Question 435: Pi Formulas

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abstract

This note presents some formulas involving pi.

1. INTRODUCTION. The number pi is defined by

$$\pi = 4\int_0^1 \sqrt{1-x^2}\,dx = 3.1415926535... \tag{1}$$

2. SOME FORMULAS INVOLVING PI.

Let $\alpha = \exp\left(-\frac{5}{2}+\frac{1}{2}\exp\left(-\frac{5}{2}+\frac{1}{2}\exp\left(-\frac{5}{2}+...\right)\right)\right)$, $A=\frac{4}{\alpha}\left(\frac{1}{\alpha}-1\right)$. then

$$\pi = \int_0^1 \sqrt{1-x^2}\,\ln\left(1+Ax^2\right)dx \tag{2}$$

Remark: $A=\left(W\left(-\frac{1}{2}e^{-5/2}\right)\right)^{-1}\left(2+\left(W\left(-\frac{1}{2}e^{-5/2}\right)\right)^{-1}\right)$, $W(x)$ is the Lambert function.

Let $\beta = \exp\left(-\frac{3}{2}-\frac{1}{2}\exp\left(-\frac{3}{2}-\frac{1}{2}\exp\left(-\frac{3}{2}-...\right)\right)\right)$, $B=\frac{4}{\beta}\left(\frac{1}{\beta}-1\right)$. then

$$\pi = \int_0^1 \sqrt{1-x^2}\,\ln\left(1+B-Bx^2\right)dx \tag{3}$$

Remark: $B=\left(W\left(\frac{1}{2}e^{-3/2}\right)\right)^{-1}\left(\left(W\left(\frac{1}{2}e^{-3/2}\right)\right)^{-1}-2\right)$, $W(x)$ is the Lambert function.

$$\pi^2=6(\ln 2)^2+24\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(\ln\frac{4}{3}-\sum_{k=0}^{2n-1}\binom{2n}{k}\frac{(-1)^k\left(1-(3/4)^{-2n+k}\right)}{2n-k}\right) \tag{4}$$

Let $Li_2(-1/2)=\sum_{n=1}^{\infty}\frac{(-2)^{-n}}{n^2}$. then

$$\begin{aligned}\pi^2&=12(\ln 2)(\ln 3)-12(\ln 2)^2-12Li_2(-1/2)\\&+24\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(\ln\frac{3}{4}-\sum_{k=0}^{2n}\binom{2n+1}{k}\frac{(-1)^k\left(1-(3/4)^{-2n+k-1}\right)}{2n-k+1}\right)\end{aligned} \tag{5}$$

Let $0<b\le a$. then

$$\begin{aligned}\frac{\pi}{2}\ln\left(\frac{2a}{\sqrt{a^2+b^2}}\right)&=\frac{1}{\sqrt{a^2+b^2}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{(2n+1)^2}\left(\frac{a^{2n+1}-b^{2n+1}}{\left(a^2+b^2\right)^n}\right)\\&+\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{(2n+1)^2}\left(\frac{ab}{a^2+b^2}\right)^{2n+1}\end{aligned} \tag{6}$$

Example: $a=2,b=1$

$$\frac{\pi}{2}\ln\frac{4}{\sqrt{5}}=\frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{(2n+1)^2}\left(\frac{2^{2n+1}-1}{5^n}\right)+\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(2/5)^{2n+1}}{(2n+1)^2} \tag{7}$$

Let $0<b\le a$. then

$$\begin{aligned}\frac{\pi}{2}\ln\left(\frac{2a}{\sqrt{a^2+b^2}}\right)&=\frac{1}{\sqrt{a^2+b^2}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{(2n+1)^2}\left(\frac{a^{2n+1}+b^{2n+1}}{\left(a^2+b^2\right)^n}\right)\\&-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\left(\frac{b}{a}\right)^{2n+1}\end{aligned} \tag{8}$$

Example: $a=2,b=1$

$$\frac{\pi}{2}\ln\frac{4}{\sqrt{5}}=\frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{(2n+1)^2}\left(\frac{2^{2n+1}+1}{5^n}\right)-\sum_{n=0}^{\infty}\frac{(-1)^n(1/2)^{2n+1}}{(2n+1)^2} \tag{9}$$

$$\pi\left(\frac{2}{3\sqrt{3}}-\frac{1}{6}\right)+\frac{1}{3\sqrt{3}}=1-2\sum_{n=1}^{\infty}\binom{2n-2}{n-1}\frac{12^{-n}}{n}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(2n-2k+1)(2k+1)} \tag{10}$$

$$\frac{2\sqrt{2}+2\left(2-\sqrt{2}\right)\sqrt{-1}}{\pi}=\int_0^1\left(\frac{1+\sqrt{-1}}{\sqrt{2}}\right)^x dx \tag{11}$$

$$\frac{\sqrt{2}}{\pi}+\frac{\left(2-\sqrt{2}\right)\sqrt{-1}}{\pi}=\int_0^{1/2}2^x F\left(-2x,1;1;\frac{1-\sqrt{-1}}{2}\right)dx \tag{12}$$

Remark: $F\equiv {}_2F_1$ is the hypergeometric function.

Let $c_n=\int_0^{1/2}2^x\left(-2x\right)_n dx$. then

$$\frac{\sqrt{2}}{\pi}=\frac{\sqrt{2}-1}{\ln 2}+\sum_{n=1}^{\infty}\frac{2^{-n}}{n!}c_n\,\mathrm{Re}\left(\left(1-i\right)^n\right) \tag{13}$$

$$\frac{2-\sqrt{2}}{\pi}=\sum_{n=1}^{\infty}\frac{2^{-n}}{n!}c_n\,\mathrm{Im}\left(\left(1-i\right)^n\right) \tag{14}$$

Remark: $i=\sqrt{-1}$, $(a)_n=a(a+1)...(a+n-1),(a)_0=1$, $\mathrm{Re}(z)$ is the real part of z , $\mathrm{Im}(z)$ is the imaginary part of z .

Let $c_n=\int_0^1(-x)_n\,dx$, $z=1-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}$. then

$$\frac{2\sqrt{2}}{\pi}=1+\sum_{n=1}^{\infty}\frac{c_n}{n!}\mathrm{Re}\left(z^n\right) \tag{15}$$

$$\frac{2\left(2-\sqrt{2}\right)}{\pi}=\sum_{n=1}^{\infty}\frac{c_n}{n!}\mathrm{Im}\left(z^n\right) \tag{16}$$

Let $c_n=\int_0^1(-x)_n\,dx$, $z=1-\frac{\sqrt{3}}{2}-\frac{i}{2}$. then

$$\frac{3}{\pi}=1+\sum_{n=1}^{\infty}\frac{c_n}{n!}\mathrm{Re}\left(z^n\right) \tag{17}$$

$$\frac{6-3\sqrt{3}}{\pi}=\sum_{n=1}^{\infty}\frac{c_n}{n!}\operatorname{Im}\left(z^n\right) \tag{18}$$

$$\pi\ln 34=\int_0^{\pi/2}\ln\left(169+126(\tan x)^2+25(\tan x)^4\right)dx \tag{19}$$

$$\pi\ln 34=\int_0^{\infty}\frac{\ln\left(169+126x^2+25x^4\right)}{1+x^2}dx \tag{20}$$

$$\pi\ln\frac{34}{13}=\int_0^{\infty}\frac{1}{169+x}\tan^{-1}\left(\sqrt{\frac{63+\sqrt{3969+25x}}{x}}\right)dx \tag{21}$$

$$\frac{\sqrt{\pi}}{4}=\int_0^{\infty}\left(1-e^{-\left(\sqrt{1+x^2}-x\right)^2}-e^{-\left(\sqrt{1+x^2}+x\right)^2}\right)dx \tag{22}$$

$$\frac{3\sqrt{\pi}}{4}=1+\int_1^{\infty}\left(1+e^{-\left(x+\sqrt{1+x^2}\right)^2}-e^{-\left(x-\sqrt{1+x^2}\right)^2}\right)dx \tag{23}$$

$$\frac{\pi^3}{8}-2G=\int_{-1/2}^{\infty}\tan^{-1}\left(\exp\left(\frac{x}{1-\sqrt{1+2x}}\right)\right)dx-\int_{-1/2}^{0}\tan^{-1}\left(\exp\left(\frac{x}{1+\sqrt{1+2x}}\right)\right)dx \tag{24}$$

Remark: $\exp(x)\equiv e^x$, G is the Catalan's constant.

Let $a=-\frac{e^2\left(e^2-2\right)}{\left(e^2-1\right)^2}$. then

$$\pi=\int_0^1\frac{\sin^{-1}x}{x\left(1+ax^2\right)}dx \tag{25}$$

Let $a=-\frac{2}{3}-\frac{1}{12}\left(26+6\sqrt{33}\right)^{1/3}+\frac{2}{3}\left(26+6\sqrt{33}\right)^{-1/3}$. then

$$\pi=\int_0^1\frac{x\sin^{-1}x}{\left(1+ax^2\right)^2}dx=\int_0^1\frac{x\cos^{-1}x}{\left(1+ax^2\right)^2}dx \tag{26}$$

Let $a=\frac{1}{2}\left(5-e^{-5+e^{-5+\cdots}}\right)=\frac{5}{2}+\frac{1}{2}W\left(-e^{-5}\right)$. then

$$\pi = \int_0^\infty \frac{(\sin ax)^2}{x^2(1+x^2)}\,dx \tag{27}$$

$$\frac{\sqrt{\pi}\left(2+\sqrt{2}\right)}{8} = \int_0^1 \sqrt{\ln\frac{2}{\sqrt{1+8x}-1}}\,dx \tag{28}$$

$$\sqrt{\pi}\left(2-\sqrt{2}\right) = \int_0^1 \left(\sqrt{\ln\frac{2}{1+\sqrt{x}}} - \sqrt{\ln\frac{2}{1-\sqrt{x}}}\right)dx \tag{29}$$

$$\frac{\pi}{2} - \frac{1}{3} - \frac{1}{12\sqrt{2\pi}}\left(\Gamma\left(\frac{1}{4}\right)\right)^2 = \int_0^1 \sin^{-1}\left(\left(\frac{4x}{1+\sqrt{1+8x}}\right)^{2/3}\right)dx \tag{30}$$

$$\frac{2}{3\sqrt{2\pi}}\left(\Gamma\left(\frac{1}{4}\right)\right)^2 - \frac{8}{3} = \int_0^1 \left(\sin^{-1}\left(\left(\frac{1+\sqrt{x}}{2}\right)^{2/3}\right) - \sin^{-1}\left(\left(\frac{1-\sqrt{x}}{2}\right)^{2/3}\right)\right)dx \tag{31}$$

Let $a \geq 0$. then

$$\sqrt{\pi} = 2\int_0^a e^{-x^2}\,dx + 4\sum_{n=0}^\infty \frac{1}{2n+1}\int_a^\infty \left(\frac{1-e^{-e^{-x^2}}}{1+e^{-e^{-x^2}}}\right)^{2n+1} dx \tag{32}$$

Let $a \geq 0$. then

$$\sqrt{\pi} = 2\int_0^a e^{-x^2}\,dx + 2\sum_{n=1}^\infty \frac{1}{n}\int_a^\infty \left(1-e^{-e^{-x^2}}\right)^n dx \tag{33}$$

Let $a > \sqrt{-\ln(\ln 2)}$. then

$$\sqrt{\pi} = 2\int_0^a e^{-x^2}\,dx + 2\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n}\int_a^\infty \left(e^{e^{-x^2}} - 1\right)^n dx \tag{34}$$

Let $0 \leq u \leq \frac{\ln 2}{2}, 0 \leq v \leq \infty, v = -\ln\left(2\sqrt{2}\sinh u\right), u = \sinh^{-1}\left(\frac{e^{-v}}{2\sqrt{2}}\right)$. then

$$\frac{\pi^2}{24} - \frac{(\ln 2)^2}{8} = uv - \int_u^{\ln 2/2} \ln\left(2\sqrt{2}\sinh x\right)dx + \int_v^\infty \sinh^{-1}\left(\frac{e^{-x}}{2\sqrt{2}}\right)dx \tag{35}$$

Let

$$f\left(x\right)=3088-2124x+12\sqrt{3}\sqrt{52500-11800x+14219x^{2}+256x^{3}} \qquad (36)$$

$$g\left(x\right)=-3088+2124x+12\sqrt{3}\sqrt{52500-11800x+14219x^{2}+256x^{3}} \qquad (37)$$

$$h\left(x\right)=\frac{2}{3}+\frac{1}{6}\sqrt[3]{f\left(x\right)}-\frac{1}{6}\sqrt[3]{g\left(x\right)} \qquad (38)$$

then

$$\frac{\pi}{5}\left(\frac{6}{\sqrt{7}}-\frac{1}{\sqrt{2}}\right)= \\ =\int_{1}^{\infty}\sqrt{-5-\sqrt{17+4h\left(x\right)}+\sqrt{42-4h\left(x\right)+10\sqrt{17+4h\left(x\right)}+8\sqrt{\left(h\left(x\right)\right)^{2}+4x-4}}}\,\frac{dx}{x^{2}} \qquad (39)$$

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