Question 443 : GOLDEN MEAN and PI

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abstract

In this note we recall some formulas for pi. they express pi in terms of the Golden Ratio.

1. Introduction.

- The number pi is defined by

$$\pi = 4\int_0^1 \sqrt{1-x^2}\, dx = 3.1415926535... \tag{1}$$

- The Golden mean is defined by

$$\phi = \frac{1+\sqrt{5}}{2} \tag{2}$$

- Pi in terms of the Golden mean (Example)

$$\pi = \frac{5\sqrt{2+\phi}}{2}\sum_{n=0}^{\infty}\phi^{-2n-1}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{n+k+1} = \frac{5\sqrt{2+\phi}}{2}\sum_{n=0}^{\infty}\frac{\phi^{-2n-1}}{(2n+1)\binom{2n}{n}} \tag{3}$$

2. Some formulas for pi

$$\pi = 4\sum_{n=0}^{\infty}\frac{c_n}{2n+1}\phi^{-2n-1} \tag{4}$$

$$c_0 = 1, c_1 = 2, c_2 = 1, c_3 = -1, c_n = -\left(c_{n-1}+c_{n-3}+c_{n-4}\right), n \geq 4 \tag{5}$$

$$\pi = 4\sum_{n=0}^{\infty}\frac{c_n}{2n+1}\phi^{-4n-2} \tag{6}$$

$$c_0 = 2, c_1 = 1, c_2 = 2, c_3 = -2, c_n = -\left(c_{n-1} + c_{n-3} + c_{n-4}\right), n \geq 4 \tag{7}$$

$$\pi = 2\sum_{n=0}^{\infty}\frac{c_n}{2n+1}\phi^{-2n-1} \tag{8}$$

$$c_0 = 3, c_1 = -3, c_2 = -2, c_3 = -3, c_4 = 3, c_5 = -3, c_n = -\left(c_{n-1} + c_{n-5} + c_{n-6}\right), n \geq 6 \tag{9}$$

$$\pi = 4\sum_{n=0}^{\infty}\frac{c_n}{2n+3}\phi^{-2n-3} \tag{10}$$

$$c_0 = 9, c_1 = 5, c_2 = 0, c_3 = -9, c_4 = 0, c_5 = 0, c_6 = 4, c_7 = 0 \tag{11}$$

$$c_n = -\left(c_{n-3} + c_{n-5} + c_{n-8}\right), n \geq 8 \tag{12}$$

$$\pi = 4\sum_{n=0}^{\infty}\frac{c_n}{2n+1}\phi^{-2n-1} \tag{13}$$

$$c_n = \left(-1\right)^n + 3\varepsilon_n\left(-1\right)^{(n-1)/3} \quad , \quad \varepsilon_n = \begin{cases} 1 & \left(n-1\right)/3 \in \mathbb{N} \cup \{0\} \\ 0 & \left(n-1\right)/3 \notin \mathbb{N} \cup \{0\} \end{cases} \tag{14}$$

$$\pi = 4\sum_{n=0}^{\infty}\frac{c_n}{2n+1}\phi^{-4n-2} \tag{15}$$

$$c_n = 2\left(-1\right)^n + 3\varepsilon_n\left(-1\right)^{(n-1)/3} \quad , \quad \varepsilon_n = \begin{cases} 1 & \left(n-1\right)/3 \in \mathbb{N} \cup \{0\} \\ 0 & \left(n-1\right)/3 \notin \mathbb{N} \cup \{0\} \end{cases} \tag{16}$$

$$\pi = 2\sum_{n=0}^{\infty}\frac{c_n}{2n+1}\phi^{-2n-1} \tag{17}$$

$$c_n = 3\left(-1\right)^n - 5\varepsilon_n\left(-1\right)^{(n-2)/5} \quad , \quad \varepsilon_n = \begin{cases} 1 & \left(n-2\right)/5 \in \mathbb{N} \cup \{0\} \\ 0 & \left(n-2\right)/5 \notin \mathbb{N} \cup \{0\} \end{cases} \tag{18}$$

$$\pi = 4\sum_{n=0}^{\infty} \frac{c_n}{2n+3}\phi^{-2n-3} \tag{19}$$

$$c_n = 9\alpha_n \left(-1\right)^{n/3} + 5\beta_n \left(-1\right)^{(n-1)/5} \tag{20}$$

$$\alpha_n = \begin{cases} 1 & n/3 \in \mathbb{N} \cup \{0\} \\ 0 & n/3 \notin \mathbb{N} \cup \{0\} \end{cases} \quad , \quad \beta_n = \begin{cases} 1 & (n-1)/5 \in \mathbb{N} \cup \{0\} \\ 0 & (n-1)/5 \notin \mathbb{N} \cup \{0\} \end{cases} \tag{21}$$

References

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