On the Universal Gravitational Constant

[Calculating its exact value]

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Abstract

The mathematical calculation of the Universal Constant of Gravitation has eluded physicists and mathematicians for in excess of three centuries and it is believed by the author that this is the first time that it has been achieved since Newton published “The Principia” in 1687. The method relies upon only two equations, the combination of which allows accurate calculation of G.

Subject Headings: Nuclear physics, Fundamental Constants, Gravitational Constant.

Introduction

The goal of this paper is to calculate as accurately as possible Newton’s Universal Gravitational constant. It is often stated that the fundamental constants are “natural” and fixed scalar values, however a more accurate definition may be that they are unchanging mathematical proportionalities and it is these proportionalities that are constant. The general approach in this paper is to analyze all of the physical constants that are used in the calculations in relation to the properties of the most fundamental atom, hydrogen, some of which have alternative simplified solutions.

§1. The Bohr radius [a_0]

It is generally accepted that the Bohr radius[1] a_0 represents the point on the axis where the attractive and repulsive force on the electron are equal and in balance. By the very nature of this statement the implication is that this point must also represent the escape velocity v_e of the electron or is indeed equivalent to the value of the Schwarzschild radius. The currently accepted value of the Bohr radius can be obtained from the more traditional version of the equation;

Value $a_0 = 5.291\,772\,109\,03 \times 10^{-11}\text{m}$

Uncertainty 1.5×10^{-10}

Calculation $a_0 = \dfrac{\hbar}{m_e c \alpha}$

It is thought that the introduction of the reduced Planck constant however, disguises somewhat the underlying equation. It will be shown that the actual equation is more accurately represented when replacing the reduced Planck constant with more fundamental values as follows;

Value $a_0 = 5.291\,772\,109\,03 \times 10^{-11}\text{m}$

Uncertainty 1.5×10^{-10}

Calculation $a_0 = \dfrac{\text{e}_0{}^2 \times 10^{-7}}{m_e \alpha^2}$

From the prior results it can be seen that the value of the Bohr radius is indeed consistent with the currently published CODATA value.

§2. Electron mass [m_e]

The final equations to calculate the Gravitational constant requires the value of the electron mass[2] the published CODATA value being;

Value $m_e = 9.109\,383\,7015 \times 10^{-31}\text{kg}$

Uncertainty 3.0×10^{-10}

Calculation $m_e = \dfrac{l_P m_P}{a_0 \alpha}$

This value can also be validated from the Planck mass, length, Bohr radius and the fine structure constants returning an identical value.

[1] *NIST: Bohr radius*

[2] *NIST: Electron mass*

§3. The Magnetic Constant [μ_0]

The magnetic constant or as it is often termed the vacuum permeability[3] is the ability of the vacuum to support a magnetic field, whose value depends only upon the value of pi. The most recent SI changes however have established the measured value as;

Value $\mu_0 = 1.256\,637\,062\,13 \times 10^{-6}\ \mathrm{N \cdot A^{-2}}$

Uncertainty 1.5×10^{-10}

Calculation $\mu_0 = 4\pi \times 1.000\,000\,000\,54 \times 10^{-7}\ \mathrm{N \cdot A^{-2}}$

The current CODATA value is not an exact multiple of pi as seen from the current published value.

§4. Speed of Light in a vacuum [c]

The currently published CODATA value for the speed of light in a vacuum[4] is found to be;

Value $c = 2.997\,924\,58 \times 10^{8}\ \mathrm{m \cdot s}$

Uncertainty Exact

Calculation $c = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}}$

The equation although not used in this context is for completeness only, but does return the exact value.

§5. The Fine Structure constant [α]

The fine structure constant[5] represents possibly one of the most accurately measured values of all. In man y circles the fine structure constant is often thought to be a mysterious value of nature, namely 1/137. Even reputable physicists such as Richard Feynman declared that "*It's a pure number that shapes the universe to an astonishing degree — "a magic number that comes to us with no understanding*". It is often forgotten or indeed overlooked that in 1916 Arnold Sommerfeld discovered that the fine structure constant is actually a dimensionless ratio between the speed of light and the orbital velocity of the ground state electron in a hydrogen atom. The currently published value of the fine structure constant being;

Value $\alpha = 7.297\,352\,5693 \times 10^{-3}$

Uncertainty 1.5×10^{-10}

Calculation $\alpha = \dfrac{e^2}{4\pi\varepsilon_0 \hbar c}$

This equation can be further simplified by the replacement of the reduced Planck and the magnetic constant with more fundamental values. When this is simplified the result is an equation showing that the hypothesis of Arnold Sommerfeld is indeed correct;

Value $\alpha = 7.297\,352\,5693 \times 10^{-3}$

Uncertainty 1.5×10^{-10}

Calculation $\alpha = \dfrac{v_o}{c}$

The equation does of course return a value that matches with CODATA perfectly.

§6. The Elementary Charge [e_0]

The elementary charge[6] is best described as the electric charge carried by a single proton or, the magnitude of the negative electric charge carried by an electron measured in Coulombs;

Value $e_0 = 1.602\,176\,634 \times 10^{-19}\ \mathrm{C}$

Uncertainty Exact

Calculation $e_0 = \alpha\sqrt{m_e a_0 \times 10^7}$

The value of the elementary charge can be obtained from the product of the fine structure constant, the mass of the electron and the orbital radius. Being that the fine structure constant can be represented as the orbital velocity divided by the orbital radius of the electron, the simplified equation is shown above. As stated the calculated value for the elementary charge is exactly in agreement with the currently published CODATA value.

§7. Orbital Speed of the Electron [v_o]

Although the orbital speed of the electron cannot be considered a constant it will be seen that it represents a value that occupies a significant role in validation of many of the physical constants. The value obtained, is commensurate with the relationship discovered by Arnold Sommerfeld. Using the equation below the orbital speed of the electron can be calculated;

[3] *NIST: Vacuum Permeability*
[4] *NIST: Speed of light in a vacuum*
[5] *NIST: Fine Structure Constant*

[6] *NIST: Elementary Charge*

Value $v_o = 2.187\ 691\ 263 \times 10^{6}\ \mathrm{m \cdot s}$

Uncertainty 1.5×10^{-10}

Calculation $v_o = \alpha c$

The value returned is the current agreed upon value.

§8. The Planck Constant [h]

It is widely believed that the Planck constant[7] is a natural constant, a scalar value from which other constants can be established and that the exact derivation of its value is unknown. The generally accepted value of the reduced Planck constant can be determined to be the angular momentum of the electron in the ground state of a hydrogen atom. Dimensional analysis does confirm that the dimensions are indeed the same.

Value $\hbar = 1.054\ 571\ 816 \times 10^{-34}\ \mathrm{J \cdot Hz^{-1}}$

Calculation $\hbar = L_e = [\mathrm{ML^2T^{-1}}]$

In 1913 Niels Bohr postulated that the angular momentum of electrons can only have certain discrete values. The equation below represents the momentum where n is a discrete positive integer value of the electron shell orbit. If the properties of the ground state electron of the hydrogen atom are substituted into the equation, whereby the value of n is 1, representing the innermost orbit and the Planck constant is replaced by its more fundamental elements;

Value $\hbar = 1.054\ 571\ 816 \times 10^{-34}\ \mathrm{J \cdot Hz^{-1}}$

Uncertainty 1.5×10^{-10}

Calculation $\hbar = \dfrac{nh}{2\pi}$

The reduced Planck constant can also be calculated using the ground state electron in the hydrogen atom;

Value $\hbar = 1.054\ 571\ 817 \times 10^{-34}\ \mathrm{J \cdot Hz^{-1}}$

Calculation $\hbar = m_e v_o a_0$

The exact value returned, confirms without contradiction that the reduced Planck constant is indeed equivalent to the angular momentum of the ground state electron in the hydrogen atom. Following convention, the Planck constant itself can be derived from its reduced value by the introduction of pi;

Value $6.626\ 070\ 15 \times 10^{-34}\ \mathrm{J \cdot Hz^{-1}}$

Calculation $h = 2\pi m_e v_o a_0$

Further correlation of the calculated value of the Planck constant can be found by using the values for the energy and frequency of the hydrogen atom in the following equation which once more confirms the value to be correct;

Value $6.626\ 070\ 15 \times 10^{-34}\ \mathrm{J \cdot Hz^{-1}}$

Calculation $h = \dfrac{U}{f}$

The results, not only agree with the currently published CODATA values, as a consequence of this it must be conceded that the Planck constant is not, as is thought by many, a fundamental constant but rather what could be better described as the angular momentum of the ground state electron in the hydrogen atom.

§9. Avogadro's Number [N_A]

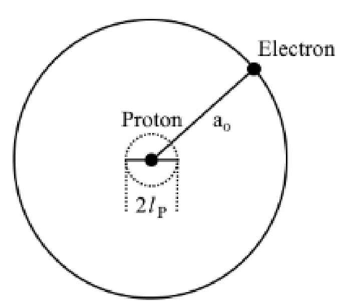
Avogadro's number[8] can be thought of as representing the number of electrons contained in one mole of any substance. Current measurement is defined as one twelfth of the mass of an unbound neutral atom of carbon-12 in its nuclear and electronic ground state at rest.

Value $6.022\ 140\ 762 \times 10^{23}\ \mathrm{mol^{-1}}$

Uncertainty Exact

Calculation $N_A = \dfrac{1}{D}$

An alternative to the current equation is to use the Bohr radius, the Planck length and Euler's number.



[7] *NIST: Planck constant*

[8] *NIST: Avogadro's number*

It can be seen from the illustration above, twice the Planck length $2l_P$ is effectively the diameter of the smallest possible spherical entity. In order to calculate Euler's number, all that is required are the hyperbolic functions of sine and cosine as shown below;

Value 2.718 281 828

Uncertainty Exact

Calculation $e = cosh(1) + sinh(1)$

It follows that the number of electrons per mole when using Euler's number results in the convergence to a value representing the proportion of electrons per mole namely, Avogadro's number;

Value $6.022\ 140\ 762 \times 10^{23}\ \text{mol}^{-1}$

Uncertainty Exact

Calculation $N_A = \frac{a_0}{2l_P e}$

It is apparent that the value returned by the equation is identical to the current published CODATA value for Avogadro's number.

§10. The Gravitational Constant [G]

To calculate the value of the Universal Gravitational Constant[9] the following method involves the combination of two equivalent equations to calculate the Planck length. The first of these equations is the instantly recognizable equation for the Planck length which itself contains Newton's constant of gravitation;

$$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616\ 255 \times 10^{-35}\ \text{m} \qquad (10.0)$$

The second equation to be used which has been shown previously to also return an identical value for the Planck length;

$$l_P = \frac{a_0}{2N_A e} = 1.616\ 255 \times 10^{-35}\ \text{m} \qquad (10.1)$$

It follows that, as the two equations are indeed equivalent they can be combined;

[9] *NIST: Universal Gravitational Constant*

$$\frac{a_0}{2N_A e} = \sqrt{\frac{\hbar G}{c^3}} \qquad (10.2)$$

Substitution, simplification and subsequent rearrangement of the above results in an equation that can be used to determine precisely the Universal Gravitational constant of Newton;

$$G = \frac{a_0 c^2}{4 m_e \alpha {N_A}^2 e^2} \qquad (10.3)$$

The calculated values of the parameters used in the equation are as follows;

Symbol	Value	Uncertainty
a_0	$5.291\ 772\ 109\ 03 \times 10^{-11}$	1.5×10^{-10}
m_e	$9.109\ 383\ 7015 \times 10^{-31}$	3.0×10^{-10}
α	$7.297\ 352\ 5693 \times 10^{-3}$	1.5×10^{-10}
c	$2.997\ 924\ 58 \times 10^{8}$	Exact
N_A	$6.022\ 140\ 762 \times 10^{23}$	Exact
e	2.718 281 828	Exact

The above values when entered result in a value of Newton's Universal Gravitational constant to an unprecedented accuracy, significantly greater than any current physical measurements made to date;

$$G = 6.674\ 787\ 645\ 64 \times 10^{-11} \qquad (10.4)$$

It follows from the above table of parameters that the uncertainty of the calculated value for the Gravitational constant itself is limited only by the accuracy of the measurement of the electron mass.

Summary

The actual value of the Gravitational constant has been ascertained mathematically, using only current CODATA published values. The result not only agrees with, but substantially exceeds, the accuracy of current measured and published data.

$$G = 6.674\ 787\ 645\ 64 \times 10^{-11}$$

The equations are uncontroversial, well understood and require nothing more than high school mathematics the result is conservatively estimated to be in excess of five σ.