The Schwarzschild solution saturates to constant velocity at far distances, the Newtonian limit is a special case only

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Abstract

The focus of this work is on an integration constant which appears when the geodesic equations for the Schwarzschild solution are solved directly. The constant has a dimension of speed-squared and is interpreted as vacuum energy. When it is set to zero, the solution is Newtonian in the far limit, otherwise it causes orbital velocities to be constant in the long range, explaining flat rotation curves. The literature is rich of examples where the four geodesic equations are solved, but the constant was not produced. This is because the direct solution one of the geodesic equations was avoided and an 'equivalent' equation was solved instead. In this work the direct solution is achieved by using an integrating factor which results in the hitherto unknown integration constant. The existence of fossile disks in the velocity profiles of galaxies is interpreted as a consequence of the vacuum energy decreasing in time. Again, in short: The geodesic equations in their geometric form are second order differential equations. They deliver additional integration constants when they are integrated to first order. Not all of the constants survive, but one of them does and when it is not deliberately set to zero it saturates velocities at far distances.

1 Introduction

On the one hand Einstein's general relativity is verified with ever increasing precision, see for example Will [1]. On the other hand there was dissatisfaction because apparantly general relativity was not able to explain the flat rotation curve of galaxies. Other theories and explanations popped up, the most prominent being dark matter, which was a very diplomatic solution: Einstein's validity was not questioned, everything follows his rules, there is just something which follows but cannot be seen.

Doubts that something might have gone wrong with the use of Einstein's equations were rare. As an example, Boccaletti et al. [2] express the discomfort they have with the traditional methods:

> *[...] we remain in a geometric context to obtain the geodesics, i.e., the orbits of a test particle in the gravitational field, without being obliged to resort to concepts copied from classical mechanics. In fact it must be remarked that very often one uses too freely in general relativity procedures which obtain a true legitimacy only for $r \to \infty$ or weak gravitational fields.*

Concerning the textbook methods for solving the geodesics this doubt is substantiated. The textbooks avoid to solve one of the geodesic equations directly but chose to solve a replacement. Because the solution returns the expected Newtonian in the far limit the procedure is taken to be legitimate. To be clear: There is nothing wrong with the solution. It is certainly correct but by the procedure the general form of the solution was lost.

Finally a word on Milgrom's MOND [3]: It is a pity that so much scientific energy was wasted in hunting ghosts in other places (like dark matter), but this is definitely not true for MOND. Although it is a theory which targets to replace Einstein's theory it uses the key ingredient needed to solve the flat rotation problem - in an empirical form, deduced from observations. So MOND was not hunting ghosts, it was seeing contours in the fog.

This work starts with the line element for the Schwarzschild solution as the only[1] input, equations 1 - 4. From them, all further results are derived with Maxima, the open source computer algebra system [9].

Of essential importance here is the package ctensor, a *'package to compute the tensors of curved space(time), most notably the tensors used in general relativity'* [10], maintained by V.Toth [11].

Then, after the geodesic equations are obtained with this package, they are solved, analyzed and plotted with the standard tools of Maxima.

2 The geodesic equations

The coordinates are the spherical coordinates

$$[t, r, \vartheta, \varphi] \tag{1}$$

The line element of the Schwarzschild solution is [4]:

$$ds^2 = -c^2\, F(r)\, dt^2 + \frac{dr^2}{F(r)} + r^2\, \left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right) \tag{2}$$

where the distortion factor F(r) is

$$F(r) = 1 - \frac{rs}{r} \tag{3}$$

and rs is defined as

$$rs = \frac{2\, GM}{c^2} \tag{4}$$

The metric tensor is calculated from the line element

$$\begin{pmatrix} -c^2\, F(r) & 0 & 0 & 0 \\ 0 & \frac{1}{F(r)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\, \sin^2\vartheta \end{pmatrix} \tag{5}$$

The Christoffel symbols and the Ricci tensor are calculated (both results are not displayed here) and then the geodesic equations.
The four geodesic equations are

$$\frac{d^2}{d\,s^2}\, t + \frac{\frac{d}{d\,s}\, r\, \left(\frac{d}{d\,r}\, F(r)\right)\, \left(\frac{d}{d\,s}\, t\right)}{F(r)} \tag{6}$$

$$\begin{aligned} &- r\, F(r)\, \left(\frac{d}{d\,s}\, \vartheta\right)^2 - \left(\frac{d}{d\,s}\, \varphi\right)^2\, r\, F(r)\, \sin^2\vartheta \\ &\qquad + \frac{c^2\, F(r)\, \left(\frac{d}{d\,r}\, F(r)\right)\, \left(\frac{d}{d\,s}\, t\right)^2}{2} - \frac{\left(\frac{d}{d\,s}\, r\right)^2\, \left(\frac{d}{d\,r}\, F(r)\right)}{2\, F(r)} + \frac{d^2}{d\,s^2}\, r \end{aligned} \tag{7}$$

$$\frac{d^2}{d\,s^2}\, \vartheta + \frac{2\, \left(\frac{d}{d\,s}\, r\right)\, \left(\frac{d}{d\,s}\, \vartheta\right)}{r} - \left(\frac{d}{d\,s}\, \varphi\right)^2\, \cos\vartheta\, \sin\vartheta \tag{8}$$

$$2\, \left(\frac{d}{d\,s}\, \varphi\right)\, \cot\vartheta\, \left(\frac{d}{d\,s}\, \vartheta\right) + \frac{2\, \left(\frac{d}{d\,s}\, \varphi\right)\, \left(\frac{d}{d\,s}\, r\right)}{r} + \frac{d^2}{d\,s^2}\, \varphi \tag{9}$$

[1] The use of observational data lateron might be seen as further inputs. But they are only used to compare.

3 The solution in the equator, $\vartheta = \pi/2$

Let's assume there exists an orbit which always remains in the equatorial plane, at $\vartheta = \pi/2$. There are arguments for this. e.g. by stating that the line element is symmetric around $\pi/2$, but here the formal procedure is preferred: If the geodesic equations have a solution with $\vartheta = \pi/2$ then the assumption was correct.

The equations are solved only up to the first integrals which is sufficient to assemble the four-velocity and show the saturation. The steps are:

Start with the goedesic equations,
substitute

$$\vartheta = \frac{\pi}{2} \tag{10}$$

It turns the third geodesic (8) into zero.
Select the first geodesic (6) and solve for t

$$\frac{d}{d\,s}\,t = \frac{kt}{F\,(r)} \tag{11}$$

select the fourth geodesic (9), set $\vartheta = \pi/2$ and solve for φ

$$\frac{d}{d\,s}\,\varphi = \frac{kp}{r^2} \tag{12}$$

where we can see that kp is nothing else than the classical angular momentum, more precisely the specific angular momentum L/m. But we want to stay clear of any classical influence and keep using kp.
Now select the second geodesic (7) - the only remaining - and solve it for r. Because the new integration constant will appear here, the derivation will be done in minute steps:

We insert (10), (11), (12) into the second geodesic (7)
and collect the derivatives on the left side:

$$r^3\,\left(\frac{d}{d\,s}\,r\right)^2\,rs - 2\,r^4\,\left(\frac{d^2}{d\,s^2}\,r\right)\,(r-rs) = c^2\,kt^2\,r^3\,rs - 2\,kp^2\,(r-rs)^2 \tag{13}$$

The integrating factor to solve this differential equation is:

$$\frac{\frac{d}{d\,s}\,r}{r^3\,(r-rs)^2} \tag{14}$$

Equation (13) is multiplied by the integrating factor (14) and both sides are integrated.
The left side is done by the product rule in reverse and is:

$$-\frac{r\,\left(\frac{d}{d\,s}\,r\right)^2}{r-rs} + klhs \tag{15}$$

The right side is:

$$-\frac{c^2\,kt^2\,rs}{r-rs} + \frac{kp^2}{r^2} + krhs \tag{16}$$

where $klhs$ and $krhs$ are the integration constants of the left hand and right hand sides, respectively. We reassemble the geodesic equation in the form (13) by setting the first integrals leftside = rightside, isolate the differential on the left side, merge the two integration constants into one constant kv, take the square root and get

$$\frac{d}{d\,s}\,r = signr\sqrt{\left(kv - \frac{kp^2}{r^2}\right)\,\left(1-\frac{rs}{r}\right) + \frac{c^2\,kt^2\,rs}{r}} \tag{17}$$

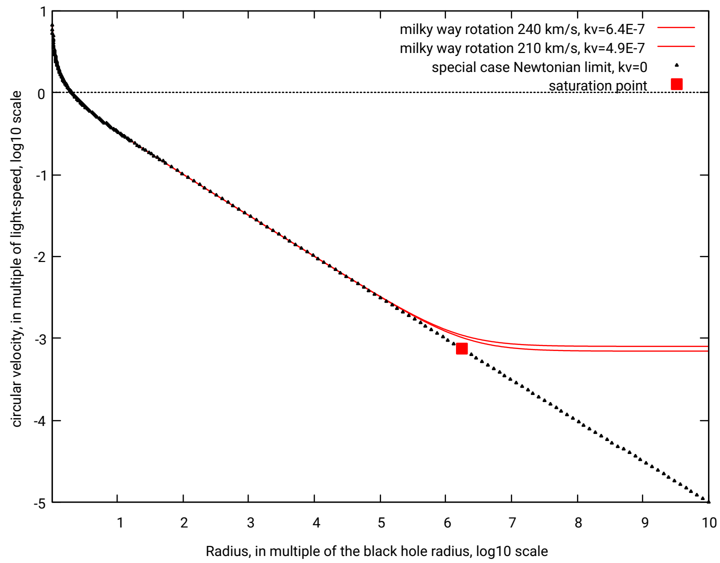

Figure 1: *The saturation of circular orbit velocities for a Schwarzschild hole depends on the integration constant kv. With smaller kv the gravitational action extends farther before it is overridden by the constant. The small dots show the special case which is Newtonian at infinity (kv=0). The lines correspond to the rotational velocity of the milky way. The square shows the saturation point, the point where the asymptotic saturation velocity crosses the Newtonian line. It defines the saturation radius as a rough measure how far the gravitational action extends. For the milky way shown in this figure the mean saturation radius of the two curves is R = 1.77E6*

where $signr = \pm 1$ is the sign choice for the square root operation we did.

As a summary and verification: The solutions (10), (11), (12) and (17) were used and when they are inserted into the four geodesic equations this results to:

$$[0, 0, 0, 0] \tag{18}$$

With the solution (17) the second geodesic equation (7) can be put into following form:

$$\frac{\left(\frac{d}{d\,s}\,r\right)^2}{F\,(r)} + \left(\frac{d}{d\,s}\,\varphi\right)^2\,r^2 = \frac{c^2\,kt^2}{F\,(r)} + kv - c^2\,kt^2 \tag{19}$$

which is useful for the discussion of circular and radial geodesic orbits. The constant kv on the right side is responsible for the saturation of velocities in both cases.

When the first term on the left side of equation (19) is set to zero we get circular orbits. The second term then describes the circular speed, squared. Figure 1 shows the saturation of circular speeds for kv values corresponding to astronomical observations.

When the second term on the left side of equation (19) is set to zero we get radial orbits. A plot for the radial velocity is ommitted because at large distances it would appear as being the same plot like Figure 1. The factor between the velocities of those two cases is

$$\sqrt{1 - \frac{1}{R}} \tag{20}$$

where R is the multiple of the black hole radius. So a plot of radial velocities would show no difference at large radii. A visible difference would be only near the black hole radius where the radial velocity remains a straight line, it does not bend up like the circular velocity does.

3.1 The four-velocity and its magnitude

As we are still in the equator, the velocity component for theta is zero. The solutions are assembled as a vector:

$$\begin{pmatrix} \frac{d}{d\,s}\,t \\ \frac{d}{d\,s}\,r \\ 0 \\ \frac{d}{d\,s}\,\varphi \end{pmatrix} = \begin{pmatrix} \frac{kt}{1-\frac{rs}{r}} \\ \sqrt{\left(kv - \frac{kp^2}{r^2}\right)\,\left(1-\frac{rs}{r}\right) + \frac{c^2\,kt^2\,rs}{r}}\;signr \\ 0 \\ \frac{kp}{r^2} \end{pmatrix} \tag{21}$$

The magnitude of this four-velocity we get by calculating its inner product, weighted by the metric, equation (5). We do this for both sides:

$$-\,c^2\,F\,(r)\,\left(\frac{d}{d\,s}\,t\right)^2 + \frac{\left(\frac{d}{d\,s}\,r\right)^2}{F\,(r)} + \left(\frac{d}{d\,s}\,\varphi\right)^2\,r^2 = kv - c^2\,kt^2 \tag{22}$$

The result is identical to equation (19), just one of the t-terms is shifted to the left side. On the left side the multiplication by the metric still can be recognized, the right side is composed of constants. So the magnitude of the four-velocity is constant - as expected for a movement along a geodesic.

But the right side is not the same as we are familiar with. It now has the constant kv added to it. Because of the dimension of this equation kv is interpreted to represent the vacuum energy.

When equation (22) is taken as the Lagrangian[2] then kv needs to be included. It now becomes visible that a method which solves the geodesics with the Lagrangian might be in danger to be incomplete. But more on this just next.

3.2 How the textbook-like derivations miss the second order integration constant

The method of textbooks can be summarized as:
The geodesic equations are of second-order form, but they can be simplified to first-order form if there exist three constants of the motion and a normalization condition on the four-velocity.

Or (to quote a more detailled formulation [5]:

> *The [...] Schwarzschild solution is static and spherically symmetric. As a result of this symmetry, the energy of the particle and the three components of its angular momentum are conserved. One can thus construct four integrals of geodesic motion that are functionally independent and mutually Poisson commute, choosing, for example, the (trivial) normalization of the four-velocity, the particle's energy, the square of its total angular momentum, and a projection of the angular momentum to an arbitrary axis. According to the Liouville's theorem, the existence of such quantities makes the geodesic motion in the spherically symmetric black hole case completely integrable.*

The problematic step is the 'normalization of the four-velocity'.
Equation (22) shows that it does not normalize to zero (for light-like particles) or -1 (resp. $-c^2$) for massive particles but has the additional constant kv - the vacuum energy. When equation (22) is taken as the Lagrangian and the geodesics are computed via Euler-Lagrange the constant disappears because of the derivatives involved in Euler-Lagrange. Therefore this method will produce the same geodesic equations.

But the Lagrangian also offers itself to substitute for the 'difficult' geodesic equation: Because when all other first integrals are already known it presents an easier way to compute the remaining first integral for the radius. In this case it ought to contain the constant.

[2] The equation is to be seen as Lagrangian + constant. The specialtity is here that the constant is not something what was imposed from outside, like a cosmological constant or similar, but a constant which emerges from the geodesic themselves.

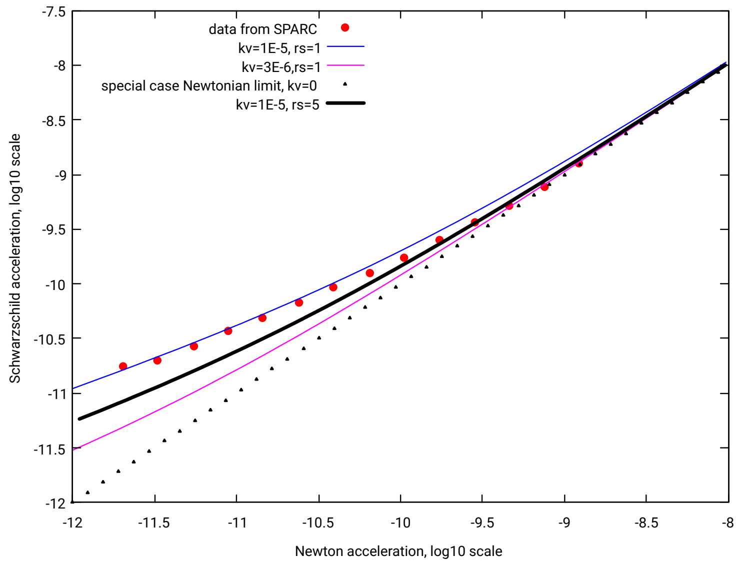

Figure 2: *The velocity saturation resembles an acceleration relation. But because the curves are determined by two parameters (rs and kv) such a presentation remains ambiguous. The big points are data from the SPARC collaboration [6] [7] which they supply on their web page [8]. The line with small dots is the identity line. It shows the special case which is Newtonian at infinity.*

So the textbook-like derivation is caught in an inescapable circle, it is indeed a trap: Starting with a Lagrangian having no vacuum energy and reusing this assumption while the geodesics are solved cannot turn up the need for an additional constant. It all looks self-consistent and assuring. Actually it is, but only for a solution which is a special case.

3.3 The acceleration relation

Although it is the velocity which saturates, let's see how this looks like when it is expressed in terms of accelerations. Considering the circular orbit, Newton would say: There is a radial acceleration which keeps the particle in the circular path.
This acceleration is

$$acc = \frac{v\_circular^2}{r} \tag{23}$$

Substituting the solution for the circular velocity - equation (19) with zero radial speed - results in:

$$acc = \frac{c^2\, kt^2\, rs}{r\,(r - rs)} + \frac{kv}{r} \tag{24}$$

The acceleration depends on two constants: On kv and on rs - the black hole radius. Therefore the acceleration relation remains ambiguous on those two constants, see Figure 2 .

4 The polar solution, $\varphi = const = \varphi_0$

There is the symmetry argument: One can always rotate the coordinate system so that the chosen polar orbit becomes equatorial. So we could save this section. But after the Lagrangian experience we prefer the formal procedure and go through the derivation again.

Start with the goedesic equations,
substitute

$$\varphi = \varphi_0 \tag{25}$$

It turns the fourth geodesic (9) into zero.
The solution for the first geodesic (6) is the same as above

$$\frac{d}{d\,s}\,t = \frac{kt}{F\,(r)} \tag{26}$$

Select the third geodesic (8), set $\varphi = \varphi_0$ and solve for ϑ

$$\frac{d}{d\,s}\,\vartheta = \frac{kth}{r^2} \tag{27}$$

Now select the second geodesic (7) - the only remaining - and solve it for r.
Insert all results we got up to now, collect the derivatives on the left side:

$$r^3\left(\frac{d}{d\,s}\,r\right)^2 rs - 2\,r^4\left(\frac{d^2}{d\,s^2}\,r\right)(r-rs) = c^2\,kt^2\,r^3\,rs - 2\,kth^2\,(r-rs)^2 \tag{28}$$

There is no φ or φ_0 dependency in any of the solutions. Furthermore, when in equation (28) kth is replaced by kp it becomes equation (13). Here we can stop the derivation. We can take the equatorial solutions and replace φ by ϑ, kp by kth.

We see that the symmetry argument was right. But following this argument would have suggested to classify the solution just to be an 'equator solution rotated into another orientation'.

Instead we consider it as a solution which may exist in addition to an equatorial disk. Like the milky way does: It has the polar structure which is orbiting at almost 90 degrees to the equatorial plane. Or one could consider the $\varphi_0 = 0$ structure as being the main disk and have other planes tilted at arbitrary angles φ.

5 Discussion

It is important to note that the curves in the figures are for single Schwarzschild holes and a galaxy has distributed matter, so the comparison with galactical data might be questioned. It is unclear how additional matter outside a central black hole influences the saturation - be it a shell or a portion of a disk. The central question is: In terms of saturation - can one replace a shell of matter by a virtual black hole mass at the center? The position taken here is that kv is an independent parameter. The gravitational action extends to infinity like we knew it, but is overridden (or dominated) by the constant kv at some distance.

An interesting case arises when matter is added in the flat portion of the rotation curve, outside the present saturation radius. With kv being an independent constant it means that adding the matter will not change anything outside the saturation radius. On the outside the additional matter will be gravitationally invisible. Of course, locally it will have an effect - like the solar system rotating around the milky way has its local gravity. But the only effect on galactical scale is that the saturation radius will shift towards a larger radius. Roughly spoken (neglecting the asymptotical transition): Adding matter far outside will change the gravitational pull only for objects inside the saturation radius.

This is equivalent to the statement that all matter outside of a saturation radius is represented by a virtual black hole at the center. But this leads also to the conclusion that the saturation radius of a galaxy does not grow with accumulation. From the beginning it is determined by its total mass, be it in the disk or in the halo.

But what if kv changes with time?
Assume a first disk in an early stage while the saturation radius was smaller (kv being larger). When kv decreases the saturation radius grows and the first disk (or the inner part of it) becomes confined to the Newtonian region. Its circular paths with flat rotation velocity will shrink to Newtonian orbits. Its innermost orbits will 'fall' to orbits having the highest velocities. So the first disk will exhibit a velocity profile which decreases radially towards the saturation velocity. The saturation radius of a galaxy is then located somewhere at the bottom of the decreasing slope. When kv is lowered the

profile will extend at the lower end of velocities because the flat bottom is lowered. There are plenty of astronomical observations which show such a fossile infant disk, see e.g. for the milky way Sofue [12], Fig.8. It shows a velocity slope from the highest value down to to the first valley at 2.5[kpc].

Roughly read from Sofue's figure the saturation radius of the milky way is $\approx$ 1.5 - 2 [kpc], at most 2.5[kpc]. From this and the definition of the saturation radius in Figure 1 the total mass of the milky way is calculated to be $\approx$ 0.89 - 1.18 E12, at most 1.48E12 solar masses.

6 Conclusions

This work shows that for a general solution of orbits the geodesic equations must be solved directly, without using a substitute authorized by classical reason. Because the differential equations are of second order, more integration constants are to be expected than from an ansatz using a first order substitute. For Schwarzschild only one of the second order constants survives, but this one is sufficient to saturate the gravitational action at long distances and causing constant rotation curves.

Considering the other successes of Einstein's general relativity it is save to conclude that the velocity saturation is the only mechanism needed to explain the flat rotation curves of galaxies. There is no need for dark matter, the flat rotation is explained by Einstein.

The existence of fossile disks in the velocity profiles of galaxies proves that the vacuum energy kv decreases in time.

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