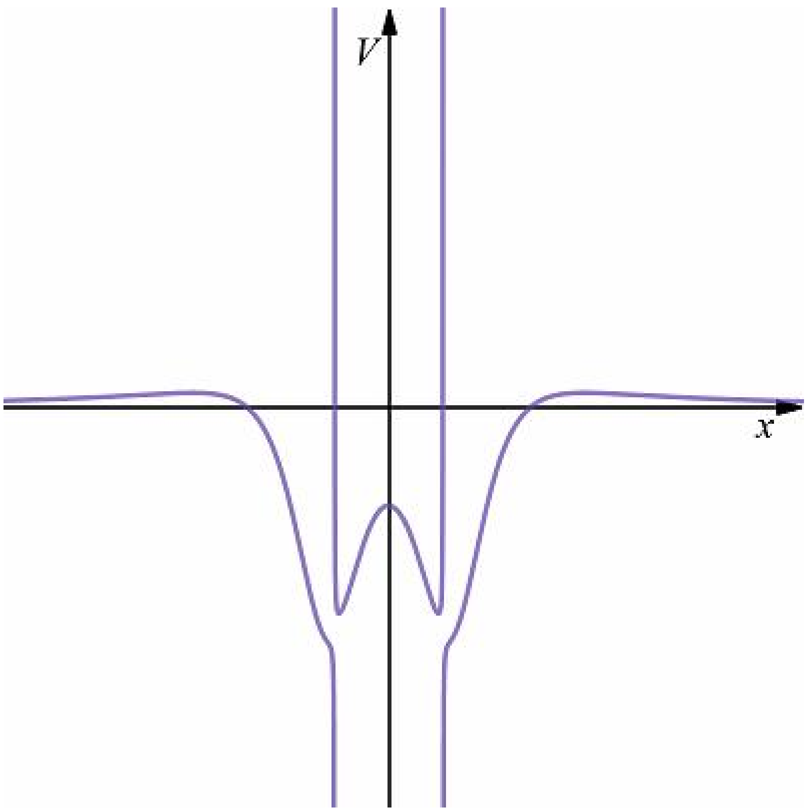

Figure 10

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