Relativistic Dirac Quantum Theory in Complex Minkowski Space and Tachyonic Signaling

Richard L Amoroso[1] & Elizabeth A Rauscher
[1]amoroso@noeticadvancedstudies.us

In this paper we develop complex solutions to the Dirac equation and discuss various implications and applications.

1. Introduction

The Dirac electron theory is unique in that it is relativistic invariant and that it predicts two states of matter having opposite charges [1-4]. In addition is the concept of a full vacuum; one in which positive energy states, such as those of a gamma ray, can activate electron-positron pair production in which an electron is kicked out of the Fermi-Dirac sea to a positive energy state, leaving an electron hole position in the negative energy sea. The two-sign solution lead to the postulate of antimatter which has been well identified. In the 1920's when Dirac developed his mathematical description of the relativistic electron, the obtainment of an antielectron or positron solution, in addition to the electron solution, did not lead immediately to hypothesis of antimatter and appeared to be an anomalous solution. In 1932 Carl Anderson discovered the positron in cloud chamber photographs leading to a good example of prediction and confirmation. With the advent of the prediction of the antiproton and its identification at the Berkeley Bevatron bubble chamber by Emilo Segre and Owen Chamberlain in 1958, the pairing of matter and antimatter lead the conundrum of why we observe more matter then antimatter in the Universe. Matter and antimatter when they collide produce massive amounts of energy, $E = mc^2$ producing high energy gamma rays through the annihilation process.

Further development of the theory led to the concept of a full vacuum termed the Fermi-Dirac sea. A gamma ray can impact a heavy nucleus producing an electron-positron pair. In the Fermi-Dirac sea model of the vacuum there are the normal positive energy states $E > 0$ and zero energy states $E = 0$ as the surface of the Sea and negative vertical energy electron states, $E < 0$. The energetic photon kicks out an electron into the positive energy states, leaving a hole in the Fermi-Dirac sea. This hole is the positron.

The Fermi-Dirac sea model has numerous applications from Feynman diagram techniques to modeling semiconductor substrates [5-7]. The presence of the full vacuum picture has been useful in describing many states of matter including particularly more exotic state of matter such as plasmas. In plasmas, the energy of the ionized plasma gas, activates the electron-positron pair production by polarizing or biasing the vacuum. Using Feynman graphical techniques, one can definitely demonstrate the actual effects of the Fermi-Dirac energy sea on such plasma dielectric constant, conductivity and other properties in medium to high temperature plasmas. The fit of these plasma parameters is to the formalism including the full vacuum picture, than just the classical or semi-classical approach [7, 8]. In this chapter, we solve the Dirac equation in the Complex Minkowski 8-Space and examine conditions in which the imaginary components of the complex 8-space contribute to small nonlinear terms in the Dirac equation. We also examine the spinor calculus and the Dirac string trick in their interpretations in M^4 and C^4 space. Historical interpretation of some of the major theories in the foundation of physics are examined.

2. The Basic Structure of Physics Theories and Their Interrelation

In the attempt to develop a unified theory, the thorny issue of quantum mechanics and relativity arises as to the manner in which to find a quantum gravity formalism. The reconciliation of two distinctly structured theories, having different domains of applicability has been a conundrum to physicists for over seventy years. The basic structure of gravity, described by general relativity is a nonlinear tensor force and the basic formalism of the quantum theory is that of linear superposition. We examine this latter issue in Chap. 10 on the consideration of additional terms that introduce small nonlinear terms in the Schrödinger equation, which are formulated in terms of the complex 8-space.

Essentially the reconciliation of the formalism of gravity and quantum mechanics is essential to develop a unification of the forces and processes of nature as a "theory of everything" (TOE), see Chap. 13. Historically the development of these two uniquely different theories has their roots in the classical Hamilton-Jacobi theory. A major link between quantum and relativistic theories is the Dirac equation [9]. In figure 1 is represented the development of physics from the past and the top of the figure to current time at the bottom of the figure. The concept of canonically conjugate or paired variables obeying an Abelian algebra was developed in the Hamilton-Jacobi classical mechanics as the (p,q) phase space variables, where p is momentum and q is a spatial dimension, x [9]. This structure is fundamental to the non-Abelian algebras of the quantum theory, exemplified by the Heisenberg uncertainty principle, $\Delta \times \Delta p \geq \hbar$. Bohr's complementary principle is funda-mental to the dual paired variables (p, x) of the quantum theory. The paired variables (E, t) for energy and the temporal dimension can also be considered for $\Delta E \Delta t \geq \hbar$. The relationship between the classical Hamilton-Jacobi theory and the quantum picture is Bohr's correspondence principle. The structure of general relativity and quantum mechanics is fundamentally very distinct. The standard quantum picture involves linear superposition where as general relativity formulates non-inertial frames or gravity which is intrinsically nonlinear. Galileo's law of fallowing bodies exemplifies the nonlinearity of gravity in a very cogent manner, that is the distance of fall, s to the time, t^2 is given as $s = ½(gt^2)$ where g is the acceleration of Earth's gravity.

The Hamiltonian equations are based on energy conservation $H = T + V$ where T is the kinetic energy and V is the potential energy. For the Lagrangian, $L = T - V$. Then $H = \frac{p^2}{2m} + \lambda q^2$ so that the equation of motion is written as $\ddot{q} + \lambda q = 0$ where p is the momentum and q is the spatial variable for the canonically conjugate variables of phase space (p,q). then the Hamiltonian expressions apply,

$$\dot{p} = \frac{\partial p}{\partial t} = -\frac{\partial H}{\partial q} \tag{1a}$$

$$\dot{q} = \frac{\partial q}{\partial t} = -\frac{\partial H}{\partial p} \tag{1b}$$

See [9-11].

The concept of energy conservation lies at the center of most major physics formulations. The conservation of total energy, E_T is expressed as the sum of the kinetic and potential energies respectively as $E_T = E + V$ in classical mechanics. The conservations principle, as the first law of thermo-dynamics, has applicability in many diverse fields of knowledge such as information theory. The Schrödinger equation is a basic expression of the quantum theory and is expressed in terms of the total energy Hamiltonian as the sum of the kinetic energy, E and potential energy, V written as $H\psi + V\psi = \psi$ where ψ is the eigenfunction and E is the eigenenergy.

Energy or energy-momentum conservation is essential to both the quantum theory and general relativity. We examine, in detail the derivation of the Schrödinger equation from its classical origins [9]. We represent the origin on developments of the non-relativistic quantum theory on the right side of figure 1.

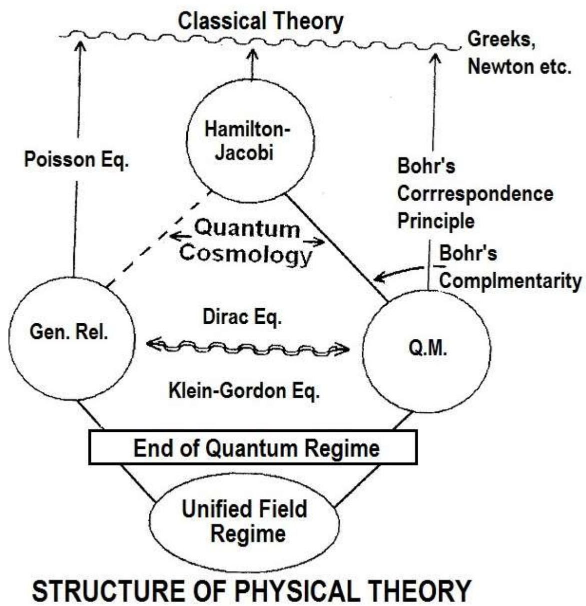

Figure 1 A schematic representation of the history and structure of the fundamental equations of physics. Earlier time to present is represented from top to bottom of the figure. Gravity may not be quantized if the quantum regime ends in a way similar to the boundary between classical mechanics and quantum mechanics.

Essentially the development of a quantum gravity theory, which forms a synergy of these two pillars of physics is fundamental to developing a unified theory; see chap.13. Supersymmetry theories, GUT and TOE theories incorporating superstring theories are approaches and attempts to unify the four force fields of the strong force, electromagnetic force, weak force (electroweak force) and gravity. We have considered the efforts of adding small nonlinear terms arising from the complexified 8D Minkowski space into the quantum picture. In Chap. 10, we examined the consequences of this approach for the structure and solutions to the Schrödinger equation. In this chapter, we examine a similar approach to complexification the Dirac equation.

In the left vertical history in Fig. 1 we represent the evolution of the structure of the general relativistic field equations from the Poisson equation. In the structure of these basic theories, conditions are required such as covariance which is basic to relativity or a relativistic quantum theory. The condition of covariance means that the equations that describe the system are constant so far as the quantities on both sides of the equation transform in the same manner covariantly. For example, the expressions of both sides of the equations must be scalars, vectors or tensors. We consider the origin of Einstein's field equations from the classical mechanics, represented on the left side of Fig. 1.

As suggested in Fig. 1, if there is a limit to the quantum regime in the same way quantum theory makes correspondence to classical mechanics; there may be no quantum gravity in the manner currently sought. Indeed, Feynman said:

> *... maybe we should not try to quantize gravity. Is it possible that gravity is not quantized and all the rest of the world is?... Now the postulate defining quantum mechanical behavior is that there is an amplitude for different processes. It cannot be that a particle which is described by an amplitude, such as an electron, has an interaction which is not described by an amplitude but by a probability...it seems that it should be impossible to destroy the quantum nature of fields. In spite of these arguments, we should like to keep an open mind. It is still possible that quantum theory does not absolutely guarantee that gravity has to be quantized.* Feynman, 1962, Lectures on Gravitation.

Gauge Theory is an approximation, which could mean there is no spin 2 graviton detectible in Minkowski space, no Higg's mechanism, no super-partners or sparticles and why no magnetic monopole has been detected. What is looked for instead are complex HD topological parameters where brane boundary conditions handle these properties in a new way as Feynman suggests.

Basic to the classical formalism of electromagnetism and relativistic physics is the Poisson equation of the form $\nabla^2\phi = 4\pi\rho$ where the divergence ∇^2 of the potential field ϕ is proportional to the energy (or energy mass) density, ρ in the space considered. The Laplace equation, $\nabla^2\phi = 0$ is written for a density free space.

From classical mechanics, we can describe the gravitational field by Poisson's equation

$$\nabla^2\phi = 4\pi\rho G \tag{2}$$

Where ϕ is the gravitational potential and ρ is the matter density for ϕ and ρ are scalars. We can generalize this equation in the linearized theory to

$$\Box\psi^{\mu\nu} = -\frac{16\pi G}{c}(\tau^{\mu\nu} + T^{\mu\nu}) \tag{3}$$

Where $\psi^{\mu\nu}$ describes the gravitational field, $\tau^{\mu\nu}$ corresponds to non-gravitational sources and the $T^{\mu\nu}$ term expresses the fact that the gravitational field can act as its own source. The $-16\pi G/c^4$ term assures that the classical limit obeys the Poisson' equation. Also, $F = c^4/G$ is the universal force [11]. In deriving Einstein's field equations, we first examine the non-relativistic limit of the linearized field equations. Assuming static conditions then $\mathrm{T}^{00} = mc^2$ the only component of the energy-momentum tensor. Neglecting $t^{\mu\nu}$ for now, we have

$$\Box\psi^{\mu\nu} \to \nabla^2\psi^{00} = -\frac{16\pi G}{c^4}\mathrm{T}^{00} = {}^{2}\!/_{F}\,\mathrm{T}^{00} = -16\pi Gm/c^2 \tag{4}$$

where T^{00} is a scalar and m is the mass having mass density ρ. To convert back into Poisson's equation, we must have $\psi^{00} = -4\phi / c^2$ where ϕ is the Newton's potential and ψ^{00} is a scalar. The 8 π and 16 π term correspond to the relativistic and non-relativistic terms, for $E = \frac{1}{2}mv^2$ or $E = mc^2$.

We can describe the gravitational field by Poisson's equation of classical mechanics $\nabla^2\phi = 4\pi\rho G$ where ϕ is the gravitational potential and ρ is the matter density. A more general form of this equation is expressed including the continuity equation (for energy, mass and charge conservation) as $\nabla^2\phi = 4\pi G(\rho + 3p / c^2)$ where *p* is momentum.

The generalized form for the above equation is given in Eq. (4) where $\psi^{\mu\nu}$ describes the gravitational field and -2 π /F insures the proper dimensionality. We now consider the solutions to Eq. (3) to demonstrate that the Poisson equation leads to Einstein's field equation solutions, we proceed as follows again utilizing Poisson's equation and the continuity equation. For

$$\nabla^2\psi = 4\pi G(\rho + 3p / c^2) \qquad (5)$$

$$\frac{d\underline{v}}{dt} = -\nabla\psi \qquad (6a)$$

then we have

$$\frac{\partial\rho}{\partial t} = -\nabla\underline{v}(\rho + p / c^{2)} \qquad (6b)$$

for $\underline{v} = \dot{S}\gamma_0$ where *S* is arbitrary term within a constant multiplication factor which depends on the time chosen so that S(t0)=1. If we define $S(t) = \frac{R(t)}{c}|C|^{1/2}$ then $S(t_0) = \frac{|C|^{1/2}}{c}$ where *R*(*t*) is the curvature of space and *C* is a constant. Using the equations for $\nabla^2\psi, \frac{d\underline{v}}{dt}$ and $\frac{\partial\rho}{\partial t}$ or Eqs. (5, 6a and 6b) or acceleration, then we have $\dot{S}^2 = \frac{8\pi G}{3}\rho S^2 - C$ and the relationship for *S*(*t*) then $\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - kc^2$ where $k = 0, \ \pm 1$ which is one of the solutions to Einstein's field equations.

It is clear that it is essential to examine the structure of the basic equations of physics that describe the micro and macro domains. Their origins from the classical Hamilton-Jacobi theory and classical concepts in general give us clues as to the manner in which to reconcile these theories and develop an approach to a unified theory. The Dirac equation stands unique in that it is relativistically Lorentz invariant. See Table 1 for force field type range and possible velocity of propagation.

Table 1 lists some types of physical phenomena, relevant forces involved and their velocity domain, *v* = 0. *v* = *c* and possibly $v = \infty$ or $v > c$ in complex 8-space as well as their theoretical speculative range. Six branches of physics are given with their forces and range. In The three domains of signal propagation as related to five branches of physics. These modes of signal propagation are manifest in other branches of physics also. We compare this to the signal propagation velocity associated with local and nonlocal phenomena.

Table 1

DOMAINS FOR $v=0,\ v=c$ AND $v=\infty$ AND BRANCHES OF PHYSICS

Branch of Physics	Domain	Type of Force	Theoretical Range
• Hamilton-Jacobi Mechanics	$v=0$	Mechanical	∞
• Electromagnetism	$v=c$	Electromagnetic	∞
• General Relativity Cosmology	$v=c$	Gravitational	∞
• Superconductivity Macro-Quantum	$v=\infty$	?	∞ ?
• Young's Double Slit Quantum Mechanics	$v=\infty$	Electromagnetic	Finite
• Bell's Nonlocality	$v=\infty$	Electromagnetic and Atomic	Infinite ?

3. The Basis and Structure of the Dirac Equation

The Dirac equation obeys the proper relativistic invariant conditions so it comprises a quantum theory that obeys relativistic constraints on the lightcone. A geometry defines a space which is an idealization of the physical 4D space of objects and momentum and locations. The lightcone with its hyperbolic topology is a covariant representation of spacetime regions. A Lie group is a topological group. For the relativistic form of the Dirac equation we use $E=(p_x^2c^2+m^2c^4)$ where E is the relativistic energy and p_x is the momentum in the x direction. We start from $E=mc^2$ and $p_x=mv_x$ so that m = E/c^2. We have $p_x/m = v_x = p_x/(E/c^2) = \dfrac{p_x c^2}{E}$. For the relativistic form of the energy

$$E=\frac{mc^2}{\sqrt{1-\beta^2}}=\frac{mc^2}{\sqrt{1-v_x^2/c^2}} \tag{7}$$

Then eliminating v_x between $v_x=\dfrac{p_c c^2}{E}$ and Eq. (7) in the form of

$$\frac{mc^2}{E}=\sqrt{1-{v_x}^2/c^2} \tag{8}$$

and then taking the inverse relation,

$$\left(\frac{E}{mc^2}\right)^2 = 1 - \frac{v_x^{\ 2}}{c^2} \ , \tag{9}$$

So that v_x is given by

$$v_x = \sqrt{(-\frac{E^2}{m^2c^4}+1)c^2} = c\sqrt{1-\frac{E^2}{m^2c^2}} \tag{10}$$

Then eliminating v_x from Eq. (10) and $v_x = p_x^2c^2/E$ so we have

$$p_x c = E\sqrt{\frac{m^2c^4 - E^2}{m^2c^4}} \tag{11}$$

Then $p^2c^2 = m^2c^4 - E^2$ so that $E = \sqrt{-p_x^{\ 2}c^2 + m^2c^4}$ so we have the usual relativistic energy equation

$$E = \sqrt{p^2c^2 + m^2c^4} \tag{12}$$

For the three components of the momentum,

$$E = \pm c\sqrt{p_x^{\ 2} + p_y^{\ 2} + p_z^{\ 2} + m^2c^2} \tag{13}$$

To derive the Dirac equation based on the two operators $p_{xop} \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial x}$ and $E_{op} \rightarrow -\frac{\hbar}{i}\frac{\partial}{\partial t}$ so that

$$-\frac{\hbar}{i}\frac{\partial}{\partial t}\psi = \pm c\sqrt{-\hbar^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + n^2c^4\psi} \tag{14}$$

where ψ is the wavefunction solution. The Hamiltonian is written as

$$H = \left(m^2c^4 + p^2c^2\right)^{1/2} \tag{15}$$

having two solutions which are given in terms of the energy equation

$$E = \pm\sqrt{m^2c^4 + p^2c^2} \tag{16}$$

which is the basic energy equation for the relativistic Dirac equation. Also, other Hamiltonian forms can be written for a charged particle in an electromagnetic field as,

$$H = [m^2c^2 + \left(cp - eA^2\right)^{1/2} + e\phi \qquad (17)$$

where A is the vector potential and ϕ is the scalar potential. Because we are dealing with a first order equation in space and time dependence, we have a square root giving two solutions, one is for the usual electron and the second is for a positive electron or positron. Dirac stuck to his two-charge solution prediction which was later verified and led to the whole concept and discovery of antimatter [13].

4. The Relativistic Dirac Equation

Proceeding from the Schrödinger equation, we express the Hamiltonian in spherical coordinates as

$$H = \frac{1}{2m}\left(p_r^2 + \frac{L^2}{r^2}\right) + V(r)\,, \qquad (18)$$

where p_r is the radial momentum $(m\dot{r})$ and L the angular momentum vector. As well known, the three components of angular momentum, derived from each other by cyclic permutation, are $L_z = xp_y - yp_x$, $L_x = yp_z - zp_y$, $L_y = zp_x - xp_z$ and $\underline{L} = \underline{r} \times \underline{\rho}$ where the total angular momentum, $L^2 = L_x^2 + L_y^2 + L_z^2$ has commutation rules $\underline{L} \times \underline{L} = i\hbar\underline{L}$ [14-16]. The SO(3) rotation generators l_1, l_2 and l_3 satisfy $l_1l_2 - l_2l_1 = l_3$, $l_2l_3 - l_3l_2 = l_1$, $l_3l_1 - l_1l_3 = l_2$; related quantum mechanically to angular momentum components L_1, L_2, L_3 with $L_x = i\hbar l_1, L_y = i\hbar l_2$ and $L_z = i\hbar l_3$ about Cartesian axes giving commutation rules $L_xL_y - L_yL_x = i\hbar L_z$, $L_yL_z - L_zL_y = i\hbar L_x$ and $L_zL_x - L_xL_z = i\hbar L_y$. Angular momentum refers to intrinsic spin about a massive particles center of mass and its magnetic moment obeys SO(3) Lie algebra which is non-Abelian acting on two component spinor wave functions $\{\psi_0(x), \psi_1(x)\} \equiv \Psi_A$; but by the uncertainty relation, $\Delta x \Delta \rho \cong \hbar$ only one set of these operators may commute at a time. Non-relativistic Fermi spin $1/2\hbar$, or simply spin ½, particles with spin angular momentum operator $s = 1/2\hbar\sigma$ can be expressed as the three anticommuting Pauli 2 x 2 spin matrices Eq. (19) satisfying $\sigma_x\sigma_y = -\sigma_y\sigma_x = i\sigma_z$ as derived empirically from the Stern-Gerlach experiments [13-18]

$$L_x = \frac{\hbar}{2}\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ L_y = \frac{\hbar}{2}\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ L_z = \frac{\hbar}{2}\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (19)$$

where the total spin operator is given as "total spin" operator, $J^2 = L_x^2 + L_y^2 + L_z^2$ commutes with all three components of L in 3D.

Spinor space and spin spaces, such as hypercharge are developed in independent topological spaces. Spinors and spin space can be complexified and occupy a hyperspace continuum. For example, the special unitary Lie groups, which are topological groups having infinitesimal elements of the Lie algebras, are utilized to represent the symmetry operations in particle physics and in infinitesimal Lorentz transformations. For example, the generators of the special unitary SU_2 group is composed of the three isospin operators, I as I_+, I_- and I_z having commutation relations $[I_+, I_-] = iI_z$. The generators of SU_3 are the three components of I, isospin, and hypercharge Y, and for other quantities which involve Y and electric charge Q. Thus, there are eight independent generators for the traceless 3 x 3 matrices of SU_3. The O_3^+ group of rotations is homomorphic to the SU_3 group. Just as in the conformal group on Minkowski space, spin space forms a two-valued representation of the Lorentz group. Note that SU_2 is the four value covering group of C(1,2), the conformal group of Minkowski space. The element of a four-dimensional space can be carried over to the complex 8-space.

For spin, n the Dirac spinor space is a covering group of SO_n where this cohomology theory will allow us to admit spin structure and can be related to the SU_2 Lie group. Now let us consider the spin conditions associated with the Dirac equation and further formulate the manner in which the Dirac "string trick" relates to the electron path having chirality [13,16,18].

Relativistic spin $1/2\,\hbar$ particles are described by Dirac's formalism for the wave equation which has been expressed by a number of notations such as

$$E\psi + c(\alpha \cdot p)\psi + mc^2\beta\psi = 0$$

or (20)

$$i\hbar\frac{\partial\psi}{\partial t} - i\hbar c\alpha \cdot \nabla\psi + mc^2\beta\psi = 0$$

for $c \neq 1$ and for the time dependent equation, which is first order in space and time with fermion particle mass, m

$$\left(-i\hbar c\alpha_\mu \frac{\partial}{\partial x_\mu} + \beta mc^2\right)\psi + \frac{i}{\hbar}\frac{\partial\psi}{\partial t} = 0. \tag{21}$$

We express the 4 x 4 Dirac α_μ and β matrices as, $\gamma_0 = \beta$, which are Hermitian and are expressed in terms of the 2 x 2 Pauli matrices, α for example

$$\gamma_0 = \beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \alpha_x = \begin{bmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{22a}$$

$$\alpha_y = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \quad \alpha_z = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{22b}$$

In the case where $m = 0$ or at very high energies, E where a particle of mass, m behaves like zero mass, only three anticommuting matrices instead of four are required. In this case the Pauli matrices are sufficient and the spinors require only 2 components which relate to the chiral representation [24]. The σ 's satisfy the equation $\sigma_x\sigma_y = -\sigma_y\sigma_x = i\sigma_z$. In general, we can write, of 2 x 2 matrices,

$$\sigma_x\sigma_y = -\sigma_y\sigma_x = i\sigma_z \tag{23a}$$

$$\sigma_y\sigma_z = -\sigma_z\sigma_y = i\sigma_x \tag{23b}$$

$$\sigma_z\sigma_x = -\sigma_x\sigma_z = i\sigma_y \tag{23c}$$

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1 \tag{23d}$$

Where $\underline{s} \times \underline{s} = is$ and $\underline{\sigma} \times \underline{\sigma} = 2i\sigma$.

The Pauli spin matrices are unitary $\sigma_x = \sigma_x^{-1}$. See Eq. (12, 19) for the 2 x 2 Dirac matrices. The Klein-Gordon equation is a 4D form where the wave function depends on (*x,y,z,t*) and is written as $\Box^2\Psi + \frac{2m}{\hbar}\Psi = 0$ where $\Box^2$ is the D'Alembertian operator, $\Box^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2 z}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2}{\partial^2 t}$ and m is the mass of the particle under consideration. Note that this equation is second order in space and time as is the classical wave equation whereas the Schrödinger equation is second order in space and first order in time in part the reason for the $i = \sqrt{-1}$ term in the equation. The first order in time term requires the *I* term in it.

We now write the Dirac equation in terms of the γ matrices. For a spin, *s*= ½ particle, the spin vector *u*(*p*) is written as $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ for spin up and spin down respectively where *p* is momentum. For a particle with mass we have $c \neq 1$. For the independent form of Eq. (21),

$$\left(-i\hbar c\alpha_\mu \frac{\partial}{\partial x_\mu} + \beta mc^2\right)\psi = 0 \tag{24}$$

for the time independent equation, and we can divide Eq. (24) by $i\hbar c$ and have,

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right)\psi = 0 \tag{25}$$

where $k_0 = p/\hbar$ or $k_0 = mc/\hbar$ and $\gamma_\mu = i\hbar c\alpha_\mu$ where indices μ run 0 to 3. The dependent Dirac equation is given in Eq. (21).

Consider spinors as basic geometrical entities that apply at a deeper level of spacetime. Spinors are complex and have real fields and real manifolds have on underlying complex nature. An essential description of nature involves complex numbers and holomorphic functions. Spinors can be mapped to twisters and visa versa. Spinors are two component entities involving the isomorphism of the conformal group and SU(2,2) which can be related to the Yang-Mills theory. The solution to the Dirac equation is in terms of spin $u(p)$ as

$$\psi = u(p)e\frac{i}{\hbar}(p \cdot \underline{x} - Et) \tag{26}$$

the Dirac spin matrices $\gamma_\mu = i\hbar c\alpha_\mu$. The spinor calculus is related to the twistor algebra, which relates a 2-space to an associated complex 8-space (see reference [25]).

An example of the usefulness of spinors is in the Dirac equation. For example, we have the Dirac spin matrices, $\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \sigma_\mu & 0 \end{pmatrix} = -i\beta\alpha_k$ where terms such as $\gamma_\mu(1-\gamma_5)$ come into the electroweak vector-axial vector formalism. The three Dirac spinors are given as,

$$L_x = \frac{\hbar}{2}\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ L_y = \frac{\hbar}{2}\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ L_z = \frac{\hbar}{2}\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{27}$$

Then $\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3 = i\gamma^0\gamma^1\gamma^2\gamma^3$ for $\gamma_0 = \beta$ is given as

$$\gamma_0 = \beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{28}$$

for trace $tr\beta = 0$, that is Eqs. (21) and (28) can be written as,

$$\gamma_0 = \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \tag{29}$$

where we have the 2 x 2 spin matrix as $I_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ for trace $I_2 = 2$. The Dirac spinors are the standard generators of the Lie algebra of SU_2. The commutation relations of the Dirac spin matrices is given as

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = ig^{\mu\nu}I \tag{30}$$

where I is the identity element and $\det\left|\gamma_{\mu\nu}\right| = \det\left|\left|g_{\mu\nu}\right|\right.$ where $g_{\mu\nu}$ is the metric tensor. The Dirac spin matrices come into use in the electroweak vector-axial vector model as $\gamma_\mu(1-\gamma_5)$ for γ_5 as,

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{31}$$

where indices run 0 to 3. We can also write,

$$\gamma_{\mu\nu}(x^5, x^\mu) = \sum_{n=-\infty}^{\infty} \gamma_{\mu\nu}^{(n)}\left(x^\nu\right)e^{inx^5} \tag{32}$$

which expresses some of the properties of the 5D Kaluza-Klein space, having $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ and γ_5.

As before stated 4 x 4 the γ matrices are Hermitian, $\gamma_\nu^* = \gamma_\mu$ and $\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu$ where $\mu \neq \nu$ and $\gamma_\mu^2 = 1$. The form of the Dirac equation in Eq. (25) is the covariant form of the wave equation. The 4-vector form for spin ½ fermions for s = ½ and $m_e = m$, the mass of the electron. The γ matrices are 4 x 4 matrices with 16 elements which obey the following relations $\left(\gamma_\mu, \gamma_\nu\right)_x = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}$ where $\delta_{\mu\nu}$ is the Kronecker delta function. The Dirac spin matrices obey Fermi-Dirac statistics, where particles such as photons obey Bose-Einstein statistics.

The γ_5 matrix is associated with a 5D metric tensor. See Chap. 11. This 5D space passes exactly one geodesic curve which returns to the same point with a continuous direction. Note that this is a similar formalism to that of the Dirac string trick 720^o path. A connection can also be made to the electromagnetic potential and the metric of the Kaluza-Klein geometry. We can express $\gamma_{\mu 5}$ in terms of a potential ϕ_μ so that

$$\gamma_{\mu 5} = \sqrt{2k\phi_\mu} \tag{33}$$

Where $k \equiv 8\pi / F$ and where $F = c^4 / G$ or the quantized cosmological force [8-10] (also see Eq. (14)). Then we have a 5-space 3-vector as,

$$\gamma_{\nu 5} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{34}$$

Through this approach, we can relate covariance and gauge invariance. See section 2.

For the covariant equation of motion in terms of ψ^ψ

$$\frac{\partial \psi^*}{\partial x_\mu}\gamma_\mu - \frac{\partial \psi^*}{\partial t}\gamma_0 + \frac{mc}{\hbar}\psi^* = 0 \qquad (35)$$

Then $\overline{\psi} = \psi * \gamma_0$ and $\gamma^* = \overline{\psi}\gamma_0$ and using Eqs. (21), (22), and (32) we can write the matrix for ψ as the complex conjugate of ψ^* for two spin states of electrons. The corresponding wave function can be written as the bispinor or 4-spinor. The 4-component function transform under rotations in exactly the same manner as the Pauli spinors. The wavefunction, ψ is four rows and one column, 4 x 1 vector matrix.

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \psi_u \\ \psi_\ell \end{pmatrix}, \qquad (36)$$

$$\psi^u = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \psi^\ell = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} \qquad (37)$$

Where the indices *u* and ℓ correspond to upper and lower respectively and are each 2-component spinors.

The γ_5 matrices are utilized in the formulation of the electroweak theory. The weak interaction Hamiltonian is formed in analogy to QED in which the Hamiltonian H is given as,

$$H_{e+m} = ie(\overline{\psi}_i \gamma_\mu \psi_j) A_\mu \qquad (38)$$

Where $\overline{\psi}_i$ is the Hermitian conjugate of ψ_j which are the eigenfunctions of the Hamiltonian and A_μ is the electromagnetic potential. In analogy to Eq. (38), the weak interaction Hamiltonian

$$H_{\text{weak}} = \overline{u}\gamma_5(1-\gamma_5)u \qquad (39)$$

where $\gamma_5\gamma_u$ is the axial vector part and the wave function is *u*.

5. The Dirac Equation in Complex 8 Space

We examine the formalism for the Dirac equation in the complex 8D space where the additional nonlinear terms arise from the imaginary components of the 8D space. The approach here is similar to that which we performed for the Schrödinger equation solved in 8D space; see Chap. 10. We proceed from the complexification of the Minkowski spacetime in which we formulated Maxwell's equations, Chaps. 5 and 6 as well as the Schrödinger equation. We identify the spinors as acting in a spin space in which spin is a conserved quantum number. Such a picture gives us understanding of the properties of spin but not its origin or source. This point is similar to that we made about charge. Physicists currently discuss the properties of charge as a conserved quantum number but the manner in which it arises is not addressed as we previously discussed. However, the origin of mass is formulated in terms of the elusive Higgs particle which may be an artifact of Gauge Theory being an approximation and might not exist.

The complex conjugate of spin space can be made since the Dirac 2 x 2 and Pauli 4 x 4 matrices are real and imaginary; hence the matrices in Eqs. (21) and (22) and their commutation relations will be effected by Eqs. (23a), (23b), (23c) and (23d). The angular momentum space will also be effected by a transformation in complex L space; see Eq. (19). Essentially formulating the Dirac equation in complex space and time utilizes the complex Minkowski formalism presented in Chap. 2. We proceed along the approach we have taken in Chap. 10 for the Schrödinger equation.

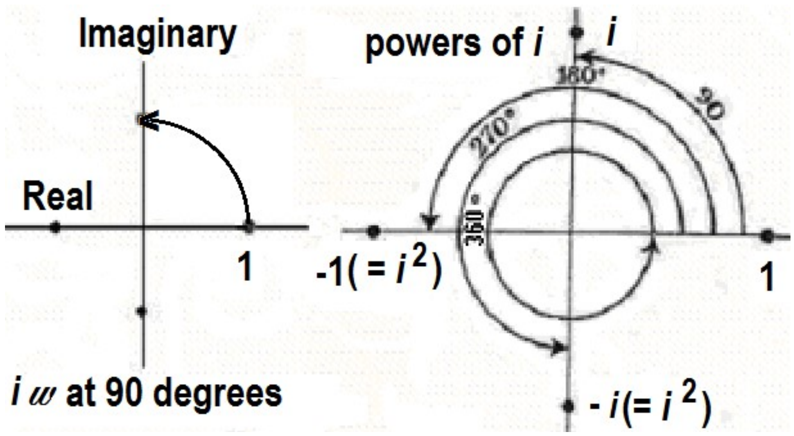

Figure 2. Through a 90° transformation Re $\rightarrow$ Im and 180° Re $\rightarrow$ -Re, for a 270° rotation Re $\rightarrow$ - Im and for a 360° rotation + Re comes back to +Re. These comprise conditions in which the 360° case is relevant to the 0° case.

5.1 COMPLEXIFYING SPIN SPACE

Complexifying spin spaces effects the Dirac spinor and Pauli matrices. These are formulated in angular momentum space, see Eq. (19). For example, the SU_3 octet with the mass splitting of the p^+ and N^0 and octet is plotted in Y spin and I_2 space. For example from Eq. (19), the Pauli matrices $\sigma_x \rightarrow i\sigma_{ix}$ $\sigma_y \rightarrow i\sigma_{iy}$ and $\sigma_z \rightarrow i\sigma_{iz}$ which satisfy the commutation relations $i\sigma_{ix} i\sigma_{iy} = -i\sigma_{iy} i\sigma_{ix} = i\sigma_{iz}$ for $i = \sqrt{-1}$ so that $-\sigma_{ix}\sigma_{iy} =$ $+\sigma_{iy}\sigma_{ix} = -\sigma_{iz}$ therefore $\sigma_{ix}\sigma_{iy} = -\sigma_{iy}\sigma_{ix} = \sigma_{iz}$, This commutation relation for the imaginary components of the σ's give a new commutation relation, that is, instead of $i\sigma_z$ we have σ_{iz}. The real components of the 2 x 2 matrices given in Eq. (19) become

$$i\sigma_{ix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad i\sigma_{iy} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad i\sigma_{iz} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \qquad (40)$$

so that now σ_{ix} and σ_{iz} become imaginary and σ_{iy} becomes real as opposed to the expression in Eq.

(19) where σ_x and σ_z are real and σ_y is imaginary. We can expand this approach to Eqs. (23a), (23), (23c) and (23d).

We can term the 4 x 4 Dirac matrices α_μ and β for Eqs. (21) and (22), as real and so is α_x and α_z but α_y is imaginary. These matrices comprise the real components of the complex 8D space. For $i\sigma_D = i\beta$ then

$$i\beta = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} \tag{41}$$

Where the trace, $tri\beta = 0$ is the real form of β.
For the imaginary part of the 4 x 4 α_μ matrices, from Eq. (22) we have,

$$i\alpha_{ix} = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix},\ i\alpha_{iy} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},\ i\alpha_{iz} = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} \tag{42}$$

Note that none of the matrix in Eq. (28) or Eq. (30) are the same as Eq. (19) or Eqs. (21) and (22). The notation for the imaginary part of $i\sigma_{ix}$ is the same as $i\sigma_{xim}$, etc. [20]. Consider Eq. (30), for γ_5, we chose the imaginary components of the γ matrices so that,

$$i\gamma_{5\mathrm{Im}} = -i\gamma_0\gamma\gamma_2\gamma_3 = -i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{43}$$

In which $\gamma_{5\mathrm{Re}} = -\gamma_{5\mathrm{Im}}$. From Eq. (34) we have the imaginary component as

$$i\gamma_{v5} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ i \end{pmatrix}. \tag{44}$$

The imaginary components of the γ matrices remain covariant under the transformation to the imaginary light cone.

Writing out the components of the γ matrices in the Dirac equation, we have

$$\left(\gamma_1 \frac{\partial}{\partial x} + \gamma_2 \frac{\partial}{\partial y} + \gamma_3 \frac{\partial}{\partial z} - \gamma_0 \frac{i}{\hbar} \frac{\partial}{\partial t}\right)\psi + \frac{mc}{\hbar}\psi = 0 \qquad (45)$$

for the time dependent form. Eq. (45) is first order in space and time. If we consider the complexification of the bispinor space and spacetime, the imaginary forms of the γ_μ functions and the spatial and temporal derivatives remain the same under a transformation, however the mass term in Eq. (45) goes from $\frac{mc}{\hbar}$ to $\frac{imc}{\hbar}$. That is the signal becomes tachyonic. Complexification produces more changes in the Schrödinger equation because it is second order in space and first order in time but since the Klein-Gordon equation and Dirac equation are the same order in space and time so that only the mass terms are effected. This holds true for the linear approximation of these equations. Nonlinearized forms can lead to distinctly different results. See next subsection and Chap. 10.

We can write the imaginary form of Eq. (45) as,

$$\left(\gamma_1 \frac{\partial}{\partial x} + \gamma_2 \frac{\partial}{\partial y} + \gamma_3 \frac{\partial}{\partial z} - \gamma_0 \frac{i}{\hbar} \frac{\partial}{\partial t}\right)\psi + \frac{+imc}{\hbar}\psi = 0 \qquad (46)$$

for the tachyonic mass, *im* summing real and imaginary components yields a factor of 2 times the components of Eq. (46) except we have the sum of tardyon and tachyon mass terms as

$$\frac{(m+im)c}{\hbar} \qquad (47)$$

and the interpretation of such a term requires further examination such as the imaginary component relating to the particle decay time of mass, *m*. Electrons are stable but other fermions, such as electron, muon and tau neutrinos, muon and tau can decay. This approach will effect our solutions to the Dirac equation; see Eqs. (36) and (37).

5.2 NONLINEAR FORMALSIM OF THE DIRAC EQUATION

In this subsection, we examine some of the properties of the Dirac equation by considering the introduction of a small nonlinear term arising from a projective geometry from the full complex 8-space, $\mathbb{C}_4$ into the 4D Minkowski space, M_4 such that the imaginary components of $\mathbb{C}_4$ are expressed in terms of a nonlinear term $g^2(\bar{\psi}\psi)\psi$ for the wave function ψ [21]. The essential properties of the complex 8-space is nonlocality and by introducing the additional imaginary components of the $\mathbb{C}_4$ space, remote spacetime connections are allowed for microscopic connections (see Chaps. 4 and 10) and macroscopic phenomena such as in Chaps. 2,5,6 and 7. It is interesting to examine the Dirac equation in this light because it is a quantum expression which is relativistically invariant.

We can write the equation of motion for a nonlinear system

$$\left(i\frac{\partial}{\partial x} - m\right)\psi + g^2(\bar{\psi}\psi)\phi = 0 \qquad (48)$$

Where $\bar{\psi}$ and ψ^* are the Hermitian and complex conjugate of ψ respectively; ψ^+ is also used for Hermitian conjugate the nonlinear term is expressed as the coupling term g^2. For the associated action variable, *S*, expressed in terms of a field $\phi(x,t)$ and its conjugate $\phi^+(x',t)$, we write

$$S = \int dt \;\; dx\left(i\phi^+\right)\phi - H \qquad (49)$$

and where $\left[\phi, \phi^+\right] = \gamma\left(x' - x\right)$ and ϕ and ϕ^+ are orthogonal to each other. The Hamiltonian, *H* for this system is given in terms of our nonlinear term g^2

$$H = \int dx\, H = \int dx \;\; \frac{\partial}{\partial x}\phi^+ \frac{\partial}{\partial x}\phi - \alpha\phi^+\phi - g^2\left(\phi^+\phi\right) \qquad (50)$$

The solutions for this equation of motion, Eq. (48) are

$$\phi(x,t) = \mathrm{A}(x,t)e^{-i\alpha t} \qquad (51)$$

$$\phi^+(x,t) = \mathrm{A}^+(x,t)e^{+i\alpha t} \qquad (52)$$

where A(*x,t*) is the wave amplitude.

We form an expression for the Dirac equation for $g^2\left(\psi^+\psi\right)$ as a small additional term as,

$$\left(i\partial_x - m\right)\psi + g^2\left(\psi^+\psi\right) = 0 \qquad (53)$$

where we use the notation, $\partial_x \equiv \partial/\partial x$. We can now write the charge density Hamiltonian as

$$H = \bar{\psi}\left(-i\gamma\frac{\partial}{\partial x} + m\right)\psi - \frac{g^2}{2}\left(\bar{\psi}\psi\right)^2 \qquad (54)$$

The Lagrangian for plane wave solution is given as

$$L = \bar{\psi}\left(i\partial_x - m\right)\psi + \frac{g^2}{2}\left(\bar{\psi}\psi\right)^2 \qquad (55)$$

where the γ Pauli spin matrices and ρ, density matrices $\rho = \psi^+\psi$. Then the lowest energy plane wave solutions are expressed in terms of spinors

$$\psi = \exp{-i\omega t}\left(\frac{u}{v}\right); \qquad (56)$$

where the spinors are $\left(\frac{u}{v}\right)$.

For the case where the coupling constant g^2 small $g \geq 0$, the attractive force for nonlinear term and ψ is the quantized Fermi field. The small nonlinear term $g^2(\psi + \psi)$ can be identified with the imaginary part of the mass, where in the linear approximation, $m_T = m = m_{Re} + im_{Im}$ where $m_T = m$ is the total mass. In Eq. (48) we associate m with the real part of the mass, m_{Re} and the additional imaginary component of the mass with m_{Im}. The imaginary component of mass may be associated with particle decay times for fermions in general.

We consider the solutions to two mass free coupled equations, where the coupling constant is expressed in terms of the nonlinear term g^2 where g^2 has two eigenvalues, g and ϕ. For our coupled equation formalism, we have wave amplitude eigenfunctions u_1 and u_2. We have considered the coupled channel formalism in nuclear physics applications with good success [22-24].

$$\frac{\partial u_1}{\partial x} + igu_1 = ig\phi u_2 \qquad (57a)$$

$$\frac{\partial u_2}{\partial x} + igu_2 = ig\phi^+ u_1 \qquad (57b)$$

The boundary conditions in the asymptotic limit on ϕ and ϕ^+ is given as ϕ and $\phi^+ \to 0,\ \lim_{x\to\infty}$.
The solutions take the form of

$$\phi(x,g) = \exp - igx + ig\int ds \quad \exp - ig(x-s)\phi(s)\phi(s,g) \qquad (58)$$

for $g^2 \to 0$, then we have,

$$\phi \to a(g)e^{-igx} \qquad (59a)$$

$$\phi^+ \to b(g)e^{igx} \qquad (59b)$$

For the case where g small perturbation expansion can be made for g^2 related to $Jm = mJm = m^*$. There is much more to explore in the richness of the Dirac theory. The Fermi-Dirac model is significant in the considerations of nonlocal coherences in plasmas and other material media and the possible relation of the vacuum concept to advanced potentials and hidden variable theories related to nonlocality such as presented in Chap. 4.

5.3 GENERALIZED WAVE EQUATIONS, CLASSICAL, QUANTUM, NONRELATIVISTIC AND RELATIVISTIC IN LINEAR AND NONLINEAR FORMS

We present a detailed comparison of the form of a number of wave equations in linear and nonlinear forms and we demonstrate their interrelationship. We summarize and discuss the structure of the

Schrödinger, Klein-Gordon and Dirac equation. The uniqueness of the properties of spin and chirality of the Dirac string trick is presented, which is unique to the Dirac formalism [25-29]. The standard wave equation is second order in space and time

$$\frac{d^2U}{dx^2} = \frac{1}{v^2}\frac{d^2U}{dt^2}, \tag{60}$$

where the amplitude, U is a function of space and time, $U(x,t)$, v is the wave velocity and the amplitude, U is expressed in terms of oscillatory solutions.

The Klein-Gordon equation is also expressed as second order in space and time as

$$\Box^2\psi + \frac{2m}{\hbar}\psi = 0 \tag{61}$$

where the D'Alembertian operator is given as

$$\Box^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{62}$$

We can write the wave mechanical treatment by revising the relativistic Klein-Gordon equation for the putative zero rest mass of the photon, m = 0 so that $\left(\nabla^2 - k^2\frac{\partial^2}{\partial t^2}\right)\phi = 0$. For $m \neq 0$, we have $m_{\mathrm{Re}} = m$ and $m_{\mathrm{Im}} = im$ terms, which may add further to the understanding of the strong force [14]. Under the action of a potential goes as 1/r for a particle of mass, m such as the pion, π particle mass $\left(\nabla^2 - k^2\frac{\partial^2}{\partial t^2} + m^2\right)\phi = 0$ then $\phi \rightarrow \left(\frac{e - cr}{r}\right)$ which yields the Yukawa potential for nuclear forces. The key is the richness of the quantum theory approach and perhaps its universality as exemplified by the Heisenberg uncertainty or indeterminacy relations and the conditions of the EPR paradox. See Chap. 4. The Sommerfeld quantization condition $\oint pdq = n\hbar$ is to the Heisenberg relations and to phase space analysis in terms of (p,q). The duality of p and x and E and t both form phase spaces. Note that we denote q generalized spatial parameter such as x and p as momentum. This phase space (p,q) approach leads to the Heisenberg indeterminacy or uncertainty principle. We may be able to relate the "phase spaces" such as (x,t), (p,E), and (x,p), (E,t), to multidimensional Fourier transforms and some physical processes [30].

The Schrödinger hypergeometric equation is formulated in terms of the second order in space and first order in time as for the potential free case,

$$\frac{\hbar\nabla^2\psi}{2m} = \frac{1}{i}\frac{\partial\psi}{\partial t} \tag{63}$$

In the case where a potential of a force is present, we have

$$\frac{\hbar\nabla^2\psi}{2m}+V\psi=\frac{1}{2}\frac{\partial\psi}{\partial t} \tag{64}$$

where we have the potential, V and $\frac{\partial\psi}{\partial t}$ is the time dependent term, where ψ is a function of the independent variable x,t as $\psi(x,t)$. For the term $\frac{1}{i}\frac{\partial\psi}{\partial t}=0$, then we have the time independent Schrödinger equation. In general, the time dependent solution is of the form

$$\psi \propto e\frac{i(kx-\omega t)}{\hbar} \tag{65a}$$

and

$$\psi^* \propto e\frac{-i(kx-\omega t)}{\hbar}. \tag{65b}$$

The quantum theory is formulated in terms of probabilities, $\psi^*\psi$ but the equations of quantum mechanics are analytic.

The Dirac equation is formulated in terms of a first order in space and time. We write the time independent Dirac equation as

$$\left(\gamma_\mu\frac{\partial}{\partial x_\mu}+\frac{mc}{\hbar}\right)\psi=0 \tag{66}$$

The γ_μ matrices are expressed in terms of the Dirac matrices, σ which are 2 x 2 matrices and the indices run 0 to 3 and the γ_μ matrices are 4 x 4 matrices. The solution to the Dirac equation takes the form $\psi=u(p)\exp\left[\frac{i}{n}(px-Et\right]$. The quantity $u(p)$ is a spinor with components $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ for spin up and spin down respectively. See Chap. 11. Since we can express the $P\mu$'s in terms of the Pauli spin matrices, α_μ which we can express in terms of the Dirac matrices, σ, we then express the Dirac equation as

$$\left(-i\hbar c\alpha_\mu\frac{\partial}{\partial x_\mu}+\beta mc^2\right)\psi=0. \tag{67}$$

The Pauli spin matrices, α_μ are expressed in terms of the Dirac 2 x 2 matrices, σ as

$$\sigma_x=\begin{vmatrix}0&1\\1&0\end{vmatrix},\quad \sigma_y=\begin{vmatrix}0&-i\\i&0\end{vmatrix} \text{ and } \sigma_z=\begin{vmatrix}1&0\\0&-1\end{vmatrix} \tag{68}$$

If we express the Klein-Gordon equation in complex 8-space, the complexification of the spatial and temporal components remain unchanged. Thus, the Klein-Gordon equation does not form extra imaginary components for the spatial and temporal second order derivatives [24]. The Dirac equation is first order in space and time. Essentially one can express the Klein-Gordon equation as a dual Dirac equation, except of course, the Dirac equation is expressed in terms of spinors, which the Klein-Gordon equation is not. Because of the electron spin symmetry conditions or the Dirac string trick in which the rotation of the system must pass through a 270° rotation [25-27,30].

The so termed Dirac string trick involves tracing the spin of an electron in space. The requirement for the electron spin and chirality to be aligned or anti-aligned along the particles direction of motion requires a 720° twist or rotation. If we rotate a 90° spin change we move from the real to imaginary axis so that a variable, ζ has a real and imaginary part, then $\zeta_{\mathrm{Re}} \to \zeta_{\mathrm{Im}}$. Through a rotation of 180° then, ζ_{Re} comes back to real again and without chirality considerations, only a phase sign charge has occurred. In the case of the Dirac spinors, symmetry requirements lead to the 270° rotation so that ζ_{Re} is now mapped into ζ_{Im} as $\zeta_{\mathrm{Re}} \to \zeta_{\mathrm{Im}}$ and hence the Dirac equation does not remain uncharged under the transformation from real spacetime, x_{Re}, t_{Re} to imaginary spacetime, x_{Im}, t_{Im}. See Fig. 2.

In the following tables we present a summary of structure of the major wave equations of physics. We enumerate a set of wave equations having classical properties. These are better linear and nonlinear equations and are classical in nature. These equations have various properties of dispersive and diffusive energy and information losses. Nonlinear terms can overcome these loss mechanisms and form coherent, non-dispersive and non-diffusive states. See Table 2.

Table 2 Linear and Nonlinear Wave Equations

- Non-dispersive – Non-diffusive wave equations $\frac{\partial\psi}{\partial t}+\frac{\partial\psi}{\partial x}=0$.
- Dispersive wave equations, where the third order term has dispersive losses $\frac{\partial\psi}{\partial t}+\frac{\partial\psi}{\partial x}=-\beta\frac{\partial^3\psi}{\partial x^3}$.
- Diffusive wave equation where the second order term has diffusional losses $\frac{\partial\psi}{\partial t}+\frac{\partial\psi}{\partial x}=\frac{\partial^2\psi}{\partial x^2}$ where β,δ are constants.
- Korteweg-deVries equation is nonlinear and is dispersive but not diffusive where the nonlinear term $\psi\partial/\partial x$ overcomes dispersive losses $\frac{\partial\psi}{\partial t}+\psi\frac{\partial\psi}{\partial x}=-\beta\frac{\partial^3\psi}{\partial x^3}$ and has soliton solutions.
- Burger's equation is nonlinear and diffusive but is not dispersive, where the nonlinear term $\psi\frac{\partial\psi}{\partial t}$ overcomes dispersive losses $\frac{\partial\psi}{\partial t}+\psi\frac{\partial\psi}{\partial x}=\delta\frac{\partial^2\psi}{\partial x^2}$.
- Nondispersive and nondiffusive, nonlinear wave equation $\frac{\partial\psi}{\partial t}+\psi\frac{\partial\psi}{\partial x}=0$.

In Table 3, we enumerate types of time dependent, time independent classical, quantum and quantum relativistic equations. All these equations are linear. If we consider a small deviation from linearity, we formulate nonlinear equations that take forms that overcome dispersive and diffusive losses. Essentially in the Everett-Graham-Wheeler Multiverse picture or in the infinite possible string theory vacuum solutions, the number of possibilities may be reduced, see [6, 23]. Selection of higher probability terms is made by inclusion of nonlinear terms in the wave equations, in some cases yielding solitary wave or soliton solutions.

Table 3 TYPES OF WAVE EQUATIONS: CLASSICAL, QUANTUM AND RELATIVISTIC

- Time dependent classical wave equation in 1D $\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$ for wave amplitude solution $u(x,t)$ and v is the classical velocity, $v << c$.
- Time dependent Klein-Gordon equation [32-35] in 3D with $m \neq 0$ $\square^2 \psi + \frac{2m}{\hbar}\psi = 0$ for $\square^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$.
- Time independent Dirac equation with $m \neq 0$. $\left(\gamma_u \frac{\partial}{\partial x_u} - \frac{mc}{\hbar}\right)\psi = 0$. The time dependent Dirac equation with $m \neq 0$: $\left(\partial_\mu \frac{\partial}{\partial x_\mu} - \frac{mc}{\hbar}\right)\psi = \frac{i}{\hbar}\frac{\partial \psi}{\partial t}$
- Time dependent Schrödinger equation $-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = -\frac{\hbar}{i}\frac{\partial \psi}{\partial t}$

  or $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ time independent Schrödinger equation for $H\psi = E\psi$ where $H = T + V$ and V is the potential energy.

In Table 4 we present the nonlinear forms of the Schrödinger and Dirac equation for both time dependent and time independent forms.

Table 4 Nonlinear Quantum Wave Equations

- Nonrelativistic nonlinear time dependent Schrödinger equation $\frac{\hbar\nabla^2\psi}{2m} + g^2\left(\psi^+\psi\right)\psi = \frac{1}{i}\frac{\partial \psi}{\partial t}$ where $g^2\left(\psi^+\psi\right)$ is the nonlinear term and ψ^+ is the Hermitian conjugate of ψ.
- Relativistic time independent Dirac equation $\left(i\frac{\partial}{\partial x} - m\right)\psi + g^2\left(\psi^+\psi\right)\psi = 0$ for the

nonlinear term $g^2\left(\psi^+\psi\right)$.

- Relativistic time dependent Dirac equation $\left(i\gamma_\mu \frac{\partial}{\partial x^u} - m\right)\psi + g^2\left(\psi^+\psi\right)\psi = \frac{i}{\hbar}\frac{\partial}{\partial t}\psi$

References and Notes

[1] Dirac, P.A.M. (1928) The quantum theory of the electron, Proc. Roy. Soc. London, Series A, 117, 778, 610.
[2] Dirac, P.A.M. (1948) Phys. Rev. 74, 817.
[3] Dirac, P.A.M. (1958) The Principles of Quantum Mechanics, 4th Edition, Oxford: Clarendon.
[4] Dirac, P.A.M. (1965) Phys. Rev. B139, 684.
[5] Feynman, R.P. (1963) *Acta. Phys. Polon*. 24, 697, Feynman developed his model when J.A. Wheeler, his Ph.D. thesis adviser suggested that all the electrons in the Universe could be considered as one electron moving rapidly forward and backward in time.
[6] Feynman, R.P. (1949) Phys. Rev. 76, 769.
[7] Rauscher, E.A. (1968) Collective electron states in quantum plasma physics, J. Plasma Phys. 2, 517.
[8] Haramein, N. & Rauscher, E.A. (2005) Collective coherent oscillating plasma models in surrounding media of black holes and vacuum structure, in R.L. Amoroso, B. Lehnert & J.P. Vigier (eds.) pp. 279-331, Orinda: The Noetic Press.
[9] Corben, H.C. & Stehle, P. (1960) Classical Mechanics, 2nd Edition, San Francisco: John Wiley and Sons.
[10] French, A.P. (1958) Principles of Modern Physics, San Francisco: John Wiley and Sons.
[11] Rauscher, E.A. (1971) A Unifying Theory of Fundamental Processes, Berkeley: UCRL-20808, LBNL-UCB Press.
[12] Westervelt, P.J. (1970) The Theory of General Relativity, Report NYO-2262-TA-218, Brown University.
[13] Rose, M.E. (1961) Relativistic Electron Theory, San Francisco: John Wiley.
[14] Messiah, A. (1999) Quantum Mechanics, Vols. 1-3, Mineola: Dover.
[15] Bohm, D. (1989) Quantum Theory, New York: Dover.
[16] Brink, D.M. & Satchler, G.R. (1968) Angular Momentum, Oxford: Oxford Univ. Press.
[17] Peebles, P.J.E. (1992) Quantum Mechanics, Princeton: Princeton Univ. Press.
[18] Schift, L.I. (1955) Quantum Mechanics, New York: McGaw Hill.
[19] Hatfield, B. (1992) Quantum Theory of Point Particles and Strings, Reading: Addison Wesley.
[20] Rauscher, E.A. (1984) The formulation of the Dirac equation in complex spacetime, Berkeley: TRL Report, PSRL-61289, June.
[21] Rauscher, E.A. (1982) Coherent, nondispersive nonlinear states for the Dirac equation, Berkeley: TRL Report, PSRL-52614.
[22] Rauscher, E.A., Rasmussen, J.O. & Harada, K. (1967) Coupled channel alpha decay beta theory applied to Po^{212m}, Nucl. Phys, A94, 331.
[23] Harada, K. & Rauscher, E.A. (1968) Unified theory of alpha decay, Phys. Rev. 169, 818.
[24] Rauscher, E.A., Seinski, A.J. & Rasmussen, J.D. (1977) Coupled channel alpha decay theory of odd-mass nuclei, Eo^{253} and Fm^{255}, Nucl. Phys, A291, 386.
[25] Dirac, P.A.M. (1935) Ann. Math. 36, 657.
[26] Dirac, P.A.M. (1952) Is There an Ether?, Nature 169, 172.
[27] Bergmann, J. (1957) Two component spinors in general relativity, Phys. Rev. 107, 624.
[28] Amoroso, R.L. (2008) Universal quantum computing: Anticipatory parameters predicting bulk implementation, in D.M. Dubois (ed.) Intl. J Computing Anticipatory Systems, 22: 254; 254-282, Proceedings CASYS07, Liege, Belgium.
[29] Amoroso, R.L. & Rauscher, E.A. (2008) Emergence of generalized F-theory 2-branes from SUSY spacetime parameters of the discrete incursive oscillator, in D.M. Dubois (ed.) Intl. J Computing Anticipatory Systems, 22: 283; 283-291. Proceedings CASYS07, Liege, Belgium.
[30] Rauscher, E.A. (1972) Generalized Heisenberg relations, Il Nuovo Cimento, 4, 757.

[31] Amoroso, R.L. & Rauscher, E.A. (2009) The Holographic Anthropic Multiverse: Formalizing the Geometry of Ultimate Reality, Singapore: World Scientific.
[32] Klein, O. (1926) *Z. Physik*, 37, 895.
[33] Gordon, W. (1926) *Z. Physik*, 38, 117.
[34] Fock, V. (1926) *Z. Physik*, 38, 2421.
[35] Fock, V. (1926) *Z. Physik*, 39, 226.