Proof of Playfair's Axiom hits a roadblock

Prashanth R. Rao

Abstract:

The Playfair's axiom is considered an equivalent of Euclid's fifth postulate or parallel postulate in Euclidean planar geometry. It states that in a given plane, with a line in the plane and a point outside the line that is also in the same plane, one and only one line passes through that point that is also parallel to the given line. Previous proofs of Euclid's postulate or the Playfair's axiom have unintentionally assumed parallel postulate to prove it. Also, these axioms have different results in hyperbolic and spherical geometries. We offer proof for the Playfair's axiom for subset of cases in the context of plane Euclidean geometry and describe another subset of cases that cannot be proven by the same approach.

Results:

Let "*l*" be a line in a plane and let A be a point in the same plane outside the line *l*. Suppose AB and AC are two lines that are both parallel to line *l* in the same plane (assumption that is being tested). Then line *l* must lie completely in one of the four sectors defined by the two lines AB and AC and not including the two lines, otherwise it would intersect with atleast one of the above two lines (AB or AC) that pass through point A (we make an assumption that the four sectors are completely defined by the two lines extending in both directions infinitely).

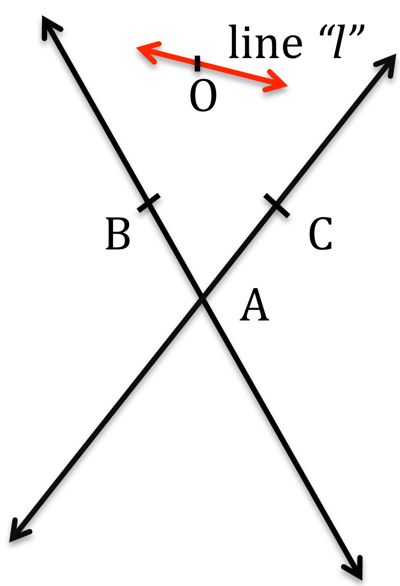

Figure 1

If line *l* lies in one of the four sectors as shown in the figure 1, then there must be a point O on line *l* that is at a finite distance from A such that segment OA is of length "s" (we make this assumption of "finite" distance) as shown in figure 2. Choose a length "r" that is greater than "s" and draw a circle with point O as its center (so segment O-A-R is one of the radii of this circle) and point A would lie within this circle. Consider the <u>two half-lines AB and AC</u>, they would each extend infinitely in one direction and therefore are longer than the diameter of the circle with center O and radius "r" and therefore must intersect the circle at points D and E, respectively (assumption that the diameter is the longest chord of a circle).

Since the line *l* is one of all possible lines that can pass through point O in the given plane, we analyze all possible lines that pass through O in the given plane for its ability to intersect the two lines AB and AC (for simplicity we will explore the ability to intersect half-lines AB and AC).

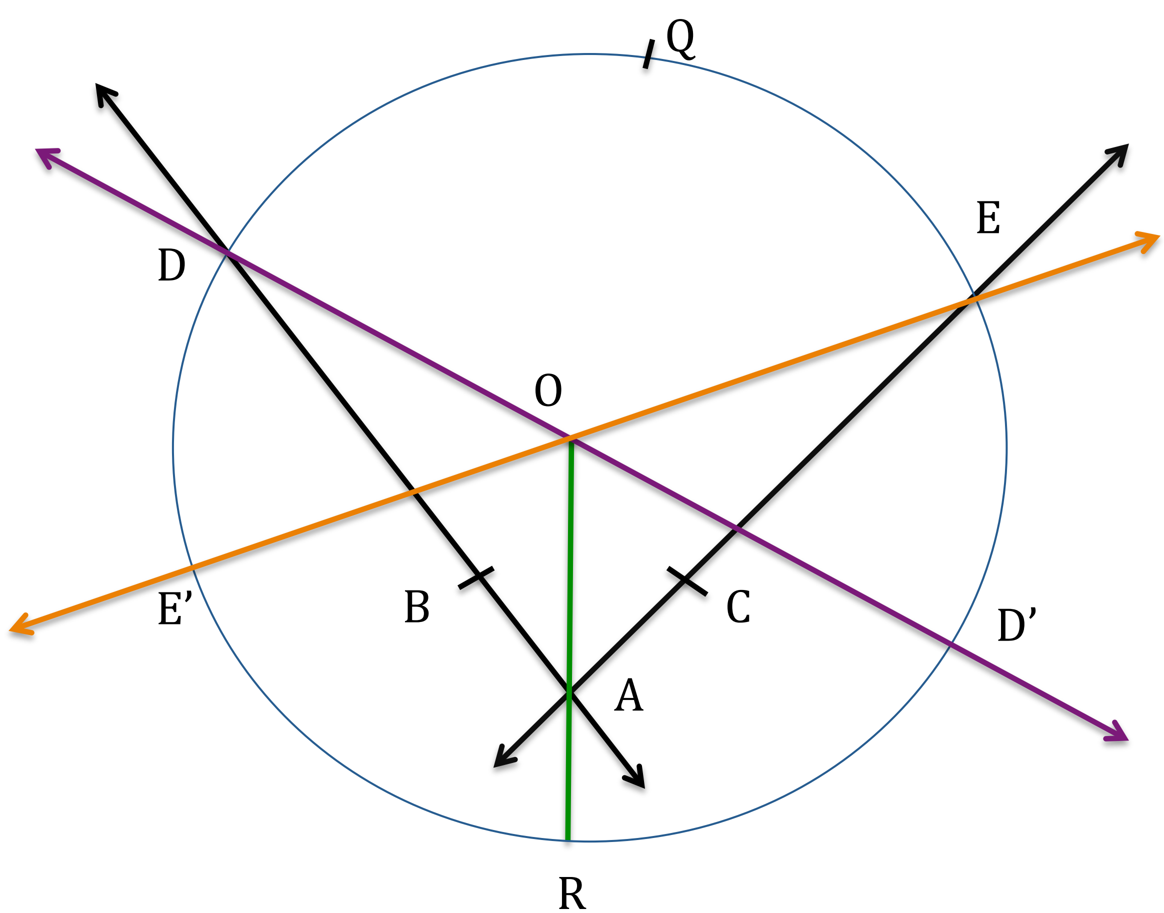

Figure 2

Consider all lines through O that would intersect anywhere along the arc DE'RD'E, such lines must intersect AB or AC or both since all points on segments AD and segment AE except D and E, lie within the circle and are each at a distance less than the radius "r" from the center O. Lines OD and OE intersect the half-lines AB and AC at D and E respectively. All lines through O that intersect the arc DQE will also intersect the arc D'RE' and will therefore intersect the lines AB and AC. Therefore all lines through O in the given plane will intersect the lines AB or AC or both. Therefore our supposition that two lines AB and AC are both parallel to a line "l" in the same plane must be incorrect. This proof works if the arc AQE is the minor arc.

<u>However, the above method would fail to prove the axiom when arc DQE is the major arc as shown in Figure 3 below and therefore making proof impossible for lines through O intersecting arc DE' or arc D'E.</u>

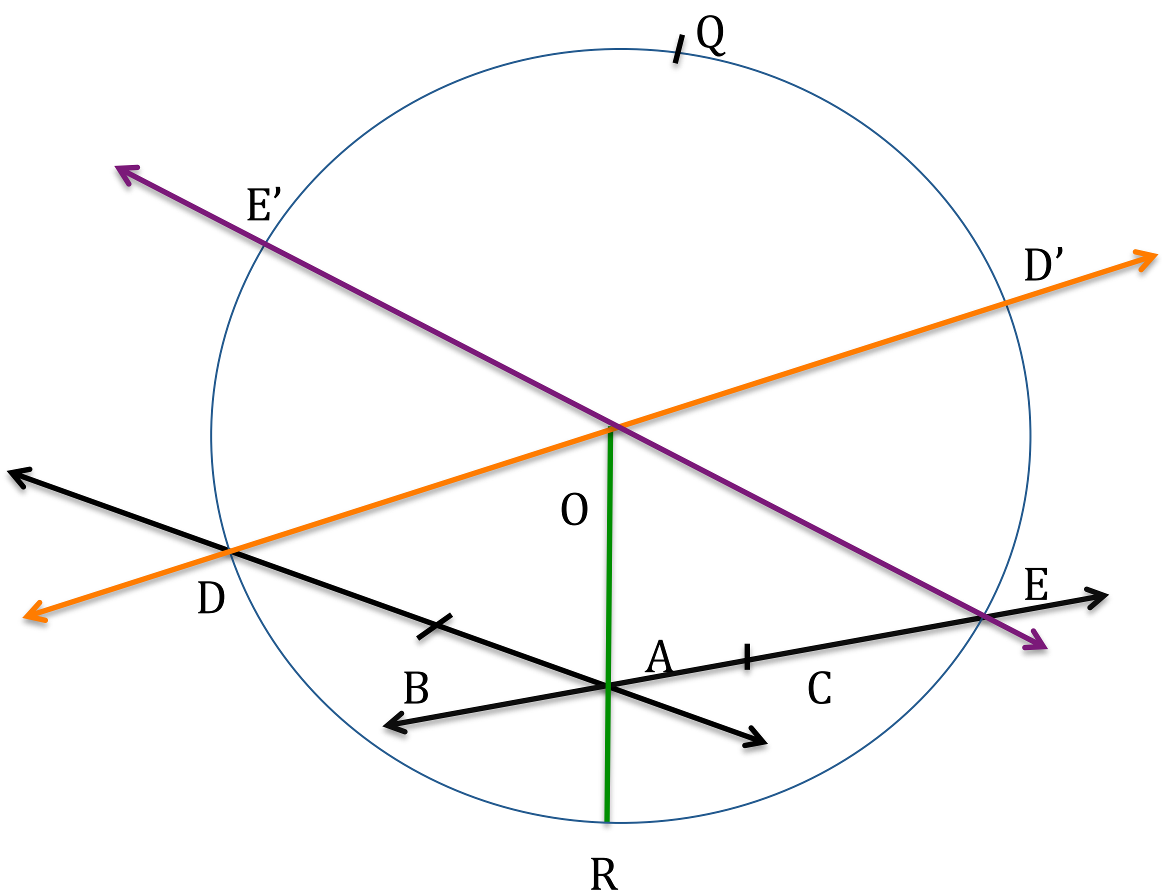

Figure 3 (impossible to prove for lines through O that intersect arc DE' or arc D'E)

Acknowledgements: I thank mathematician (Prof. PD) for illuminating me on the past related results in the field such as Beltrami-Klein models that suggest the failure of this axiom in revised Euclidean geometry.