A class of converted target functions preserving the set of minimizers

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Abstract

We show that conversion of target function using any increasing function U: $\mathbf{R} \to \mathbf{R}$, U = U(u) preserves the set of minimizers of the original minimization problem.

Keywords: minimization, target function, conversion, minimizers

1. Introduction

The purpose of this paper is to find a class of converted target functions, preserving the set of minimizers of the original minimization problem.

We show that conversion of target function using any increasing function U: $\mathbf{R} \to \mathbf{R}$, U = U(u) preserves the original set of minimizers.

That conversion gives a possibility to obtain desired properties of the minimization problem(e.g., faster minimization), which can consider it as a flexible and effective tool.

2. U-equivalent minimization

The following results give us a possibility to change the properties of the objective function with preservation of the set of minimizers of the original problem.

Theorem 1. *Let* O *be the minimization problem:*

$$O = \{\textit{minimize} \;\; g(x) \;\; \textit{subject to} \;\; x \in G\}, \;\; g\colon X \to \mathbf{R}, \;\; G \subseteq X.$$

Let E *be the minimization problem:*

$$E = \{minimize \ \ U(g(x)) \ \ subject \ \ to \ \ x \in G\}, \ G \subseteq X,$$

where $U: \mathbf{R} \to \mathbf{R}$, $U = U(u)$ *is any increasing function.*

Let M_O *be a set of minimizers of problem* O *and*

let M_E *be a set of minimizers of problem* E.

Then:

$$M_O = M_E \, (\text{argmin}(O) = \text{argmin}(E)) .$$

Proof. If $x_0 \in M_O$ then $g(x_0) \le g(x)$ for any $x \in G$. Hence, $U(g(x_0)) \le U(g(x))$ for any $x \in G$, since function U is the increasing function and therefore $x_0 \in M_E$ and $M_O \subseteq M_E$. If $x_0 \in M_E$ then we have: $U(g(x_0)) \le U(g(x))$ for any $x \in G$ and therefore $g(x_0) \le g(x)$ for any $x \in G$, as otherwise there exists $y_0 \in G$ such that $g(x_0) > g(y_0)$ and since function U is the increasing function it would mean that $U(g(x_0)) > U(g(y_0))$ in contradiction to the original supposition that $U(g(x_0)) \le U(g(x))$ for any $x \in G$. So, since $g(x_0) \le g(x)$ for any $x \in G$ then $x_0 \in M_O$ and $M_E \subseteq M_O$ and finally: $M_O = M_E$. □

Definition 1. *We say that the minimization problem:*

$$E = \{minimize \ \ U(g(x)) \ \ subject \ \ to \ \ x \in G\},$$

is U*–equivalent to the minimization problem:*

$$O = \{minimize \ \ g(x) \ \ subject \ \ to \ \ x \in G\}, \ g: X \to \mathbf{R}, \ G \subseteq X,$$

where $U: \mathbf{R} \to \mathbf{R}$, $U = U(u)$ *is some increasing function.*

Corollary 1. *If* E *is* U*-equivalent to* O *then* E *and* O *have the same set of minimizers: argmin(O) = argmin(E).*

Proof. It follows from Theorem 1 and Definition 1. □

Thus, using U-equivalence we can convert original minimization problem into minimization problem that has objective function with desired properties, so that both problems, - the original one, and U-equivalent have the same set of minimizers and share the same feasible set.

Hence, as a result of the U-equivalent conversion the original feasible set and the original set of minimizers remain unchanged, whereas the objective function is being changed to obtain desired properties (e.g., faster minimization), which can consider it(U-equivalence) as a flexible and effective tool.

U-equivalent conversion can be considered as unary operation defined on the set of minimization problems, having the same feasible set.

Example 1. Let us consider the following integer minimization problem:

$$\begin{aligned} &\text{minimize} \quad xy \\ &\text{subject to} \quad xy \geq N, \\ &\qquad 2 \leq x \leq N-1, \\ &\qquad N/(N-1) \leq y \leq N/2, \\ &\qquad x \in \mathbf{N},\ y \in \mathbf{N},\ N \in \mathbf{N}. \end{aligned} \tag{1}$$

Let $\Omega := \{ (x, y) \in \mathbf{R}^2 \mid xy \geq N,\ 2 \leq x \leq N-1, N/(N-1) \leq y \leq N/2, x \in \mathbf{R},\ y \in \mathbf{R} \}$ for a given $N \in \mathbf{N}$.

Hence, $\Omega^I := \Omega \cap \mathbf{Z}^2$ is a feasible set of the problem (1).

Suppose, the problem (1) is the original minimization problem. Let q be e^u-equivalent to the problem (1). The objective function of the problem (1) is xy, whereas the objective function of q is $f(x, y) = e^{xy}$. Both problems, due to the Theorem 1 have the same set of minimizers (and each such minimizer is a solution of the integer factorization problem).

Note that if N is not a prime, the minimum, $q = e^N$.

References

[1] S. Boyd and L. Vandenberghe. Convex Optimization. Cambridge University Press, 2004.