Discussion about the Interaction between Current Elements
Proposal of a new Force Law

Óscar Sánchez Uriarte[1] and Julio Sánchez Cánovas
[1]Dpto. de Ingeniería Eléctrica
Universidad de Valladolid, Valladolid, España
osanchez@eii.uva.es[1]

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1 Abstract

In this article, a new force law between current elements is proposed which holds the 3rd Newton's Law and concides with known experimental measurements avoiding the contradictions of Ampère and Grassman expressions. Likewise, an interaction expression between point charges is postulated which satisfies action reaction principle and is Galileo Invariant. This opens the way to a revision of the concept of magnetic field and a further study of the interaction between moving charged bodies.

Keywords: Grassman, Ampère, Whittaker, Maxwell, Lorentz Force, Third Newton's Law, Action-Reaction Principle, Magnetic Field.

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2 Introduction

After the publication, July 21st 1820 [15], of Oesterd's obsevations about the effects of a current on a magnetized needle, Jean-Baptiste Biot and Felix Savart show, October the 30th 1820 [6], their results about the force on a magnet precticed by a stationary current in a wire.

Ampère extends this phenomenon to the interaction between currents. He assumes the impossibility of making and insloated measurement of the influence of a piece of a circuit on another piece of the same circuit and, from the Newtonian point of view, establishes his law of interaction between current elements. His works are published between 1820 and 1825 [1].

Grassman, in his work titled "A new Theory of Electrodynamics" (1845) [17], criticizes Ampère's Law pointing that the force of interaction vanishes when the current elements form a determinated angle with the line that links them both. He postulates a new expression compatible with Biot-Savart law and, knowing that his expression does not satisfy the Action-reaction principle, asserts that the action all over the circuit must be consider for evaluating the force of a part of the circuit to another properly and giving the same results as Ampère's Law. He also prouposes some experiments for trying to discern the validity of one or other expression.

Maxwell analizes in "A Treatise on Electricity and Magnetism" (1865) [8], several force laws within current elements which give the same results being integrated on closed paths. Moreover, from Ampère's Law, he developes the different expressions of Gauss and Weber who were looking for justifying the intereaction between currents starting from known electrical phenomena.

Whittaker, analizing Ampère and Grassman expresions in his book "Theories of Aether and Electricity" (1910) [21], shows that the first expression is condicioned for giving a force in the direction of the linking line among current elements and adds a second term to Grassman expression for satisfying 3rd Newton's law and whose contribution vanishes integrating its action on a closed path. The expression which he obtains coincides with one considered by Maxwell [12].

We can think that the Ampère's belief that the force direction was on the liking line one prevented him to get the correct solution. The expression of Maxwell and Whittaker safisfies the 3rd Newton's Law without that condition:

$$d^2\mathbf{F_{ba}} = \frac{\mu_0}{4\pi}\frac{I_b I_a}{|\mathbf{r_{ba}}|^2}\left[d\mathbf{l_b}(d\mathbf{l_a}\cdot\mathbf{r_{ba}}) + d\mathbf{l_a}(d\mathbf{l_b}\cdot\mathbf{r_{ba}}) - \mathbf{r_{ba}}(d\mathbf{l_a}\cdot d\mathbf{l_b})\right] \tag{1}$$

Despite of obeying the Action-Reaction principle and predicting the experimental results, this expression has not received any attention and the dispute has been reduced within Ampère and Grassman (equivalent to Biot-Savart Law or Lorentz Force) expressions. Dispute which is not settled, nowadays, as some authors consider that Ampère's Law could explain some experimental issues and, furthermore, discard the necesity of using the Special Theory of Relativity [2][16][20][7].

The different experiences for supporting the Ampère expression are, basically, "Ampère's boat" modifications [9]. On the other hand, experiences based on capacitor discharges or "wire explosions" are proposed where some authors assert the existence of longitudinal forces [4][19][11]. It is not necessary to call into question those evidences as the discharges give autoinduction and oscillating phenomena where those forces would be justified. Summing up, no experience is decisive.

In this article a way of obtaining the Maxwell-Whittaker is discribed, based on the experimental observations and considering the 3rd Newton's Law. Likewise, we transform the Maxwell-Whittaker expression taking the field concept and a possible law, Galileo invariant, of interaction between charges is postulated.

3 Experimental Observations

A typical example of the experimental montage for proving the validity of the Ampère or Grassman expression consist in the measurement of the practised force of one part of the circuit on another[2][11].

The circuit is divided in two parts: A and B. The eletrical continuity is guaranteed by two mercury contacts $C1$ and $C2$. A force F appears on part B. The measurement method is not shown.

The obtained values of F coincide, despite of experimental errors, with the theorical values obtained with Ampère's Law:

$$\mathbf{F_{AB}} = \frac{\mu_0}{4\pi}\int_A\int_B I^2\left[3\left(d\mathbf{r_b}\cdot\frac{\mathbf{r_{ab}}}{|\mathbf{r_{ab}}|}\right)\left(d\mathbf{r_a}\cdot\frac{\mathbf{r_{ab}}}{|\mathbf{r_{ab}}|}\right) - 2\left(d\mathbf{r_a}\cdot d\mathbf{r_b}\right)\right]\frac{\mathbf{r_{ab}}}{|\mathbf{r_{ab}}|^3} \tag{2}$$

On the other hand, they also coincide with the values obtained appliying the Lorentz's Law:

$$\mathbf{F_{AB}} = \int_B I d\mathbf{r_i}\wedge\frac{\mu_0}{4\pi}\oint_{A+B}\frac{I\left(d\mathbf{r_j}\wedge\mathbf{r_{ij}}\right)}{|\mathbf{r_{ij}}|^3} \tag{3}$$

However, for obtaining the same results appliying Lorenz's Law, the second integral has been extended to the total circuit $A+B$[10].

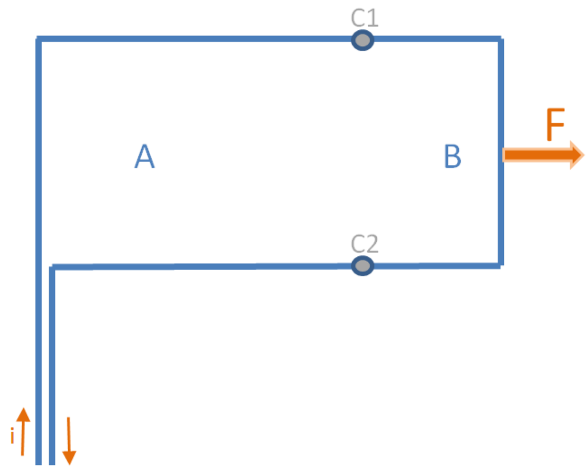

Figure 1: Circuit Scheme

Summarizing, for obtaining the force of A performed on B applying Ampère's force we would consider the action of each element of A on each element of B and, on the other hand, appliying Lorentz's force we would consider the action of the whole circuit on B. Following this procedure, we would obtain the same result compatible with the experimental measurements.

In these terms, the experience does not allow to make a distinction between Ampère's and Lorentz's Law. If we consider, as we do, the validity of the 3rd Newton's Law we would consider the Ampère's expression as the correct one from a physical point of view and Lorentz's one would be relegated to a "mathematical artifice".

4 Neither Ampère nor Grassmann: Deduction of a new expression

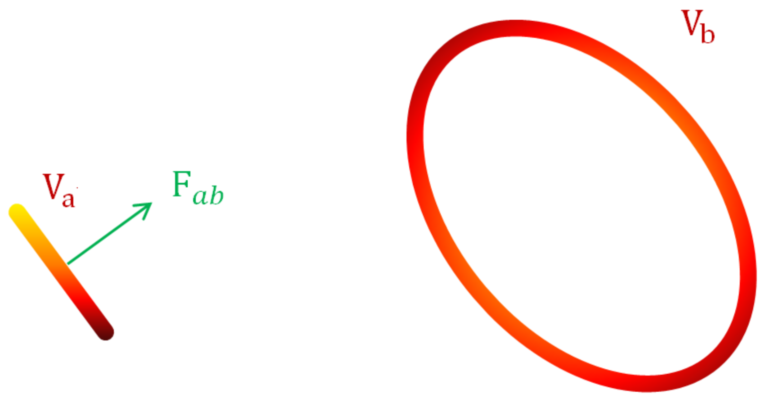

Figure 2: Two closed circuits interacting

Experimentally, we could not evaluate the force practised by an element on another of the same circuit. However, we can measure the force of a closed circuit, contained in a volume V_b, on an element of another circuit contained in a volume V_a. From this moment, we would consider a "Grassman force" type:

$$\mathbf{F_{ba}} = \int_{V_a} \mathbf{J_a} dV_a \wedge \frac{\mu_0}{4\pi} \int_{V_b} \frac{\mathbf{J_b} \wedge \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_b \tag{4}$$

We would admit, based on the experimental observations, that the same type of force on a volumen element is given:

$$d\mathbf{F_{ba}} = \mathbf{J_a} dV_a \wedge \frac{\mu_0}{4\pi} \int_{V_b} \frac{\mathbf{J_b} \wedge \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_b \tag{5}$$

Now, we will deduce an expression of the force between two volumen elements dV_a and dV_b, both electrically neutral:

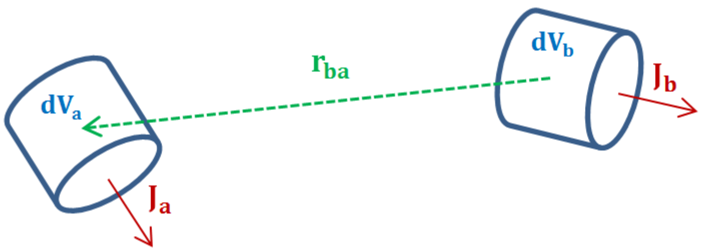

Figure 3: Two neutral volumen elements interacting

The previous considerations could imply that:

$$d^2\mathbf{F_{ba}} = \mathbf{J_a} dV_a \wedge \frac{\mu_0}{4\pi} \frac{\mathbf{J_b} \wedge \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_b \tag{6}$$

However, the force on the element dV_b by dV_a:

$$d^2\mathbf{F_{ab}} = \mathbf{J_b} dV_b \wedge \frac{\mu_0}{4\pi} \frac{\mathbf{J_a} \wedge \mathbf{r_{ab}}}{|\mathbf{r_{ab}}|^3} dV_a \tag{7}$$

which collide with the 3rd Newton's Law as it does not always hold:

$$d^2\mathbf{F_{ba}} = -d^2\mathbf{F_{ab}} \tag{8}$$

Let's consider the experimental evidences and imposing the 3rd Newton's Law. Without loss of generality, we are able to suppose:

$$d^2\mathbf{F_{ba}} = \mathbf{J_a} dV_a \wedge \frac{\mu_0}{4\pi} \frac{\mathbf{J_b} \wedge \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_b + \mathbf{B_{1a}} + \mathbf{B_{2a}} \tag{9}$$

$$d^2\mathbf{F_{ab}} = \mathbf{J_b} dV_b \wedge \frac{\mu_0}{4\pi} \frac{\mathbf{J_a} \wedge \mathbf{r_{ab}}}{|\mathbf{r_{ab}}|^3} dV_a + \mathbf{A_{1b}} + \mathbf{A_{2b}} \tag{10}$$

Through expression (8) we can say:

$$\mathbf{B_{1a}} = -\mathbf{J_b} dV_b \wedge \frac{\mu_0}{4\pi} \frac{\mathbf{J_a} \wedge \mathbf{r_{ab}}}{|\mathbf{r_{ba}}|^3} dV_a \tag{11}$$

and

$$\mathbf{A_{1b}} = -\mathbf{J_a} dV_a \wedge \frac{\mu_0}{4\pi} \frac{\mathbf{J_b} \wedge \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_b \tag{12}$$

We would obtain:

$$d^2\mathbf{F_{ba}} = \mathbf{J_a} dV_a \wedge \frac{\mu_0}{4\pi} \frac{\mathbf{J_b} \wedge \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_b + \mathbf{J_b} dV_b \wedge \frac{\mu_0}{4\pi} \frac{\mathbf{J_a} \wedge \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_a + \mathbf{B_{2a}} \tag{13}$$

$$d^2\mathbf{F_{ab}} = \mathbf{J_b} dV_b \wedge \frac{\mu_0}{4\pi} \frac{\mathbf{J_a} \wedge \mathbf{r_{ab}}}{|\mathbf{r_{ab}}|^3} dV_a + \mathbf{J_a} dV_a \wedge \frac{\mu_0}{4\pi} \frac{\mathbf{J_b} \wedge \mathbf{r_{ab}}}{|\mathbf{r_{ab}}|^3} dV_b + \mathbf{A_{2b}} \tag{14}$$

Now, we have to determine $\mathbf{B_{2a}}$ and $\mathbf{A_{2b}}$, we know by (5):

$$\int_{V_b} \left[\mathbf{J_a} dV_a \wedge \frac{\mu_0}{4\pi} \frac{\mathbf{J_b} \wedge \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_b + \mathbf{J_b} dV_b \wedge \frac{\mu_0}{4\pi} \frac{\mathbf{J_a} \wedge \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_a + \mathbf{B_{2a}} \right] = \mathbf{J_a} dV_a \wedge \frac{\mu_0}{4\pi} \int_{V_b} \frac{\mathbf{J_b} \wedge \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_b \tag{15}$$

Using the vectorial identity: $\mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$ we can obtain:

$$\int_{V_b} \mathbf{J_b} dV_b \wedge \frac{\mu_0}{4\pi} \frac{\mathbf{J_a} \wedge \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_a = \frac{\mu_0}{4\pi} \mathbf{J_a} dV_a \int_{V_b} \frac{\mathbf{J_b} \cdot \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_b - \int_{V_b} \frac{\mu_0}{4\pi} \frac{(\mathbf{J_a} \cdot \mathbf{J_b})\mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_a dV_b \tag{16}$$

In Appendix 1 is demostrated that, if J_b is contained in the volume V_b and $\nabla \cdot \mathbf{J_b} = 0$ in every point of V_b:

$$\int_{V_b} \frac{\mathbf{J_b} \cdot \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_b = 0 \tag{17}$$

So, we obtain:

$$\int_{V_b} \mathbf{J_b} dV_b \wedge \frac{\mu_0}{4\pi} \frac{\mathbf{J_a} \wedge \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_a = -\int_{V_b} \frac{\mu_0}{4\pi} \frac{(\mathbf{J_a} \cdot \mathbf{J_b})\mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_a dV_b \tag{18}$$

and equation (15) is satisfied if:

$$\mathbf{B_{2a}} = \int_{V_b} \frac{\mu_0}{4\pi} \frac{(\mathbf{J_a} \cdot \mathbf{J_b})\mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_a dV_b \tag{19}$$

Analogously:

$$\mathbf{A_{2b}} = \int_{V_b} \frac{\mu_0}{4\pi} \frac{(\mathbf{J_a} \cdot \mathbf{J_b})\mathbf{r_{ab}}}{|\mathbf{r_{ab}}|^3} dV_a dV_b \tag{20}$$

And:

$$\mathbf{B_{2a}} = -\mathbf{A_{2b}} \tag{21}$$

We have no experimental evidences which justify that the interaction between elements is:

$$d^2\mathbf{F_{ba}} = \mathbf{J_a} dV_a \wedge \frac{\mu_0}{4\pi} \frac{\mathbf{J_b} \wedge \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_b + \mathbf{J_b} dV_b \wedge \frac{\mu_0}{4\pi} \frac{\mathbf{J_a} \wedge \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_a + \frac{\mu_0}{4\pi} \frac{(\mathbf{J_a} \cdot \mathbf{J_b})\mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_a dV_b \tag{22}$$

Nevertheless, action-reaction principle holds and the expression is compatible with the experimental results for any volume V_b.

If we were dealing with current elements we could write the previous expression as:

$$d^2\mathbf{F_{ba}} = I_a d\mathbf{l_a} \wedge \frac{\mu_0}{4\pi}\frac{I_b(d\mathbf{l_b} \wedge \mathbf{r_{ba}})}{|\mathbf{r_{ba}}|^3} + I_b d\mathbf{l_b} \wedge \frac{\mu_0}{4\pi}\frac{I_a(d\mathbf{l_a} \wedge \mathbf{r_{ba}})}{|\mathbf{r_{ba}}|^3} + \frac{\mu_0}{4\pi}\frac{I_a I_b(d\mathbf{l_a} \cdot d\mathbf{l_b})\mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} \tag{23}$$

Using the same vectorial identity $\mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$ as before:

$$d^2\mathbf{F_{ba}} = \frac{\mu_0}{4\pi}\frac{I_a I_b}{|\mathbf{r_{ba}}|^3}\left[d\mathbf{l_b}(d\mathbf{l_a} \cdot \mathbf{r_{ba}}) + d\mathbf{l_a}(d\mathbf{l_b} \cdot \mathbf{r_{ba}}) - \mathbf{r_{ba}}(d\mathbf{l_a} \cdot d\mathbf{l_b})\right] \tag{24}$$

Obtaining Maxwell-Whittaker expression.

5 Interaction between point charges

As we have seen until now, the interaction between current elements would be given by:

$$d^2\mathbf{F_{ba}} = \mathbf{J_a} dV_a \wedge \frac{\mu_0}{4\pi}\frac{\mathbf{J_b} \wedge \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_b + \frac{\mu_0}{4\pi}\mathbf{J_a} dV_a \frac{\mathbf{J_b} \cdot \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_b \tag{25}$$

instead of:

$$d^2\mathbf{F_{ba}} = \mathbf{J_a} dV_a \wedge \frac{\mu_0}{4\pi}\frac{\mathbf{J_b} \wedge \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_b \tag{26}$$

From the equation (26) below, has been extrapolated, without experimental evidence, an equation which describes the interaction between charges, accepted until today:

$$\mathbf{F_{ba}} = q_a \mathbf{v_a} \wedge \frac{\mu_0}{4\pi}\frac{q_b(\mathbf{v_b} \wedge \mathbf{r_{ba}})}{|\mathbf{r_{ba}}|^3} \tag{27}$$

We could extrapolate Maxwell-Whittaker equation (25), postulating that the interaction between charges is due to their relative velocity:

$$\mathbf{F_{ba}} = q_a(\mathbf{v_a} - \mathbf{v_b}) \wedge \frac{\mu_0}{4\pi}\frac{q_b(\mathbf{v_b} - \mathbf{v_a}) \wedge \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} + \frac{\mu_0}{8\pi}\frac{q_a q_b\left[(\mathbf{v_a} - \mathbf{v_b}) \cdot (\mathbf{v_b} - \mathbf{v_a})\right]\mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} \tag{28}$$

From this expression, Maxwell-Whittaker equation (25) is deduced in the Appendix 2.

6 Conclusions

As a result of this work, we are able to say that neither Ampère's and Grassman's formulas describe properly the interaction between current elements, nor Lorentz's Law the interaction between point charges as being an extrapolation of Grassman's formula. Thus, we propose a Force Law for current elements (22) and for point charges (28), which are free of the problems carried by the previous expressions and satisfy 3rd Newton's Law.

On the other hand, a revision of the concept of magnetic field is proposed since Biot-Savart Law, which describe the magnetic field created by a current element, is not consistent with the proposed expression.

Appendix 1

Let V_b a region of the space where the current density is confined and $\nabla \cdot \mathbf{J_b} = 0$. Then:

$$\int_{V_b} \frac{\mathbf{J_b} \cdot \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_b = 0 \tag{29}$$

Proof:

We can rewrite the previous integral as:

$$-\int_{V_b} \mathbf{J_b} \cdot \nabla \left(\frac{1}{|\mathbf{r_{ba}}|} \right) dV_b \tag{30}$$

Using the vectorial identity: $\nabla \cdot (\psi \mathbf{A}) = \nabla \psi \cdot \mathbf{A} + \psi (\nabla \cdot \mathbf{A})$ the previous integral results:

$$-\int_{V_b} \nabla \cdot \left(\mathbf{J_b} \frac{1}{|\mathbf{r_{ba}}|} \right) dV_b + \int_{V_b} \frac{1}{|\mathbf{r_{ba}}|} (\nabla \cdot \mathbf{J_b}) dV_b \tag{31}$$

We said that $\nabla \cdot \mathbf{J_b} = 0$ so the second term of the sum vanishes. The first integral, using the divergence theorem, results:

$$\oint_{\partial V_b} \frac{1}{|\mathbf{r_{ba}}|} \mathbf{J_b} \cdot d\mathbf{S} = 0 \tag{32}$$

which is zero as all the density current is contained in the volume V_b and there is no flux through the surface that enclose it.

So, the integral (29) is equal to zero.

Appendix 2

In this appendix, Maxwell-Whittaker expression will be obtained from the equation (28).

Let be two conductors, electrically neutral, of volumes V_a and V_b motionless respect each other. The interaction between two elements of the conductors will be due to their moving charges:

$$d^2\mathbf{F_{ba}} = dq_a(\mathbf{v_a} - \mathbf{v_b}) \wedge \frac{\mu_0}{4\pi} \frac{dq_b(\mathbf{v_b} - \mathbf{v_a}) \wedge \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} + \frac{\mu_0}{8\pi} \frac{dq_a dq_b \left[(\mathbf{v_a} - \mathbf{v_b}) \cdot (\mathbf{v_b} - \mathbf{v_a})\right] \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} \tag{33}$$

Rewriting the expression below in terms of charge density:

$$d^2\mathbf{F_{ba}} = \rho_a dV_a(\mathbf{v_a} - \mathbf{v_b}) \wedge \frac{\mu_0}{4\pi} \frac{\rho_b dV_b(\mathbf{v_b} - \mathbf{v_a}) \wedge \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} + \frac{\mu_0}{8\pi} \frac{\rho_a \rho_b dV_a dV_b \left[(\mathbf{v_a} - \mathbf{v_b}) \cdot (\mathbf{v_b} - \mathbf{v_a})\right] \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} \tag{34}$$

Suppose, now, that we integrate the contribution of a volume element ΔV_b to another ΔV_a which both contain several moving particles inside but they are small enough from the vector $\mathbf{r_{ba}}$.

$$\mathbf{F_{ba}} = \int_{\Delta V_a} \int_{\Delta V_b} \rho_a dV_a(\mathbf{v_a} - \mathbf{v_b}) \wedge \frac{\mu_0}{4\pi} \frac{\rho_b dV_b(\mathbf{v_b} - \mathbf{v_a}) \wedge \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} + \int_{\Delta V_a} \int_{\Delta V_b} \frac{\mu_0}{8\pi} \frac{\rho_a \rho_b dV_a dV_b \left[(\mathbf{v_a} - \mathbf{v_b}) \cdot (\mathbf{v_b} - \mathbf{v_a})\right] \mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} \tag{35}$$

Developing the vectorial product of the first term and the scalar product of the second one, we get:

$$\begin{aligned}\mathbf{F_{ba}} = \frac{\mu_0}{4\pi}\int_{\Delta V_a}\int_{\Delta V_b} \frac{\rho_a\rho_b}{|\mathbf{r_{ba}}|^3}\left[\mathbf{v_a}\wedge(\mathbf{v_b}\wedge\mathbf{r_{ba}}) - \mathbf{v_a}\wedge(\mathbf{v_a}\wedge\mathbf{r_{ba}}) - \mathbf{v_b}\wedge(\mathbf{v_b}\wedge\mathbf{r_{ba}}) + \mathbf{v_b}\wedge(\mathbf{v_a}\wedge\mathbf{r_{ba}})\right] dV_a dV_b \\ + \frac{\mu_0}{8\pi}\int_{\Delta V_a}\int_{\Delta V_b} \rho_a\rho_b \frac{\left[2(\mathbf{v_a}\cdot\mathbf{v_b}) + |\mathbf{v_a}|^2 + |\mathbf{v_b}|^2\right]\mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_a dV_b\end{aligned} \tag{36}$$

Since we consider $\mathbf{r_{ba}}$ constant because we are integrating in very small volumes, the terms which do not depend on both velocities vanishes because of the electrical neutrality of the conductors.

$$\int_{\Delta V_a} \rho_a dV_a = 0\,, \int_{\Delta V_b} \rho_b dV_b = 0 \tag{37}$$

Then:

$$\mathbf{F_{ba}} = \frac{\mu_0}{4\pi}\int_{\Delta V_a}\int_{\Delta V_b} \frac{\left[\rho_a\mathbf{v_a}\wedge(\rho_b\mathbf{v_b}\wedge\mathbf{r_{ba}}) + \rho_b\mathbf{v_b}\wedge(\rho_a\mathbf{v_a}\wedge\mathbf{r_{ba}})\right]}{|\mathbf{r_{ba}}|^3} + \frac{\mu_0}{4\pi}\int_{\Delta V_a}\int_{\Delta V_b} \frac{\rho_a\rho_b(\mathbf{v_a}\cdot\mathbf{v_b})\mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_a dV_b \tag{38}$$

As $\mathbf{J_i} = \rho_i\mathbf{v_i}$, we obtain:

$$\mathbf{F_{ba}} = \int_{\Delta V_a}\int_{\Delta V_b} \left\{\mathbf{J_a} dV_a \wedge \frac{\mu_0}{4\pi}\frac{\mathbf{J_b}\wedge\mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_b + \mathbf{J_b} dV_b \wedge \frac{\mu_0}{4\pi}\frac{\mathbf{J_a}\wedge\mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_a + \frac{\mu_0}{4\pi}\frac{(\mathbf{J_a}\cdot\mathbf{J_b})\mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_a dV_b\right\} \tag{39}$$

and:

$$d^2\mathbf{F_{ba}} = \mathbf{J_a} dV_a \wedge \frac{\mu_0}{4\pi}\frac{\mathbf{J_b}\wedge\mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_b + \mathbf{J_b} dV_b \wedge \frac{\mu_0}{4\pi}\frac{\mathbf{J_a}\wedge\mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_a + \frac{\mu_0}{4\pi}\frac{(\mathbf{J_a}\cdot\mathbf{J_b})\mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} dV_a dV_b \tag{40}$$

Which is Maxwell-Whittaker expression. In terms of linear current elements:

$$d^2\mathbf{F_{ba}} = I_a d\mathbf{l_a} \wedge \frac{\mu_0}{4\pi}\frac{I_b(d\mathbf{l_b}\wedge\mathbf{r_{ba}})}{|\mathbf{r_{ba}}|^3} + I_b d\mathbf{l_b} \wedge \frac{\mu_0}{4\pi}\frac{I_a(d\mathbf{l_a}\wedge\mathbf{r_{ba}})}{|\mathbf{r_{ba}}|^3} + \frac{\mu_0}{4\pi}\frac{I_a I_b(d\mathbf{l_a}\cdot d\mathbf{l_b})\mathbf{r_{ba}}}{|\mathbf{r_{ba}}|^3} \tag{41}$$

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