Question 440 : Nested Radicals and Trigonometric Formulas

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abstract

This note presents some trigonometric formulas that involving nested radicals.

1. INTRODUCTION. An formula

$$\sqrt[3]{7+7\sqrt[3]{7+7\sqrt[3]{7+\ldots}}} = \sqrt{7-\frac{7}{\sqrt{7-\frac{7}{\sqrt{7-\ldots}}}}} + \left\{1+\frac{1}{7}\left(1+\frac{1}{7}(1+\ldots)^3\right)^3\right\} \tag{1}$$

2. SOME FORMULAS FOR PI.

$$\pi = 6\arcsin\left(\frac{\sqrt{21}}{14}\sqrt{7-\frac{7}{\sqrt{7-\frac{7}{\sqrt{7-\ldots}}}}}\right) - 2\arcsin\left(\frac{1}{\sqrt{28}}\right) \tag{2}$$

$$\pi = 6\arcsin\left(\frac{\sqrt{21}}{14}\left\{1+\frac{1}{7}\left(1+\frac{1}{7}(1+\ldots)^3\right)^3\right\}\right) + 2\arcsin\left(\frac{1}{\sqrt{28}}\right) \tag{3}$$

$$\pi = 3\arcsin\left(\frac{\sqrt{21}}{14}\sqrt[3]{7+7\sqrt[3]{7+7\sqrt[3]{7+\ldots}}}\right) - \arcsin\left(\frac{9}{2\sqrt{21}}\right) \tag{4}$$

$$\pi = 3\arcsin\left(\frac{\sqrt{21}}{14}\sqrt{7-\frac{7}{\sqrt{7-\frac{7}{\sqrt{7-\ldots}}}}}\right) + 3\arcsin\left(\frac{\sqrt{21}}{14}\left\{1+\frac{1}{7}\left(1+\frac{1}{7}(1+\ldots)^3\right)^3\right\}\right) \tag{5}$$

3. SOME TRIGONOMETRIC FORMULAS.

$$35+21\sqrt[3]{7+7\sqrt[3]{7+7\sqrt[3]{7+\ldots}}}=-128\left(\sin\frac{4\pi}{7}\right)^6\cos\frac{8\pi}{7}=-128\left(\cos\frac{\pi}{14}\right)^6\cos\frac{8\pi}{7}=$$
$$=128\left(\cos\frac{\pi}{14}\right)^6\cos\frac{\pi}{7}=128\left(\cos\frac{\pi}{14}\right)^6\sin\frac{5\pi}{14}=128\left(\sin\frac{4\pi}{7}\right)^6\sin\frac{5\pi}{14} \tag{6}$$

$$35-21\left\{1+\frac{1}{7}\left(1+\frac{1}{7}(1+\ldots)^3\right)^3\right\}=-128\left(\sin\frac{2\pi}{7}\right)^6\cos\frac{4\pi}{7}=$$
$$=-128\left(\cos\frac{3\pi}{14}\right)^6\cos\frac{4\pi}{7}=128\left(\sin\frac{2\pi}{7}\right)^6\sin\frac{\pi}{14}=128\left(\cos\frac{3\pi}{14}\right)^6\sin\frac{\pi}{14} \tag{7}$$

$$-35+21\sqrt{7-\cfrac{7}{\sqrt{7-\cfrac{7}{\sqrt{7-\ldots}}}}}=128\left(\sin\frac{8\pi}{7}\right)^6\cos\frac{2\pi}{7}=128\left(\sin\frac{\pi}{7}\right)^6\cos\frac{2\pi}{7}=$$
$$=128\left(\sin\frac{\pi}{7}\right)^6\sin\frac{3\pi}{14}=128\left(\cos\frac{5\pi}{14}\right)^6\sin\frac{3\pi}{14}=128\left(\cos\frac{5\pi}{14}\right)^6\cos\frac{2\pi}{7} \tag{8}$$

$$5+21\sqrt[3]{7+7\sqrt[3]{7+7\sqrt[3]{7+\ldots}}}=52\cos\frac{\pi}{7}+32\cos\frac{2\pi}{7}+12\cos\frac{3\pi}{7}+2\cos\frac{4\pi}{7}=$$
$$=10\sin\frac{\pi}{14}+32\sin\frac{3\pi}{14}+52\sin\frac{5\pi}{14} \tag{9}$$

$$-5+21\left\{1+\frac{1}{7}\left(1+\frac{1}{7}(1+\ldots)^3\right)^3\right\}=32\cos\frac{\pi}{7}-2\cos\frac{2\pi}{7}+52\cos\frac{4\pi}{7}-12\cos\frac{5\pi}{7}=$$
$$=-52\sin\frac{\pi}{14}+10\sin\frac{3\pi}{14}+32\sin\frac{5\pi}{14} \tag{10}$$

$$-5+21\sqrt{7-\frac{7}{\sqrt{7-\frac{7}{\sqrt{7-\ldots}}}}}=52\cos\frac{2\pi}{7}-32\cos\frac{4\pi}{7}+12\cos\frac{6\pi}{7}-2\cos\frac{8\pi}{7}= \qquad (11)$$

$$=32\sin\frac{\pi}{14}+52\sin\frac{3\pi}{14}-10\sin\frac{5\pi}{14}$$

4. EQUATIONS.

$$u=\sqrt[3]{7+7\sqrt[3]{7+7\sqrt[3]{7+\ldots}}}\Rightarrow u^3-7u-7=0 \qquad (12)$$

$$v=1+\frac{1}{7}\left(1+\frac{1}{7}\left(1+\frac{1}{7}(1+\ldots)^3\right)^3\right)^3\Rightarrow v^3-7v+7=0 \qquad (13)$$

$$w=\sqrt{7-\frac{7}{\sqrt{7-\frac{7}{\sqrt{7-\ldots}}}}}\Rightarrow w^3-7w+7=0 \qquad (14)$$

5. FORMULAS WITH u,v,w .

$$\begin{cases}\frac{1}{6}+\frac{7}{30}u-\frac{2}{5}v-\frac{1}{15}w=\sin\frac{\pi}{14}\\ -\frac{1}{6}-\frac{1}{15}u+\frac{7}{30}v+\frac{2}{5}w=\sin\frac{3\pi}{14}\\ \frac{1}{6}+\frac{2}{5}u-\frac{1}{15}v-\frac{7}{30}w=\sin\frac{5\pi}{14}\end{cases} \qquad (15)$$

$$\begin{cases}\frac{1}{6}u-\frac{1}{6}v+\frac{1}{3}w=\sin\frac{\pi}{14}+\sin\frac{3\pi}{14}\\ \frac{1}{3}u+\frac{1}{6}v+\frac{1}{6}w=\sin\frac{3\pi}{14}+\sin\frac{5\pi}{14}\\ \frac{1}{6}u+\frac{1}{3}v-\frac{1}{6}w=-\sin\frac{\pi}{14}+\sin\frac{5\pi}{14}\end{cases} \qquad (16)$$

$$\frac{u}{w}=2\cos\frac{\pi}{7}\quad,\quad\frac{w}{v}=2\cos\frac{2\pi}{7}\quad,\quad\frac{v}{u}=2\cos\frac{3\pi}{7} \qquad (17)$$

$$u=2\left(\sin\frac{3\pi}{14}+\sin\frac{5\pi}{14}\right),\ v=2\left(-\sin\frac{\pi}{14}+\sin\frac{5\pi}{14}\right),\ w=2\left(\sin\frac{\pi}{14}+\sin\frac{3\pi}{14}\right) \tag{18}$$

$$u=v+w \quad , \quad u^2-vw=7 \quad , \quad uvw=7 \tag{19}$$

$$u^2+v^2=7+uv \quad , \quad u^2+w^2=7+uw \quad , \quad v^2+vw+w^2=7 \tag{20}$$

$$(u+1)(v-1)(w-1)=1 \quad , \quad \frac{1}{v}+\frac{1}{w}-\frac{1}{u}=1 \tag{21}$$

$$\begin{cases} \sqrt[3]{u+1}=\sqrt[3]{v-1}+\sqrt[3]{w-1} \\ \sqrt{1+\frac{1}{u}}=\sqrt{1-\frac{1}{v}}+\sqrt{1-\frac{1}{w}} \end{cases} \tag{22}$$

$$\sqrt[3]{\sin\frac{\pi}{7}\sin\frac{2\pi}{7}-\frac{1}{4}}+\sqrt[3]{\sin\frac{\pi}{7}\sin\frac{3\pi}{7}-\frac{1}{4}}=\sqrt[3]{\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}+\frac{1}{4}} \tag{23}$$

$$\sqrt{4-\csc\frac{\pi}{7}\csc\frac{2\pi}{7}}+\sqrt{4-\csc\frac{\pi}{7}\csc\frac{3\pi}{7}}=\sqrt{4+\csc\frac{2\pi}{7}\csc\frac{3\pi}{7}} \tag{24}$$

$$\pi=4\arctan\frac{1}{v}+4\arctan\frac{1}{w}-4\arctan\frac{1}{u}-4\arctan\frac{1}{15} \tag{25}$$

$$\sqrt[6]{1-\frac{7}{4u^2}}+\sqrt[6]{1-\frac{7}{4v^2}}-\sqrt[6]{1-\frac{7}{4w^2}}=\sqrt[3]{\frac{3\sqrt[3]{7}-5}{2}} \tag{26}$$

$$u=\frac{\sqrt{7}}{2}\csc\frac{8\pi}{7} \quad , \quad v=\frac{\sqrt{7}}{2}\csc\frac{4\pi}{7} \quad , \quad w=\frac{\sqrt{7}}{2}\csc\frac{2\pi}{7} \tag{27}$$

$$\pi=4\arctan\frac{1}{u}+4\arctan\left(\frac{11}{3}-\frac{2}{3}\sqrt[3]{7+21\sqrt[3]{7+21\sqrt[3]{7+\ldots}}}\right) \tag{28}$$

References

1. Ramanujan, S. : Modular Equations and Approximations to π . Quart. J. Pure. Appl. Math. 45, 350 – 372, 1913-1914.
2. Ramanujan, S. : Notebooks (2 Volumes) , Tata Institute of Fundamental Research , Bombay , 1957.
3. Ramanujan, S. : The Lost Notebook and Other Unpublished Papers, Narosa, New Delhi, 1988.